Note on the Bertotti–Robinson electromagnetic universe[a)]

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It is shown in a concise manner using Debever's vectorial formalism that the Bertotti–Robinson solution is the most general conformally flat solution of the source free Einstein–Maxwell equations for nonnull electromagnetic fields.

1. INTRODUCTION

This note concerns a solution of the source free Einstein–Maxwell equations, which, with a suitable choice of units, may be written as follows:

$$R_{ab}=F_{ac}F_b{}^c-\tfrac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{1.1a}$$

$$F_{ab;}{}^b=F_{[ab;c]}=0. \tag{1.1b}$$

Robinson[1] presented the following solution to these equations:

$$ds^2=(\lambda x^1dx^0)^2+2dx^0dx^1-(dx^2)^2-(\cos\lambda x^2dx^3)^2, \tag{1.2a}$$

$$F_{ab}=\sqrt{2}\lambda(\delta^{01}_{ab}\cos\mu+\delta^{23}_{ab}\cos\lambda x^2\sin\mu). \tag{1.2b}$$

He observed that the electromagnetic field (1.2b) is covariantly constant and that under the change of coordinates $\lambda x^i=(t-r,1/r,\pi/2-\theta,\psi)$ the metric (1.2a) takes the form

$$ds^2=(\lambda r)^{-2}[dt^2-dr^2-r^2(d\theta^2+\sin^2\theta d\psi^2)], \tag{1.3}$$

showing that the space is conformally flat. Bertotti[2] found independently the same solution in a different coordinate system as a solution to equations (1.1) for a covariantly constant electromagnetic field in the presence of a cosmological constant. Eardley[3] has characterized the solution as the only solution of equations (1.1) admitting a covariantly constant nonnull electromagnetic field. Cahen and Leroy[4] obtained a solution equivalent to (1.2a) as a limiting case of Petrov type N solutions of Eqs. (1.1) for nonnull electromagnetic fields. The space–time with metric (1.2a) has been characterized by Cahen and McLenaghan[5] as the only conformally flat space–time with a covariantly constant Riemann tensor and $R_{ab}R^{ab}=R=0$. The solution also appears among the Schrödinger separable solutions of the Einstein–Maxwell equations found by Carter[6] and in the list of space-times with local isotropy given by Cahen and Defrise.[7] The solution is a metric product of two two-dimensional spaces of constant curvature and hence admits a six-parameter group of motions. The properties of this solution have also been investigated by Lindquist,[8] Lovelock,[9] and Dolan.[10]

In a recent article Tariq and Tupper[11] characterize the the solution (1.2) as the most general conformally flat solution of the source free Einstein–Maxwell equations for a nonnull electromagnetic field. It is the purpose of this note to present an alternative proof of this result, which seems to be more direct, using the vectorial formalism of Debever.[12]

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2. NOTATION AND CONVENTIONS

Let θ^i $(i=0,1,2,3)$ denote a tetrad of null 1-forms with θ^0 and θ^3 real and $\theta^1=\overline{\theta^2}$ complex. In the tetrad the metric of V_4 has the form

$$ds^2=2\theta^0\theta^3-2\theta^1\theta^2. \tag{2.1}$$

The absence of torsion of the pseudo-Riemannian connection is expressed by the equation

$$d\theta^i+\omega^i{}_j\wedge\theta^j=0, \tag{2.2}$$

where $\omega^i{}_j$ denote the 1-form valued components of the connection. A basis of the space of self-dual 2-forms is given by

$$Z^1=\theta^2\wedge\theta^3,\quad Z^2=\theta^0\wedge\theta^1,\quad Z^3=\tfrac{1}{2}(\theta^0\wedge\theta^3-\theta^1\wedge\theta^2). \tag{2.3}$$

The metric in this space is

$$\gamma^{\alpha\beta}=2\delta_1^{(\alpha}\delta_2^{\beta)}-\tfrac{1}{2}\delta_3^\alpha\delta_3^\beta\quad(\alpha,\beta=1,2,3). \tag{2.4}$$

In terms of the basis (2.3) Eq. (2.2) has the form

$$dZ^\alpha+\sigma^\alpha{}_\beta\wedge Z^\beta=0. \tag{2.5}$$

The quantities $\sigma^\alpha{}_\beta$ are related to the $\omega^i{}_j$ by

$$\sigma^2{}_2=-\omega^2{}_2-\omega^3{}_3,\quad \sigma^2{}_3=2\omega^1{}_3,\quad \sigma^1{}_3=2\omega^3{}_1. \tag{2.6}$$

To the 1-form valued matrix $\sigma^\alpha{}_\beta$ we may associate the vectorial 1-form

$$\sigma^\alpha=\tfrac{1}{2}\epsilon^{\alpha\beta\gamma}\gamma_{\beta\delta}\sigma^\delta{}_\gamma, \tag{2.7}$$

where $\epsilon^{\alpha\beta\gamma}$ is the three-dimensional Levi-Civita symbol. The components of σ^α with respect to the basis $\{\theta^i\}$ are denoted by $\sigma^\alpha{}_i$. They are proportional to the NP spin coefficients.[13] The vectorial curvature 2-form Σ_α is defined by the relations

$$\Sigma_1=d\sigma^2-\sigma^2\wedge\sigma^3,\quad \Sigma_2=d\sigma^1+\sigma^1\wedge\sigma^2,$$
$$\Sigma_3=-2d\sigma^3-\sigma^1\wedge\sigma^2. \tag{2.8}$$

Writing Σ_α in the basis $\{Z^\alpha\}$ yields

$$\Sigma_\alpha=(C_{\alpha\beta}-\tfrac{1}{6}R\gamma_{\alpha\beta})Z^\beta+E_{\alpha\bar\beta}\overline{Z}^\beta, \tag{2.9}$$

where the trace-free symmetric tensor $C_{\alpha\beta}$ corresponds to the Weyl tensor, the Hermitian tensor $E_{\alpha\bar\beta}$ to the trace-free Ricci tensor, and R denotes the curvature scalar. The components of $C_{\alpha\beta}$ and $E_{\alpha\bar\beta}$ are proportional to the NP scalars.[14]

Bianchi's identities have the form

$$(DC_{\alpha\beta}-\tfrac{1}{6}\gamma_{\alpha\beta}dR)\wedge Z^\beta+DE_{\alpha\bar\beta}\wedge\overline{Z}^\beta=0, \tag{2.10}$$

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where

$$DC_{\alpha\beta}=dC_{\alpha\beta}-C_{\gamma\beta}\sigma^{\gamma}{}_{\alpha}-C_{\alpha\gamma}\sigma^{\gamma}{}_{\beta}, \tag{2.11}$$

$$DE_{\alpha\bar\beta}=dE_{\alpha\bar\beta}-E_{\gamma\bar\beta}\sigma^{\gamma}{}_{\alpha}-E_{\alpha\bar\gamma}\bar\sigma^{\gamma}{}_{\beta}. \tag{2.12}$$

Finally the Einstein–Maxwell field equations may be written as

$$E_{\alpha\bar\beta}=-2F_{\alpha}\bar F_{\beta}, \tag{2.13}$$

$$R=0, \tag{2.14}$$

$$d\overset{+}{F}=0, \tag{2.15}$$

where

$$\overset{+}{F}=F_{\alpha}Z^{\alpha} \tag{2.16}$$

is the self-dual part of the electromagnetic field 2-form.

3. DERIVATION OF THE RESULT

We choose the 1-forms θ^0 and θ^3 to be proportional to the characteristic 1-forms of the nonnull electromagnetic field. With this choice

$$\overset{+}{F}=F_3Z^3, \tag{3.1}$$

and the Einstein–Maxwell equations take the form

$$R=E_{\alpha\bar\beta}=0\quad(\alpha=1,2,3,\ \beta=1,2), \tag{3.2a}$$

$$E_{3\bar3}=-2F_3\bar F_3. \tag{3.2b}$$

The tetrad is determined by the above choice modulo the transformations

$$\theta^{0\prime}=e^{a}\theta^0,\quad \theta^{1\prime}=e^{ib}\theta^1,\quad \theta^{3\prime}=e^{-a}\theta^3. \tag{3.3}$$

The assumption of conformal flatness is equivalent to

$$C_{\alpha\beta}=0\quad(\alpha,\beta=1,2,3). \tag{3.4}$$

With the help of Bianchi's identities (2.10), (2.11), and (2.12) we obtain

$$\sigma^2\wedge\bar Z^3=\sigma^1\wedge\bar Z^3=dE_{3\bar3}\wedge\bar Z^3=0. \tag{3.5}$$

It follows that

$$\sigma^1=\sigma^2=0, \tag{3.6}$$

$$E_{3\bar3}=-k, \tag{3.7}$$

where $k>0$ is a real constant.

The conditions (3.6) allow a first form of the metric to be obtained. From Eqs. (2.2), (2.6), and (2.7) we obtain

$$\begin{aligned}d\theta^0&=-\tfrac12(\sigma^3+\bar\sigma^3)\wedge\theta^0,\\ d\theta^1&=-\tfrac12(\sigma^3-\bar\sigma^3)\wedge\theta^1,\\ d\theta^3&=\tfrac12(\sigma^3+\bar\sigma^3)\wedge\theta^3.\end{aligned} \tag{3.8}$$

It follows that

$$d\theta^i\wedge\theta^i=0\quad(i=0,1,3). \tag{3.9}$$

This implies that there exists a system of coordinates $(u,z,\bar z,v)$, real valued functions e and g, and a complex valued function F such that

$$\theta^0=e\,du,\quad \theta^1=F\,dz,\quad \theta^3=g\,dv. \tag{3.10}$$

We now use the remaining tetrad freedom (3.3) to set

$$\theta^0=du,\quad \theta^1=f\,dz,\quad \theta^3=g\,dv, \tag{3.11}$$

where f is a real valued function of the coordinates. From (3.8) and (3.11) we deduce

$$\sigma^3{}_1=-\bar\sigma^3{}_2=f^{-2}f_z,\quad \sigma^3{}_0=g^{-1}g_u,\quad \sigma^3{}_3=0, \tag{3.12}$$

$$f_u=f_v=g_z=0. \tag{3.13}$$

Thus the general form of the metric satisfying (3.6) is

$$ds^2=2g\,du\,dv-2f^2\,dz\,d\bar z, \tag{3.14}$$

where $f=f(z,\bar z)$ and $g=g(u,v)$ are real valued functions which without loss of generality can be taken to be positive. In order to determine these functions, it remains to solve the field equations (2.8) which on account of (3.2), (3.4), and (3.7) reduce to

$$4g^{-1}(\log g)_{uv}=-k, \tag{3.15}$$

$$8f^{-2}(\log f)_{z\bar z}=-k. \tag{3.16}$$

By means of the substitution $g=e^{m\theta}$, where $m=-\frac14 k$, Eq. (3.15) becomes

$$\theta_{uv}=e^{m\theta}. \tag{3.17}$$

This equation has the general solution (Forsyth[15])

$$e^{m\theta}=2m^{-1}(\phi+\psi)^{-2}\phi_u\phi_v, \tag{3.18}$$

where ϕ and ψ are arbitrary functions of u and v respectively. If we now choose ϕ and ψ as coordinates, we have

$$\theta^0\theta^3=-8k^{-1}(\phi+\psi)^{-2}\,d\phi\,d\psi, \tag{3.19}$$

which after a further coordinate transformation becomes

$$\theta^0\theta^3=(1+\tfrac18 kuv)^{-2}du\,dv. \tag{3.20}$$

In order to solve Eq. (3.16), we make the substitution $f^2=e^{-kw}$. The transformed equation is

$$4w_{z\bar z}=e^{-kw}. \tag{3.21}$$

It has the general solution (Forsyth[16])

$$e^{w}=\tfrac18 k(1+X\bar X)^2(X_z\bar X_{\bar z})^{-1}, \tag{3.22}$$

where X is an arbitrary analytic function of z. On choosing X as a new coordinate we have

$$\theta^1\theta^2=8k^{-1}(1+X\bar X)^{-2}dX\,d\bar X, \tag{3.23}$$

which by means of a simple coordinate transformation takes the form

$$\theta^1\theta^2=(1+\tfrac18 kz\bar z)^{-2}dz\,d\bar z. \tag{3.24}$$

It remains to solve Maxwell's equations (2.15). It follows from (2.5), (3.6), and (3.12) that $dZ^2=0$. Thus from (2.16) and (3.1) we conclude that $F_3=\text{const}$. With the help of (3.2b) and (3.7) we may write

$$F_3=\sqrt{(k/2)}\,e^{i\phi}, \tag{3.25}$$

where ϕ is a real constant.

4. CONCLUSION

In a conformally flat space–time in which the source free Einstein–Maxwell equations (1.1) are satisfied for a nonnull electromagnetic field there exists a system of coordinates $(u,z,\bar z,v)$ with respect to which the metric and electromagnetic field have the form

$$ds^2 = 2(1+\tfrac{1}{8}kuv)^{-2}du dv - 2(1+\tfrac{1}{8}kz\bar{z})^{-2}dz d\bar{z}, \qquad (4.1)$$

$$\overset{+}{F} = (\tfrac{1}{8}k)^{1/2}e^{i\phi}[(1+\tfrac{1}{8}kuv)^{-2}du\wedge dv - (1+\tfrac{1}{8}kz\bar{z})^{-2}dz\wedge d\bar{z}], \qquad (4.2)$$

where $k>0$ and ϕ are arbitrary constants. By means of the coordinate transformation

$$u=\sqrt{2}x^0, \quad v = x^1[\sqrt{2}(1-\tfrac{1}{2}\lambda^2 x^0 x^1)]^{-1},$$

$$z = \sqrt{2}\lambda^{-1}e^{i\lambda x^2}\cot(\tfrac{1}{2}\lambda x^2 - \tfrac{1}{4}\pi),$$

where $\lambda = \frac{1}{2}\sqrt{k}$, we recover Robinson's metric (1.2a) and electromagnetic field (1.2b).

Note added in proof. Dr. H. Stephani has kindly drawn the authors' attention to a paper[17] where the Bertotti–Robinson electromagnetic universe is characterized in the same way as Tariq and Tupper[18] using properties of an embedding in a flat six-dimensional space.

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Correlation functions and higher order coherence in inelastically scattered quantum radiation

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The concept of coherence is analyzed in a system of interacting radiation and matter. Using the projection technique, the reduced field density operator is found, and with it the first and higher order field correlation functions are computed. It is proved that the inelastically scattered field is, for the most part, incoherent in any order at any time interval after collision with an atomic ensemble, except some specific time intervals and positions, where the order of coherence is determined by the atomic correlations considered.

Correlations play a fundamental role in the concept of higher order coherence in the Glauber sense.[1] The present paper reports some results in exploring this concept of coherence in a system of interacting radiation and matter.

The question is whether an initially coherent laser beam preserves its coherence (and if so, to what order) after being inelastically scattered by an atomic ensemble. The answer—as can be expected—given by this paper is, in general, also negative. The scattered field is incoherent to any order in most of the time intervals after the collision at most of the points different from the origin of the scattering system.

The scattered field is coherent, however, in some specific time intervals and positions; the order of coherence in these cases is determined by the atomic correlations due—in the first approximation—to the dipole–dipole interactions among the particles in the scattering system. The analysis of these specific cases presents considerable interest.

Let us consider an ensemble of N identical three-level atoms (as particle system), ground state $|g\rangle$, intermediate state $|h\rangle$, final state $|f\rangle$, interacting with incident (exciting) coherent radiation (we shall only discuss the electric component)

$$\bar{E}_i(\bar{r};t)=i\left(\frac{\hbar\omega_k}{2\epsilon_g V}\right)^{1/2}\hat{e}^{(\lambda)}[a_k(0)e^{i\varphi}-a_k^+(0)e^{-i\varphi}], \qquad (1)$$

where we use Glauber's notations[1,2] and $\varphi=(\bar{k}.\bar{r}-\omega_k t)$.

The scattered field—after resonant Raman scattering the atomic ensemble—is

$$\bar{E}_s(\bar{r}_j;t_j)=i\sum_k\left(\frac{\hbar\omega_k}{2\epsilon_g V}\right)^{1/2}\hat{e}^{(\lambda')}[a_k(0)e^{i\varphi_j}-a_k^+(0)e^{-i\varphi_j}], \qquad (2)$$

with

$$\varphi_j=(\bar{k}_\circ\bar{r}_j-\omega_k t_j).$$

The necessary and sufficient conditions for coherence in the above-mentioned sense are connected with some properties—namely normalization and factorization properties—of the jth order field correlation functions,

$$G^{(j)}(x_1;\ldots;x_{2j}) =\mathrm{tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_j)E^{(+)}(x_{j+1})\cdots E^{(+)}(x_{2j})\}, \qquad (3)$$

where $x_j=(\bar{r}_j;t_j)$ and $E^{(-)}(x_j)$, $E^{(+)}(x_y)$ are respectively the negative and positive frequency parts of $\bar{E}(\bar{r}_j;t_j)$ and ρ is the density operator of the whole system in the Heisenberg representation. We further suppose that all of our detectors are fitted with polarizers and record only photons polarized parallel to an arbitrary unit vector $\hat{e}$. Thus

$$E^{(-)}(x_j)=\hat{e}.\bar{E}^{(-)}(x_j),\quad E^{(+)}(x_j)=\hat{e}.\bar{E}^{(-)}(x_j), \qquad (4)$$

However, to find the correlation functions $G^{(j)}$, we do not need the density operator of the whole system. All the information needed is contained in the reduced field density operator $\sigma(t)$.

To find $\sigma(t)$ we use the projection technique,[3]

$$i\hbar\frac{\partial\rho(t)}{\partial t}=\mathcal{H}\rho(t), \qquad (5)$$

with $\mathcal{H}$ the Liouville operator for the whole Hamiltonian

$$H=H_F+H_A+V, \qquad (6)$$

where

$$H_F=\hbar\sum_k\omega_k a_k^+a_k, \qquad (7)$$

and

$$H_A=\sum_{l=g}^{f}\epsilon_l\mathcal{N}_l,\quad \mathcal{N}_l=\sum_{m=1}^{N}|l\rangle\langle l|_m, \qquad (8)$$

are the field and atomic Hamiltonians respectively.

If we let $\bar{\mu}_{hf}=\langle h|e\bar{r}|f\rangle$ be the dipole matrix element between the atomic states $|h\rangle$ and $|f\rangle$, and we let μ_{hf} be the component of the dipole moment in the direction of the electric field, the interaction Hamiltonian can be written as

$$V=\sum_k\sum_m i\hbar\mu_{hf}[a_k^+|h\rangle\langle f|_m-a_k|f\rangle\langle h|_m], \qquad (9)$$

where the μ_{gh} and μ_{gf} are missing, because the field induces radiative transitions between the states $|h\rangle$ and $|f\rangle$ only. Converting (5) into the interaction picture and then successively applying the tr_A and $\mathcal{D}$ operations on it we have

$$\frac{\partial\sigma(t)}{dt}=-i\,\mathrm{tr}_A\gamma f(H_A)\sigma(t) -\mathrm{tr}_A\gamma\int_0^t d\tau\, e^{-i\mathcal{D}\gamma\tau}\mathcal{D}\gamma f(H_A)\sigma(\tau), \qquad (10)$$

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where γ is the Liouville operator for $V(t)$, $\mathcal{D}=[1-f(H_A)\,\mathrm{tr}_A]$ is a projector, with $f(H_A)=e^{-\beta H_A}/\mathrm{tr}_A\, e^{-\beta H_A}$, and tr_A means trace over the atomic ensemble. This non-Markoffian equation (10) is valid for all times and for any orders in $V(t)$, and has in the second term on the right-hand side a generalized collision operator containing the memory of the system.

The solution of (10) up to the second order terms in $V(t)$ is performed elsewhere[4] and results in a reduced density operator in the Heisenberg representation

$$\sigma(a_k, a_k^+, t)=\pi\sum_k\sum_{k^2} e^{-(F/4)k^2 t} e^{-[(B-F)/F]a_k a_k^+} \times \mathcal{J}_0(k(a_k a_k^+)^{1/2}), \quad (11)$$

where $k^2=2(n_x+n_y)$, $n_x, n_y=0,1,2,\cdots$, $\mathcal{J}_0(k(a_k a_k^+)^{1/2})$ are the zeroth order Bessel functions, and B and F are

$$B=\hbar^2\mu_{hf}\mu'_{fh}\,2\pi g(\omega_k)(gh),$$
$$F=\hbar^2\mu_{hf}\,\mu'_{fh}\,2\pi g(\omega_k)(gf), \quad (12)$$

with $g(\omega_k)$ a weight function[5] and (gh), (gf) the atomic correlation functions

$$(gh)=\mathrm{tr}_A\sum_{\substack{m=1\\l=1}}^{N}\left(|h\rangle\langle f|_m|f\rangle\langle h|_l\frac{e^{-\beta H_A}}{\mathrm{tr}_A e^{-\beta H_A}}\right),$$
$$(gf)=\mathrm{tr}_A\sum_{\substack{m=1\\l=1}}^{N}\left(|h\rangle\langle f|_m\frac{e^{-\beta H_A}}{\mathrm{tr}_A e^{-\beta H_A}}|f\rangle\langle h|_l\right), \quad (13)$$

respectively.

Inserting (11) and (2) into (3) we have

$$G^{(j)}(x_1;\ldots;x_{2j})=f(x_1;\ldots;x_{2j})\sum_{k^2}\sum_{n=1}^{N}\frac{(n+j-1)!}{(n-1)!} \times e^{-[(B-F)/F](n+1)}\,\mathcal{J}_0(k\sqrt{n+1}), \quad (14)$$

the jth order field correlation functions, where

$$f(x_1;\ldots;x_{2j})=\frac{\pi\hbar^j}{2^j}(\omega_{k_1}\cdots\omega_{k_{2j}})^{1/2} \times\exp[i(\varphi_{j+1}+\cdots+\varphi_{2j}-\varphi_1-\cdots-\varphi_j)] \times e^{-(k^2/4)Ft}. \quad (15)$$

The necessary conditions for coherence in the Glauber's sense[1] are that the normalized correlation functions all have unit absolute magnitude

$$|g^{(j)}(x_1;\ldots;x_{2j})|=\frac{G^{(j)}(x_1;\ldots;x_{2j})}{\Pi_{i=1}^{2j}\{G^{(1)}(x_i;x_i)\}^{1/2}}=1, \quad (16)$$

while the sufficient conditions are related to the factorization property

$$G^{(j)}(x_1;\ldots;x_{2j}) =\mathcal{E}^*(x_1)\cdots\mathcal{E}^*(x_j)\mathcal{E}(x_{j+1})\cdots\mathcal{E}(x_{2j}), \quad (17)$$

where $\mathcal{E}(x_j)$ are complex functions.

The normalized field correlation functions (16) computed with (14) and (15),

$$g^{(j)}(x_1;\ldots;x_{2j})=\sum_{k^2}\sum_n \exp\left(-\frac{k^2}{8}F\{(t_{j+1}+\cdots+t_{2j}) -3(t_1+\cdots+t_j)\}\right) \times\exp[-i(\varphi_{j+1}+\cdots+\varphi_{2j}-\varphi_1-\cdots-\varphi_j)] \times\frac{n(n+1)\cdots(n+j-1)}{n^i} \times\frac{e^{(j-1)[(B-F)/F](n+1)}}{\mathcal{J}_0^{(i-1)}}, \quad (18)$$

have in general an absolute magnitude different from one, and the $G^{(j)}(x_1;\ldots;x_{2j})$ field correlation functions don't factorize into a product of complex eigenvalues as it is required by (17). Thus for the most part of the space–time after collision, neither conditions (16) nor (17) are met in this resonant Raman scattering problem.

Therefore, the conclusion—as it was expected—is that the scattered radiation at most of the time intervals after t_1 (incident time) and at most of the points different than r_1 (incident position) *is in general incoherent*.

The scattered field is coherent, however, in some time intervals and positions other than $(\bar{r}_1;t_1)$ depending on the specific B and F values of the atomic correlations considered. This can be seen by analyzing the behavior of the normalized field correlation functions (18).

When the conditions $(t_{j+1}+\cdots+t_{2j})-3(t_1+\cdots+t_j)$ and $(\varphi_{j+1}+\cdots+\varphi_{2j})=(\varphi_1+\cdots+\varphi_j)$ are fulfilled in the space–time after collision, the factor

$$\sum_n\frac{n(n+1)\cdots(n+j-1)}{n^j}\times\frac{e^{(j-1)[(B-F)/F](n+1)}}{\mathcal{J}_0^{(j-1)}},$$

can be equal to one, in the event that $B\propto F$. This means that in some specific space–time points the scattered field is coherent to some degree in the Glauber's sense.

When and where such coherent scattered radiation can be found is to be determined by a thorough analysis of our higher order correlation functions. Such an analysis, and the detailed calculations of the correlation functions involved will be published elsewhere.

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The reduced field density operator for an inelastically scattered quantum radiation

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The projection technique used for setting a non-Markoffian differential equation for the reduced field density operator is discussed, and a detailed solution of the equation is found.

In a recent paper[1] the author found the first and higher order field correlation functions for an inelastically scattered quantum radiation, and discussed some of its coherence properties. In order to extract the relevant information from the jth order correlation functions,

$$G^{(j)}(x_1;\cdots;x_{2j}) = \mathrm{tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_j) \times E^{(+)}(x_{j+1})\cdots E^{(+)}(x_{2j})\}, \qquad (1)$$

we needed the reduced field density operator in the Heisenberg representation. This task was accomplished by the so-called projection technique.[2] The outline of this technique and the detailed calculations related to the reduced field density operator are the topics of this paper.

Let us consider the same ensemble of N identical three level atoms interacting with a coherent radiation, as in the previous paper.[1] The considered field, atomic, and interaction Hamiltonians are

$$H_F = \hbar\sum_k \omega_k a_k^\dagger a_k, \qquad (2)$$

$$H_A = \sum_l \epsilon_l \mathcal{N}_l, \quad \mathcal{N}_l = \sum_{m=1}^{N} |l\rangle\langle l|_m, \qquad (3)$$

$l = g, h, f$ meaning ground, intermediate, and final states, respectively, and

$$V = \sum_k\sum_m i\hbar\mu_{hf}[a_k^\dagger|h\rangle\langle f|_m - a_k|f\rangle\langle h|_m], \qquad (4)$$

where

$$\mu_{hf} = \left(\frac{\omega_k}{2\hbar\epsilon_g v}\right)^{1/2}\bar{\mu}_{hf}\cdot\hat{e}$$

with

$$\bar{\mu}_{hf} = \langle h|e\bar{r}|f\rangle,$$

being the dipole matrix element between the atomic states $|h\rangle$ and $|f\rangle$, and $\hat{e}$ the electric field polarization vector.

The density operator $\rho(t)$ satisfies the equation

$$i\hbar\frac{\partial\rho(t)}{\partial t} = \mathcal{H}\rho(t), \qquad (5)$$

where $\mathcal{H}$ is the Liouville operator for the Hamiltonian of the entire system

$$H = H_A + H_F + V, \qquad (6)$$

meaning

$$\mathcal{H}\rho = [H,\rho]. \qquad (7)$$

In the interaction picture (5) takes the form[3]

$$i\hbar\frac{\partial\chi(t)}{\partial t} = \gamma\chi(t), \qquad (8)$$

where γ is the Liouville operator for the time dependent interaction operator

$$V(t) = \exp[(i/\hbar)(H_F + H_A)t]\,V\exp[-(i/\hbar)(H_F+H_A)t]. \qquad (9)$$

The reduced field density operator is now

$$\sigma(t) = \mathrm{tr}_A\chi(t), \qquad (10)$$

where tr_A means trace over the atomic ensemble. At $t=0$ before the interaction is turned on, the system is still considered split into an independent radiation and an atomic ensemble, thus $\chi(0)$ factorizes in direct product

$$\chi(0) = \sigma(0)f_0(H_A),$$

where

$$f_0(H_A) = \frac{e^{-\beta H_A}}{\mathrm{tr}_A e^{-\beta H_A}} \qquad (11)$$

assuming the atomic ensemble in thermal equilibrium. After the system is coupled, we can write the density operator of the whole system in a convenient form

$$\chi(t) = f(H_A)\sigma(t) + \eta(t), \qquad (12)$$

where

$$\eta(t) = \mathcal{D}\chi(t) \qquad (13)$$

with $\mathcal{D} = \mathcal{D}^2 = [1 - f(H_A)\mathrm{tr}_A]$ a projector in the operator space of the entire system.

The following rules apply[2]:

$$\begin{aligned}
&\sigma(t) = \sigma^\dagger(t), && \chi(t) = \chi^\dagger(t),\\
&\mathcal{D}\chi(t) = \eta(t), && \mathrm{tr}_A\chi(t) = \sigma(t),\\
&\mathcal{D}\eta(t) = \eta(t), && \mathrm{tr}_A\eta(t) = 0,\\
&\mathcal{D}f(H_A)\sigma(t) = 0, && \mathrm{tr}_A f(H_A) = 1,\\
&\mathcal{D}\mathcal{H}_0 = \mathcal{H}_0\mathcal{D}, && \mathrm{tr}_A\mathcal{H}_A f(\mathcal{H}_A) = 0,\\
& && \mathrm{tr}_A\mathcal{H}_A\eta(t) = 0.
\end{aligned} \qquad (14)$$

To obtain the equation of motion for $\sigma(t)$ let us apply the tr_A and $\mathcal{D}$ operations on (8) successively. We have

$$i\hbar\frac{\partial\sigma(t)}{\partial t} = \mathrm{tr}_A\gamma f(H_A)\sigma(t) + \mathrm{tr}_A\gamma\eta(t), \qquad (15)$$

$$i\hbar\frac{\partial\eta(t)}{\partial t} = \mathcal{D}\gamma f(H_A)\sigma(t) + \mathcal{D}\gamma\eta(t), \qquad (16)$$

a coupled system of differential equations. Solving (16)

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with the initial conditions $\eta(0)=\mathcal{D}\chi(0)=0$ and substituting the solution in (16) we obtain

$$\frac{d\sigma(t)}{dt}=-i\mathrm{tr}_A\gamma f(H_A)\sigma(t) - \mathrm{tr}_A\gamma\int_0^t d\tau \exp(-i\mathcal{D}\gamma\tau)\mathcal{D}\gamma f(H_A)\sigma(\tau), \qquad (17)$$

a non-Markoffian equation of motion for the reduced field density operator. (17) is valid for all times and for any orders in $V(t)$ and has a generalized collision operator containing the memory of the system in the second term on the right-hand side.

In solving (17) we can easily remove the first order term in $V(t)$ because the rules set in (14) and a simple renormalization of the unperturbed energy levels, which leads us to $\mathrm{tr}_A\gamma f(H_A)=0$, $\sigma(t)$ being already independent of the tr_A operation. The second order term in $V(t)$ contained in the collision operator of (17) is

$$\mathrm{tr}_A\gamma\int_0^t d\tau\,\mathcal{D}\gamma f(H_A)\sigma(t) = \mathrm{tr}_A\{V(t)\int_0^t d\tau\, V(\tau)f(H_A)\sigma(\tau) - V(t)\int_0^t d\tau f(H_B)\sigma(\tau)V(\tau)\}, \qquad (18)$$

where $V(t)$ is defined in (9). Inserting (9) into (18) we have

$$\mathrm{tr}_A\gamma\int_0^t d\tau\mathcal{D}\gamma f(H_A)\sigma(\tau) = -\hbar^2\mu_{hf}\mu'_{fh}\sum_{k,k'}\sum_{m,l}\mathrm{tr}_A\{[|h\rangle\langle f|_m a_k^+(t)-|f\rangle\langle h|_m a_k(t)] \times\int_0^t d\tau[(|h\rangle\langle f|_l a_{k'}^+(\tau)-|f\rangle\langle h|_l a_{k'}(\tau))f(H_A)\sigma(\tau) - f(H_A)\sigma(\tau)(|h\rangle\langle f|_l a_{k'}^+(\tau)-|f\rangle\langle h|_l a_{k'}(\tau))]\}, \qquad (19)$$

where $a_k(t)=a_k(0)\exp[-i(\omega_k-\omega_s)t]$ with $\hbar\omega_s=\epsilon_h-\epsilon_f$. Taking into account all the relevant commutation rules, namely,

$$[a_k,a_{k'}^+]=\delta_{kk'},\quad \mathrm{tr}_A|h\rangle\langle f|=\langle f|h\rangle=\delta_{fh},\quad \sum|h\rangle\langle h|=1,$$

the only nonzero terms in (19) are

$$\mathrm{tr}_A\gamma\int_0^t d\tau\mathcal{D}\gamma f(H_A)\sigma(\tau) = -\hbar^2\mu_{hf}\mu'_{fh}\sum_{k,k'}\sum_{m,l}\mathrm{tr}_A \times\{|h\rangle\langle f|_m|f\rangle\langle h|_l f(H_A)\int_0^t a_k^+(t)a_{k'}(\tau)\sigma(\tau)\,d\tau - |h\rangle\langle f|_m f(H_A)|f\rangle\langle h|_l\int_0^t a_k^+(t)\sigma(\tau)a_{k'}(\tau)\,d\tau + |f\rangle\langle h|_m|h\rangle\langle f|_l f(H_A)\int_0^t a_k(t)a_{k'}^+(\tau)\sigma(\tau)\,d\tau - |f\rangle\langle h|_m f(H_A)|h\rangle\langle f|_l\int_0^t a_k(t)\sigma(\tau)a_{k'}^+(\tau)\,d\tau\}. \qquad (20)$$

The summation over all the k' modes—which are assumed to be closely spaced with a density $g(\omega_{k'})$—can be converted into integration,

$$\sum_{k'}\{\cdots\} \Rightarrow \int_{-\infty}^{+\infty} g(\omega_{k'})\{\cdots\}\,d\omega_{k'}.$$

Performing the indicated integrals and inserting (20) into (17) we have for the reduced field density operator

$$\frac{d\sigma(t)}{dt}=\sum_k B[a_k(0)\sigma(t)a_k^+(0)-a_k^+(0)a_k(0)\sigma(t)] + F[a_k^+\sigma(t)a_k - a_k a_k^+\sigma(t)] \qquad (21)$$

with

$$B=\hbar^2\mu_{hf}\mu'_{fh}2\pi g(\omega_k)(gh),\qquad F=\hbar^2\mu_{hf}\mu'_{fh}2\pi g(\omega_k)(gf). \qquad (22)$$

The (gh) and (gf) are the atomic correlation functions

$$(gh)=\mathrm{tr}_A\sum_{m,l}^{N}\left(|h\rangle\langle f|_m|f\rangle\langle h|_l\frac{\exp(-\beta H_A)}{\mathrm{tr}_A\exp(-\beta H_A)}\right),\qquad (gf)=\mathrm{tr}_A\sum_{m,l}^{N}\left(|h\rangle\langle f|_m\frac{\exp(-\beta H_A)}{\mathrm{tr}_A\exp(-\beta H_A)}|f\rangle\langle h|_l\right). \qquad (23)$$

Performing the indicated traces, the numerical values of (gh) and (gf) can be easily obtained by a relatively simple computer program for any value of N, thus B and F are well-defined numbers.

Using the P representation[4] for $\sigma(t)$,

$$\sigma(a,a^+,t)=\int\bar{\sigma}^{(a)}(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha|\frac{d^2\alpha}{\pi}, \qquad (24)$$

we find from (21) a Fokker–Planck type differential equation for the antinormal associated functions $\bar{\sigma}^{(a)}(\alpha,\alpha^*,t)$,

$$\frac{\partial\sigma^{(a)}}{\partial t}=\left\{G\frac{\partial}{\partial\alpha}\alpha - F\frac{\partial}{\partial\alpha^*}\alpha^* + F\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}\bar{\sigma}^{(a)}, \qquad (25)$$

and introducing new variables $\alpha=(x+iy)$, $\alpha^*=(x-iy)$, (25) becomes

$$\frac{\partial\bar{\sigma}^{(a)}}{\partial t}=\left\{(G-F)\left[1+\frac{1}{2}x\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial y}\right] + \frac{F}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}\bar{\sigma}^{(a)}. \qquad (26)$$

The solution of (26) is sought in the form of[5]:

$$\bar{\sigma}^{(a)}=e^{-\lambda t}e^{-\mu(x^2+y^2)/2}N(x,y). \qquad (27)$$

Thus for $N(x,y)$ we immediately have

$$\frac{\partial^2 N}{\partial x^2}+\frac{\partial^2 N}{\partial y^2}+k^2N=0 \qquad (28)$$

with $\mu=2(G-F)/F$ and the condition that $k^2=4\lambda/F$ should be an integer. Letting $N=X(x)Y(y)$, the general solution of (28) is

$$N(x,y)=\int_{-\infty}^{+\infty}e^{i[\mu x+(k^2+\mu^2)^{1/2}y]}\alpha\mu,$$

which leads us to

$$N(x,y)=\pi\mathcal{J}_0(kr). \qquad (29)$$

Therefore, the antinormal associated function $\bar{\sigma}^{(a)}(\alpha,\alpha^*,t)$ is

$$\bar{\sigma}_{k^2}^{(a)}=\exp[-(F/4)k^2t]\exp\{[-(G-F)/F](x^2+y^2)\}\times\pi\mathcal{J}_0(k,r) \qquad (30)$$

with $r=(x^2+y^2)^{1/2}$, $\lambda=(F/2)(m_x+m_y)=(F/4)k^2$, $m_x,m_y=0,1,2,\cdots$, and $\mathcal{J}_0(kr)$ the zeroth order Bessel functions.

Finally from (30), because of (24), we obtain the sought solution of (21) by replacing all the α by a_k and all the α^* by a_k^+, and putting them in antinormal order

$$\sigma(a_k, a_k^+, t) = \sum_{k^2} \sum_{k} e^{[-(F/4)k^2 t]} e^{\{[-(G-F)/F]\} a_k a_k^+} \times \mathcal{J}_0(k(a_k a_k^+)^{1/2}). \quad (31)$$

This is the reduced field density operator for the system under consideration up to the second order terms in $V(t)$.

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A consequence of the invariance of Biot's variational principle in thermal conduction[a)]

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The invariance of Biot's variational principle in thermal conduction suggests that this variational principle can be formulated in terms of a self-similar variational principle. This work presents such a derivation and applies it to a thermodynamics example.

In a series of papers which have been recently summarized in a book,[1] Biot has derived a variational principle ("BVP") for thermal conduction systems and has applied it to several problems.[1] The principle is quite general in philosophy, and one could easily think of finding applications for it in such diverse fields as thermodynamics, plasma physics, distributed transmission line calculations, fluid dynamics, etc. We therefore feel compelled to examine some of the implications of it to ascertain if some important computational techniques can be uncovered. It turns out that the property of *invariance* allows one to cast the original BVP which is in spatial and temporal dependent variables x and t repectively into a self-similar BVP which is in terms of the self-similar variable $\xi = \xi(x, t)$.

Following Biot,[1] we write the law of conservation of energy as

$$c\theta = -H_x, \tag{1}$$

where c is the heat capacity per unit volume, θ is the temperature, and H is the heat displacement. We use subscript notation for differentiation. Equation (1) must also be satisfied for any variation

$$c\delta\theta = -(\delta H)_x. \tag{2}$$

Heat conduction is given by

$$\theta_x + (1/k)H_t = 0, \tag{3}$$

where k is the thermal conductivity.

Multiply (3) by the variation δH and integrate over the volume of the medium (we shall assume a semi-infinite volume which extends from $x=0$ to $x=\infty$):

$$\int_0^\infty [\theta_x + (1/k)H_t]\delta H\, dx = 0. \tag{4}$$

Integrating the first term by parts yields

$$\int_0^\infty [-\theta\delta H_x + (1/k)H_t\delta H]dx = -\theta\delta H\Big|_0^\infty. \tag{5}$$

Eliminate θ between (1) and (5):

$$\int_0^\infty \{(1/c)H_x\delta H_x + (1/k)H_t\delta H\}dx = (1/c)H_x\delta H\Big|_0^\infty. \tag{6}$$

This is the BVP.

The BVP is amenable to treatment by self-similar techniques, in particular to a transformation with a one-parameter Lie group defined by

$$H = a^\alpha \bar{H}, \quad x = a^\beta \bar{x}, \quad t = a^\gamma \bar{t}, \tag{7}$$

where a is the parameter and α, β, and γ are constants which are to be determined. It is known that the invariants of this group are identical to the self-similar variables[2-4]

$$\phi(\xi) = H(x,t)/t^{\alpha/\gamma} \quad \text{and} \quad \xi = x/t^{\beta/\gamma}. \tag{8}$$

Substituting (7) into the BVP (6), we obtain

$$a^{2\alpha-\beta}\int_0^\infty \bar{H}_{\bar{x}}\delta\bar{H}_{\bar{x}}d\bar{x} + a^{2\alpha+\beta-\gamma}\int_0^\infty \bar{H}_{\bar{t}}\delta\bar{H}\, d\bar{x} = a^{2\alpha-\beta}\bar{H}_{\bar{x}}\delta\bar{H}\Big|_0^\infty, \tag{9}$$

where we have assumed c and k are constants. This is probably not a severe limitation as we have treated nonlinear and inhomogeneous partial differential equations previously.[5,6] The variables have also been normalized such that c and k become one. We now make use of the *invariance* property of the BVP. The transformation is invariant under (7) if

$$2\alpha - \beta = 2\alpha + \beta - \gamma \tag{10}$$

or $\beta/\gamma = \frac{1}{2}$.

Substituting the invariants (8) into (6) and including the results of (10), we write

$$\int_0^\infty \phi_\xi\delta\phi_\xi d\xi + \int_0^\infty [(\alpha/\gamma)\phi - \xi\phi_\xi/2]\delta\phi d\xi = \phi_\xi\delta\phi\Big|_0^\infty. \tag{11}$$

The first term can be integrated by parts to yield

$$\int_0^\infty [\phi_{\xi\xi} + \xi\phi_\xi/2 - (\alpha/\gamma)\phi]\delta\phi d\xi = [\phi\delta\phi_\xi - \phi_\xi\delta\phi]\Big|_0^\infty. \tag{12}$$

In transforming from the BVP to the self-similar BVP, we note that there has been a "consolidation" of boundary conditions, namely $H(x, t=0)$ and $H(x=\infty, t)$ have consolidated to $\phi(\xi=\infty)$. For the case where the variation is zero at the boundaries, the resulting Euler—Lagrange equation is found:

$$\phi_{\xi\xi} + \xi\phi_\xi/2 - (\alpha/\gamma)\phi = 0. \tag{13}$$

The solution of this is known and can be written in terms of complementary error functions of various order.[7] The order is determined by the parameter α/γ.

The final solution can be written by combining (8),

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(10), and this solution of (13). The choice of α/γ will be dictated by, say, a second boundary condition at $x=0$ or a conservation law. For example, the requirement that $H(x,t)$ be time independent at $x=0$ specifies from (8) that $\alpha/\gamma=0$ and the final result is

$$H(x,t)=A\,\mathrm{erfc}(x/2\sqrt{t}). \tag{14}$$

The requirement that $\theta(x,t)$ be time independent at $x=0$ specifies from (1) and (8) that $\alpha/\gamma=\frac{1}{2}$, and the final results is in terms of the integral of (14). Finally for cases where a conservation law such as $\int H dx=\text{const}$ or $\int \theta dt=\text{const}$ must be satisfied, we note that this conservation law must be invariant under transformation.[8] Applying (7) to either of these, we find that $\alpha/\gamma=-\frac{1}{2}$ and the final result is

$$H=(2A/\sqrt{\pi t})\epsilon^{-x^2/4t}. \tag{15}$$

In conclusion, we have shown that the invariance of the BVP allows one to cast the original BVP into its self-similar form *immediately*. This application of self-similarity to variational calculations does not seem to have been noted previously. We observe that by applying the techniques at this variational stage of a calculation reduces the computational work in self-similar calculations.

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A kinetic formulation of the three-dimensional quantum mechanical harmonic oscillator under a random perturbation

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The behavior of a three-dimensional, nonrelativistic, quantum mechanical harmonic oscillator is investigated under the influence of three distinct types of randomly fluctuating potential fields. Specifically, kinetic (or transport) equations are derived for the corresponding stochastic Wigner equation (the exact equation of evolution of the phase-space Wigner distribution density function) and the stochastic Liouville equation (correspondence limit approximation) using two closely related statistical techniques, the first-order smoothing and the long-time Markovian approximations. Several physically important averaged observables are calculated in special cases. In the absence of a deterministic inhomogeneous potential field (randomly perturbed, freely propagating particle), the results reduce to those reported previously by Besieris and Tappert.

1. INTRODUCTION

In a previous paper,[1] referred to in the sequel as Paper I, kinetic equations were derived for the stochastic Wigner equation (the exact equation of evolution of the phase-space Wigner distribution density function) and the stochastic Liouville equation (correspondence limit approximation) associated with the quantized nonrelativistic motion of a particle described by a stochastic Schrödinger equation having a deterministic background potential field independent of the space and time coordinates. It is our purpose in this paper to lift the latter restriction and investigate specifically the behavior of a three-dimensional quantum mechanical harmonic oscillator experiencing a random perturbation.

Consider the stochastic Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t;\alpha)=H_{\rm op}\left(\mathbf{x},-i\hbar\frac{\partial}{\partial\mathbf{x}},t;\alpha\right)\psi(\mathbf{x},t;\alpha),\qquad t>t_0,\quad \mathbf{x}\in R^3, \tag{1.1a}$$

$$H_{\rm op}\left(\mathbf{x},-i\hbar\frac{\partial}{\partial\mathbf{x}},t;\alpha\right)=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x},t;\alpha), \tag{1.1b}$$

$$\psi(\mathbf{x},t_0;\alpha)=\psi_0(\mathbf{x}). \tag{1.1c}$$

Here, the Hamiltonian $H_{\rm op}$ is a self-adjoint, stochastic operator depending on a parameter $\alpha\in A$; (A,F,P) being an underlying probability measure space. In addition, $\psi(\mathbf{x},t;\alpha)$, the complex random wavefunction, is an element of an infinitely dimensional vector space H, and $V(\mathbf{x},t;\alpha)$ is the potential field which is assumed to be a real, space- and time-dependent random function.

In the course of this work we shall deal explicitly with the following three distinct categories of the potential field:

(i) $V(\mathbf{x},t;\alpha)=\frac{1}{2}kx^2+\delta V(\mathbf{x},t;\alpha)$, (1.2a)

(ii) $V(\mathbf{x},t;\alpha)=\frac{1}{2}kx^2[1+\delta G(t;\alpha)]$, (1.2b)

(iii) $V(\mathbf{x},t;\alpha)=\frac{1}{2}k[\mathbf{x}-\mathbf{a}\delta H(t;\alpha)]^2$, (1.2c)

where $x=|\mathbf{x}|$, k is a positive real constant number, and $\mathbf{a}$ is a fixed vector quantity. The first category corresponds to a linear harmonic oscillator immersed in a zero-mean, space- and time-dependent, random potential field $\delta V(\mathbf{x},t;\alpha)$; the second one is the case of a harmonic oscillator whose frequency is modulated by the zero-mean, time-dependent, random field $\delta G(t;\alpha)$; finally, the third type of potential is associated with a harmonic oscillator whose equilibrium position is perturbed via the zero-mean, time-dependent, random function $\delta H(t;\alpha)$. (This is also closely linked to the Brownian motion arising from a randomly forced harmonic oscillator.)

The random quantum mechanical harmonic oscillator problem corresponding to potential fields of types (ii) and (iii) has already been investigated extensively by several workers under specific restrictive assumptions regarding the random processes $\delta G(t;\alpha)$ and $\delta H(t;\alpha)$. We cite here the early treatment of the Brownian motion of a quantum oscillator by Schwinger,[2] and the quantum theory of a randomly modulated harmonic oscillator by Crosignani *et al.*[3] and Mollow.[4] A more complete account of the statistical analysis of the quantum mechanical oscillator, with applications to quantum optics, can be found in the recent review article by Agarwal.[5]

Besides its generic significance in quantum mechanics, the random harmonic oscillator is of fundamental importance in other physical areas since it provides a dynamic model incorporating salient features common to all of them. For example, Schrödinger-like equations of the form (1.1) and (1.2) play a significant role in plane and beam electromagnetic and acoustic wave propagation. They are usually derived from a scalar Helmholtz equation within the framework of the parabolic (or small-angle) approximation. Statistical analyses of optical wave propagation in randomly perturbed lenslike media have been undertaken by Vorob'ev,[6] Papanicolaou *et al.*,[7] McLaughlin,[8] Beran and Whitman,[9] and Chow.[10] Along the same vein, starting from a space–time parabolic approximation to the full wave equation, Besieris and Kohler[11] have recently considered the problem of underwater sound wave propagation in the presence of a randomly perturbed parabolic sound speed profile.

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It is our intent in this paper to present a unified stochastic kinetic analysis of the random harmonic oscillator, which is equally applicable to the three types of potential field in (1.2), without imposing physically unjustifiable restrictions on the random processes δV, δG, and δH. Special emphasis will be placed on the additional effects contained in our formulation as compared with previously reported results. Finally, it should be pointed out that although the discussion in this paper is restricted to the quantum mechanical random harmonic oscillator, the main results are also applicable to other physical problems by virtue of the statements made in the previous paragraph.

2. THE STOCHASTIC WIGNER DISTRIBUTION FUNCTION

The phase-space analog of the equal-time, two-point density function for a pure state,

$$\rho(\mathbf{x}_2, \mathbf{x}_1, t; \alpha) = \psi^*(\mathbf{x}_2, t; \alpha)\psi(\mathbf{x}_1, t; \alpha), \tag{2.1}$$

is provided by the Wigner distribution function which is defined as follows[12]:

$$f(\mathbf{x}, \mathbf{p}, t; \alpha) = (2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times \rho(\mathbf{x} + \tfrac{1}{2}\mathbf{y}, \mathbf{x} - \tfrac{1}{2}\mathbf{y}, t; \alpha). \tag{2.2}$$

This quantity is real, but not necessarily positive everywhere. It can be shown (cf. Appendix A; also Ref. 13), in general, that $|f(\mathbf{x}, \mathbf{p}, t; \alpha)| \leq (\hbar\pi)^{-3}$ for any realization $\alpha \in A$. Provided that $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ is normalized (to unity), this means that the Wigner distribution function is different from zero in a region of which the volume in phase space is at least equal to $(\hbar\pi)^3$. Hence, $f(\mathbf{x}, \mathbf{p}, t;\alpha)$ can never be sharply localized in $\mathbf{x}$ and $\mathbf{p}$. This situation is a reflection of the uncertainty principle.[14]

The total wave energy and wave action are given in terms of the Wigner distribution function as follows:

$$E = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} H(\mathbf{x}, \mathbf{p}, t; \alpha) f(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{2.3a}$$

$$A = \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p} f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.3b}$$

Here, $H(\mathbf{x}, \mathbf{p}, t; \alpha)$ is the Weyl transform of the operator H_{op} and is given explicitly as

$$H(\mathbf{x}, \mathbf{p}, t; \alpha) = \frac{1}{2m} p^2 + V(\mathbf{x}, t; \alpha), \quad p \equiv |\mathbf{p}|. \tag{2.4}$$

The total wave energy is not conserved since the potential field is assumed to be time dependent. On the other hand, the total wave action is conserved because of the self-adjointness of the Hamiltonian operator, a property satisfied by the three types of potential fields in (1.2).

The time evolution of the Wigner distribution function is governed by the equation

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t; \alpha) = Lf(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{2.5a}$$

$$Lf(\mathbf{x}, \mathbf{p}, t; \alpha) = -\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t; \alpha) + \theta f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.5b}$$

The potential-dependent term on the right-hand side of (2.5b) can be cast into the following three useful representations:

(i)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \int_{R^3} d\mathbf{p}' K(\mathbf{x}, \mathbf{p} - \mathbf{p}', t; \alpha) f(\mathbf{x}, \mathbf{p}', t; \alpha),$$
$$K(\mathbf{x}, \mathbf{p}, t; \alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times [V(\mathbf{x} - \tfrac{1}{2}\mathbf{y}, t; \alpha) - V(\mathbf{x} + \tfrac{1}{2}\mathbf{y}, t; \alpha)]; \tag{2.6a}$$

(ii)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3} \int_{R^3} d\mathbf{y} \exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times \rho(\mathbf{x} + \tfrac{1}{2}\mathbf{y}, \mathbf{x} - \tfrac{1}{2}\mathbf{y}, t; \alpha) \times [V(\mathbf{x} - \tfrac{1}{2}\mathbf{y}, t; \alpha) - V(\mathbf{x} + \tfrac{1}{2}\mathbf{y}, t; \alpha)]; \tag{2.6b}$$

(iii)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = V(\mathbf{x}, t; \alpha) \frac{2}{\hbar} \times \sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial \mathbf{x}} \cdot \frac{\overrightarrow{\partial}}{\partial \mathbf{p}}\right)\right] f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.6c}$$

We shall refer to the exact equation of evolution of $f(\mathbf{x}, \mathbf{p}, t; \alpha)$ as the *stochastic Wigner equation.*

It is seen from (2.6c) that in the correspondence limit ($\hbar \to 0$),

$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \frac{\partial}{\partial \mathbf{x}} V(\mathbf{x}, t; \alpha) \cdot \frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t; \alpha) + O(\hbar^2). \tag{2.7}$$

Within the limits of this approximation, we shall refer to (2.5) as the *stochastic Liouville equation.*

We shall next list the specific realization of $\theta f(\mathbf{x}, \mathbf{p}, t; \alpha)$ corresponding to the three choices of the potential field $V(\mathbf{x}, t; \alpha)$ in (1.2):

(i)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{\partial}{\partial \mathbf{x}}\delta V(\mathbf{x}, t; \alpha)\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha) + O(\hbar^2); \tag{2.8a}$$

(ii)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + kG(t; \alpha)\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha); \tag{2.8b}$$

(iii)
$$\theta f(\mathbf{x}, \mathbf{p}, t; \alpha) = \left(k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} - k\delta H(t; \alpha)\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{p}}\right) \times f(\mathbf{x}, \mathbf{p}, t; \alpha). \tag{2.8c}$$

It should be noted that the last two expressions for $\theta f(\mathbf{x}, \mathbf{p}, t; \alpha)$ are exact. This is due to the special forms of the representations for $V(\mathbf{x}, t; \alpha)$ in (1.2b) and (1.2c).

3. GENERAL EQUATIONS FOR THE MEAN WIGNER DISTRIBUTION FUNCTION

The stochastic Wigner distribution function f and the operator L [cf. Eq. (2.5a)] are next separated into mean and fluctuating parts:

$$f(\mathbf{x}, \mathbf{p}, t; \alpha) = E\{f(\mathbf{x}, \mathbf{p}, t; \alpha)\} + \delta f(\mathbf{x}, \mathbf{p}, t; \alpha), \tag{3.1a}$$

$$L = E\{L\} + \delta L. \tag{3.1b}$$

On the basis of the first-order smoothing approximation,[17–19] one obtains the following general kinetic equation for the ensemble average of f:

$$\left(\frac{\partial}{\partial t} - E\{L\}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_0^t d\tau\, E\{\delta L(t)\exp[\tau E\{L\}]\delta L(t-\tau)\}E\{f(\mathbf{x},\mathbf{p},t-\tau;\alpha)\}. \tag{3.2}$$

In deriving (3.2) it has been assumed that $\delta f(\mathbf{x},\mathbf{p},0;\alpha) = 0$ and that $E\{L\}$ is independent of the time variable. [The latter condition is satisfied for the three types of potential fields prescribed in (1.2)]. This kinetic equation is uniformly valid in time. The right-hand side of (3.2) contains generalized operators (nonlocal, with memory) in phase space.

Various levels of simplification can be obtained by introducing additional constraints. For example, the long-time Markovian results in the simpler kinetic equation

$$\left(\frac{\partial}{\partial t} - E\{L\}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_0^\infty d\tau\, E\{\delta L(t)\exp[\tau E\{L\}]\delta L(t-\tau)\} \times\exp[-\tau E\{L\}]E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{3.3}$$

This particular functional form is due to Van Kampen.[20] It should be pointed out, however, that this expression is identical to Eq. (3.3) of Paper I. A detailed discussion of the long-time Markovian approximation can be found in Refs. 20 and 21. Here, we mention simply that in addition to the usual assumptions entering into the first-order smoothing approximation (cf. Refs. 17–19), the derivation of (3.3) presupposes that $E\{f\}$ vary slowly on the scale of the correlation time of δL.

Having established an expression for the mean Wigner distribution function by solving either of the above kinetic equations, physical observables, such as the average probability density, the average probability current density, the centroid of a wavepacket, the spread of a wavepacket, etc., can be found by taking appropriate phase-space moments (cf. Paper I).

If the mean Wigner distribution function is normalized to unity, i.e.,

$$\int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}\, E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = 1, \tag{3.4}$$

the following general relationship holds:

$$D^2(t) \equiv (2\pi\hbar)^3 \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}\, [E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}]^2 \leq 1. \tag{3.5}$$

[A proof of (3.5) is outlined in Appendix B.] Equality holds if and only if $E\{f\}$ is a "pure" state. Otherwise, $E\{f\}$ is said to represent a "mixed" state, and D (which we shall call the *degree of coherence*) is less than unity.

4. KINETIC THEORY FOR THE STOCHASTIC WIGNER EQUATION

The results of the previous section are specialized here to the stochastic Wigner equation (2.5) corresponding to the potential field given in (1.2a), viz., $V(\mathbf{x},t;\alpha) = \frac{1}{2}kx^2 + \delta V(\mathbf{x},t;\alpha)$. It is convenient to use for this purpose the representation (2.6a) for $\theta f(\mathbf{x},\mathbf{p},t;\alpha)$.

A. The first-order smoothing approximation

The mean and fluctuating parts of the operator L in (2.5) are given explicitly as follows:

$$E\{L\} = -\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} + k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}, \tag{4.1}$$

$$\delta L = \int_{R^3} d\mathbf{p}'\, \delta K(\mathbf{x},\mathbf{p}-\mathbf{p}',t;\alpha)(\cdot), \tag{4.2a}$$

$$\delta K(\mathbf{x},\mathbf{p},t;\alpha) = (i\hbar)^{-1}(2\pi\hbar)^{-3}\int_{R^3} d\mathbf{y}\exp(i\mathbf{p}\cdot\mathbf{y}/\hbar) \times[\delta V(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha) - \delta V(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)]. \tag{4.2b}$$

Introducing (4.1) and (4.2a) in (3.2), we determine the following equation for the ensemble average of the Wigner distribution function within the framework of the first-order smoothing approximation:

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \Theta E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \tag{4.3a}$$

$$\begin{aligned}\Theta E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} &= \int_0^t d\tau \int_{R^3} d\mathbf{p}' \int_{R^3} d\mathbf{p}''\, E\{\delta K(\mathbf{x},\mathbf{p}-\mathbf{p}',t;\alpha) \\ &\times\delta K[\mathbf{x}\cos\omega_0\tau - (\mathbf{p}'/m\omega_0)\sin\omega_0\tau, \mathbf{x}m\omega_0\sin\omega_0\tau \\ &+ \mathbf{p}''\cos\omega_0\tau - \mathbf{p}'', t-\tau;\alpha]\}E\{f[\mathbf{x}\cos\omega_0\tau \\ &- (\mathbf{p}'/m\omega_0)\sin\omega_0\tau, \mathbf{p}'', t-\tau]\},\end{aligned} \tag{4.3b}$$

where $\omega_0 = (k/m)^{1/2}$. In deriving this equation we have made use of the well-known propagator property

$$\exp\left[\tau\left(-\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} + k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\right]g(\mathbf{x},\mathbf{p}) = g[\mathbf{x}\cos\omega_0\tau - (\mathbf{p}/m\omega_0)\sin\omega_0\tau, \mathbf{x}m\omega_0\sin\omega_0\tau + \mathbf{p}\cos\omega_0\tau]. \tag{4.4}$$

For the sake of simplicity, we shall assume that $\delta V(\mathbf{x},t;\alpha)$ [which enters into (4.3b) via the defining equation (4.2b)] is a spatiallv homogeneous, wide-sense stationary random process, viz.,

$$\Gamma(\mathbf{y},\tau) = E\{\delta V(\mathbf{x},t;\alpha)\ \delta V(\mathbf{x}-\mathbf{y},t-\tau;\alpha)\}. \tag{4.5}$$

The correlation function is even in both $\mathbf{y}$ and τ. In our subsequent work we shall require the spectrum [i.e., the space–time Fourier transform of $\Gamma(\mathbf{y},\tau)$], viz., $\hat{\Gamma}(\mathbf{p},u) = F_4\{\Gamma(\mathbf{y},\tau)\}$. It is related to the space–time Fourier transform of $\delta V(\mathbf{x},t;\alpha)$, viz., $\delta\hat{V}(\mathbf{p},u) = F_4\{\delta V(\mathbf{x},t;\alpha)\}$ in the following manner:

$$E\{\delta\hat{V}(\mathbf{p},u)\,\delta\hat{V}(\mathbf{p}',u')\} = \delta(\mathbf{p}+\mathbf{p}')\delta(u+u')\hat{\Gamma}(\mathbf{p},u). \tag{4.6}$$

It should be noted that $\hat{\Gamma}(\mathbf{p},u)$ is real, nonnegative, and even in both $\mathbf{p}$ and u.

The operator Θ on the right-hand side of (4.3a) can now be evaluated explicitly. The resulting kinetic equation for the mean Wigner distribution function assumes the following form:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} \\ &= \frac{2}{\hbar^2}\int_{R^3} d\mathbf{p}' \int_0^t d\tau\, Q(\mathbf{x},\mathbf{p},\mathbf{p}',\tau)\Big(E\Big\{f\Big[\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}') \\ &\times\frac{1}{m\omega_0}\sin\omega_0\tau, -\tfrac{1}{2}(\mathbf{p}-\mathbf{p}') + \mathbf{x}m\omega_0\sin\omega_0\tau + \tfrac{1}{2}(\mathbf{p}+\mathbf{p}') \\ &\times\cos\omega_0\tau, t-\tau;\alpha\Big]\Big\} - E\Big\{f\Big[\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\end{aligned}$$

$$\times \frac{1}{m\omega_0}\sin\omega_0\tau, + \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') + \mathbf{x} m\omega_0 \sin\omega_0\tau + \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')$$

$$\times\cos\omega_0\tau, t-\tau, \alpha\Big]\Big\}\Big), \tag{4.7a}$$

$$Q(\mathbf{x},\mathbf{p},\mathbf{p}',\tau) = (2\pi\hbar)^{-3}\int_{R^3} d\mathbf{y}\,\Gamma(\mathbf{y},\tau)$$

$$\times \cos\Big[\mathbf{y}\cdot(\mathbf{p}-\mathbf{p}')/\hbar + (\mathbf{p}-\mathbf{p}') \times\Big(\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\frac{1}{m\omega_0}\sin\omega_0\tau - \mathbf{x}\Big)\Big/\hbar\Big]. \tag{4.7b}$$

This rather formidable integrodifferential equation constitutes a uniform approximation, valid for any value of time, from which short and long time limiting cases can be considered. (The latter will be dealt with in detail in the following subsection.) The right-hand side of (4.7) contains a generalized operator (nonlocal, with memory) in phase space due to the presence of random fluctuations in the potential field, as well as to the interaction of these random inhomogeneities with the deterministic profile of the potential field. No special assumptions concerning the scale lengths of the potential fluctuations have been made in deriving (4.7). The only condition (which is implicit in the first-order smoothing approximation) is that the potential fluctuations be sufficiently small. Finally, it should be noted that in the limit $\omega_0 \to 0$ (absence of deterministic inhomogeneities), (4.7) coincides with Eq. (4.5) of Paper I.

B. The long-time Markovian approximation

By imposing additional restrictions, the kinetic equation (4.7) can be simplified considerably. The long-time Markovian approximation [cf. Eq. (3.3)] yields the following expression:

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}$$

$$= \int_{R^3} d\mathbf{p}'\, W(\mathbf{x},\mathbf{p},\mathbf{p}')[E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\} - E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}], \tag{4.8a}$$

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\int_0^\infty d\tau\,\tilde{\Gamma}(\mathbf{p}-\mathbf{p}',\tau)\cos\Big[(\mathbf{p}-\mathbf{p}')\cdot\Big(\mathbf{x}\cos\omega_0\tau - \tfrac{1}{2}(\mathbf{p}+\mathbf{p}')\frac{1}{m\omega_0}\sin\omega_0\tau - \mathbf{x}\Big)\Big/\hbar\Big], \tag{4.8b}$$

where $\tilde{\Gamma}(\mathbf{p},\tau)$ is the spatial Fourier transform of the correlation function $\Gamma(\mathbf{y},\tau)$.

Equation (4.8) has the form of a radiation transport equation, or a Boltzmann equation for waves (quasiparticles in phase space). The expression for the transition probability [cf. Eq. (4.8b)] is space-dependent (in contradistinction to the case of a potential field having a constant deterministic part), and obeys the principle of detailed balance, viz., $W(\mathbf{x},\mathbf{p},\mathbf{p}') = W(\mathbf{x},\mathbf{p}',\mathbf{p})$. The latter implies conservation of probability (total mean action).

The integration over τ on the right-hand side of (4.8b) can be carried out explicitly resulting in the following more revealing form for the transition probability:

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \sum_{n=-\infty}^{\infty} W_n(\mathbf{x},\mathbf{p},\mathbf{p}'), \tag{4.9a}$$

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar} J_n\Big(\frac{a}{\hbar}\Big)\cos\Big(\frac{b}{\hbar} - n\frac{\pi}{2}\Big)\Big\{\cos\Big[n\Big(\delta+\frac{\pi}{2}\Big)\Big] \times\hat{\Gamma}(\mathbf{p}-\mathbf{p}',n\hbar\omega_0) - \sin\Big[n\Big(\delta+\frac{\pi}{2}\Big)\Big] \hat{\Gamma}_H(\mathbf{p}-\mathbf{p}',n\hbar\omega_0)\Big\}, \tag{4.9b}$$

$$a = \{[\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')]^2 + [(p^2-p'^2)/(2m\omega_0)]^2\}^{1/2}, \tag{4.9c}$$

$$b = \mathbf{x}\cdot(\mathbf{p}-\mathbf{p}'), \tag{4.9d}$$

$$\delta = \tan^{-1}[-2m\omega_0 b/(p^2-p'^2)]. \tag{4.9e}$$

J_n in (4.9b) denotes an ordinary Bessel function of the nth order, and $\hat{\Gamma}(\mathbf{p}-\mathbf{p}',n\hbar\omega_0)$ is the Hilbert transform of the spectrum $\hat{\Gamma}(\mathbf{p}-\mathbf{p}',n\hbar\omega_0)$ with respect to the second argument, viz.,

$$\hat{\Gamma}_H(\cdot,n\hbar\omega_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty} d\omega\,\frac{\hat{\Gamma}(\cdot,\omega)}{\omega - n\hbar\omega_0}. \tag{4.10}$$

The representation of the transition probability W in (4.9a) as an infinite sum is a manifestation of the discrete nature of the quantum mechanical stochastic harmonic oscillator. The term W_n, for example, can be interpreted as the transition probability of the scattering event that changes the energy of the particle by an amount equal to $n\hbar\omega_0$.

If the correlation function $\Gamma(\mathbf{y},\tau)$ decreases rapidly in τ, so does the spectrum $\hat{\Gamma}(\mathbf{p},u)$ in u, and its Hilbert transform $\hat{\Gamma}_H(\mathbf{p},u)$ with respect to its second argument. Under these conditions, since the Bessel functions and the sinusoidal terms in (4.9b) are bounded, it is possible to approximate the transition probability W in (4.9a) by a sum of the first few terms, i.e.,

$$W = \sum_{n=-N}^{N} W_n, \tag{4.11}$$

where the integer N can be estimated from our knowledge of the correlation time of the random process $\delta V(\mathbf{x},t;\alpha)$.

It is clear from (4.8b) that in the limiting case $\omega_0 \to 0$ (stochastically perturbed free particle),

$$W(\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\int_0^\infty d\tau\,\tilde{\Gamma}(\mathbf{p}-\mathbf{p}',\tau)\cos\Big[\tau\Big(\frac{p^2}{2m} - \frac{p'^2}{2m}\Big)\hbar\Big], \tag{4.12}$$

which, upon integration, yields the following expression for the transition probability,

$$W(\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}\hat{\Gamma}\Big(\mathbf{p}-\mathbf{p}', \frac{p^2}{2m} - \frac{p'^2}{2m}\Big) \tag{4.13}$$

[cf. Eq. (4.7), Paper I]. The same result can be also obtained from (4.9) provided that the operations $\lim_{\omega_0\to 0}$ and infinite summation are not interchanged.

We shall close this subsection with the following remark: If the lower limit in the integral on the right-hand side of (4.8b) were replaced by $-\infty$ (this corresponds to the specification of initial data at $t=-\infty$ instead of $t=0$), the expression for W_n in (4.9b) would be modified as follows:

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{4\pi}{\hbar} J_n\left(\frac{a}{\hbar}\right)\cos\left(\frac{b}{\hbar} - n\frac{\pi}{2}\right)\cos\left[n\left(\delta + \frac{\pi}{2}\right)\right] \times \hat{\Gamma}(\mathbf{p}-\mathbf{p}', n\hbar\omega_0). \quad (4.14)$$

The terms in (4.9b) proportional to the Hilbert transform $\hat{\Gamma}_H(\mathbf{p}-\mathbf{p}', n\hbar\omega_0)$, which are absent in (4.14), can be interpreted as representing the effect of "switching on" the interaction between the random fluctuations of the potential field and the inhomogeneous deterministic background at the finite time $t=0$. In the special case of a potential field with a constant deterministic part, one has the relationship

$$W^{\mathrm{LTMA}}_{t_0=0} = \tfrac{1}{2} W^{\mathrm{LTMA}}_{t_0=-\infty} \quad (4.15)$$

for the transition probabilities corresponding to initial data prescribed at $t_0=0$ and $t_0=-\infty$, respectively. (LTMA is an abbreviation for the term long-time Markovian approximation.)

C. Kinetic equations in special cases

We shall derive here the explicit form of the kinetic equation in the long-time Markovian approximation limit for several special types of the random function $\delta V(\mathbf{x}, t; \alpha)$.

Case (i): $\delta V(\mathbf{x}, t; \alpha)$ has δ-function correlations in time.

Let $\Gamma(\mathbf{y},\tau) = \gamma(\mathbf{y})\delta(\tau)$. It follows, then, that $\hat{\Gamma}(\mathbf{p}, u) = \hat{\gamma}(\mathbf{p})$, where $\hat{\gamma}(\mathbf{p})$ is the Fourier transform of $\gamma(\mathbf{y})$. The transport equation (4.8) specializes in this case to

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_{R^3} d\mathbf{p}'\, W(\mathbf{p},\mathbf{p}')[E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\} - E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}], \quad (4.16a)$$

$$W(\mathbf{p},\mathbf{p}') = \frac{1}{\hbar^2}\hat{\gamma}(\mathbf{p}-\mathbf{p}'). \quad (4.16b)$$

The right-hand side of (4.16a), with W given in (4.16b), is identical to Eq. (5.1) of Paper I. It is, therefore, due solely to the random fluctuations of the potential field. The terms in the more general kinetic equation (4.8) arising from the interaction of the deterministic profile and the random fluctuations of the potential field are completely eliminated in this special case.

The spectrum $\hat{\gamma}(\mathbf{p})$ is real, nonnegative, and even. As a consequence, the transition probability $W(\mathbf{p},\mathbf{p}')$ [cf. Eq. (4.16b)] is real, nonnegative, and obeys the (detailed balance) property $W(\mathbf{p},\mathbf{p}') = W(\mathbf{p}',\mathbf{p})$. The latter implies conservation of probability (total mean action). On the strength of the principle of detailed balance, together with the nonnegativity of the transition probability, it follows, also, that the degree of coherence introduced in Sec. 3 is a monotonically decreasing function of time, viz., $(d/dt)D(t) \leq 0$.[22]

The *scattering rate* (also called the *extinction coefficient* or *collision frenquency*) is defined in general as

$$\nu(\mathbf{p}) = \int_{R^3} d\mathbf{p}'\, W(\mathbf{p},\mathbf{p}'). \quad (4.17)$$

In the case under consideration here, the scattering rate is independent of $\mathbf{p}$ and is given by

$$\nu = \frac{1}{\hbar^2}\gamma(0). \quad (4.18)$$

Using this result, (4.16) can be rewritten in the following form:

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{1}{\hbar^2}\gamma(0)\right) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \frac{1}{\hbar^2}\int_{R^3} d\mathbf{p}'\,\hat{\gamma}(\mathbf{p}-\mathbf{p}') E\{f(\mathbf{x},\mathbf{p}',t;\alpha)\}. \quad (4.19)$$

Starting from the convolution-type integro-differential equation (4.19), with the prescribed initial condition $E\{f(\mathbf{x},\mathbf{p},0;\alpha)\} = f_0(\mathbf{x},\mathbf{p})$, it is possible to determine a Green's function $G(\mathbf{x},\mathbf{x}',\mathbf{p},\mathbf{p}',t)$ such that[24]

$$E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \int_{R^3} d\mathbf{x}' \int_{R^3} d\mathbf{p}'\, G(\mathbf{x},\mathbf{x}',\mathbf{p},\mathbf{p}',t) f_0(\mathbf{x}',\mathbf{p}'). \quad (4.20)$$

This is a useful expression because, for specific statistics $\gamma(\mathbf{y})$ [or, equivalently, $\hat{\gamma}(\mathbf{p})$] and initial data $f_0(\mathbf{x},\mathbf{p})$, physically important averaged observables can be found directly from (4.20) by taking phase-space moments, without having to solve first explicitly for the mean Wigner distribution function. (This procedure is illustrated in Appendix C.)

It can be shown by means of the Donsker–Furutsu–Novikov[26–28] functional method that for a potential field fluctuation $\delta V(\mathbf{x}, t; \alpha)$ which constitutes a δ-correlated (in time), homogeneous, wide-sense stationary, Gaussian random process, the kinetic equation (4.16) for the mean Wigner distribution function is the *exact* statistical equation. (The proof will not be presented here since it is similar to that given in the Appendix of Paper I.)

Case (ii): $\delta V(\mathbf{x}, t; \alpha)$ has no time dependence.

Assuming that $\Gamma(\mathbf{y},\tau) = \gamma(\mathbf{y})$, we have $\hat{\Gamma}(\mathbf{p}, u) = \hat{\gamma}(\mathbf{p})\delta(u)$. The transition probability W becomes

$$W(\mathbf{x},\mathbf{p},\mathbf{p}') = \frac{2}{\hbar^2}\hat{\gamma}(\mathbf{p}-\mathbf{p}')\int_0^\infty d\tau \cos\left(\frac{a\sin(\omega_0\tau+\delta) - b}{\hbar}\right), \quad (4.21)$$

where a, b, and δ are defined in Eqs. (4.9c)–(4.9e). It must be pointed out that the condition for the applicability of the long-time Markovian approximation [i.e., $E\{f\}$ should vary slowly on the scale of the correlation time of $\delta V(\mathbf{x}, t; \alpha)$] is clearly violated in this case. In this sense, (4.21) should be considered only as a formal result. Finally, in the limit as $\omega_0 \to 0$, (4.21) reduces to Eq. (5.4) of Paper I, viz.,

$$W(\mathbf{p},\mathbf{p}') = \frac{2\pi}{\hbar}\hat{\gamma}(\mathbf{p}-\mathbf{p}')\delta\left(\frac{p^2}{2m} - \frac{p'^2}{2m}\right). \quad (4.22)$$

Case (iii): $\delta V(\mathbf{x}, t; \alpha)$ has δ-function correlations in space.

Let $\Gamma(\mathbf{y},\tau) = (2\pi\hbar)^3\delta(\mathbf{y})\gamma(\tau)$. It follows, then, that $\hat{\Gamma}(\mathbf{p}, u) = \hat{\gamma}(u)$, where $\hat{\gamma}(u)$ denotes the time Fourier transform of $\gamma(\tau)$. The mean Wigner distribution function evolves in time according to (4.8a), with the transition probability given by

$$W(\mathbf{x},\mathbf{p},\mathbf{p}')=\sum_{n=-\infty}^{\infty} W_n(\mathbf{x},\mathbf{p},\mathbf{p}'), \tag{4.23a}$$

$$W_n(\mathbf{x},\mathbf{p},\mathbf{p}')=\frac{2\pi}{\hbar} J_n\left(\frac{a}{\hbar}\right)\cos\left(\frac{b}{\hbar}-n\frac{\pi}{2}\right)\left\{\cos\left[n\left(\delta+\frac{\pi}{2}\right)\right] \times \hat{\gamma}(n\hbar\omega_0)-\sin\left[n\left(\delta+\frac{\pi}{2}\right)\right]\hat{\gamma}_H(n\hbar\omega_0)\right\}. \tag{4.23b}$$

$\hat{\gamma}_H(n\hbar\omega_0)$ stands for the Hilbert transform of the temporal spectrum $\hat{\gamma}(n\hbar\omega_0)$ [cf., also, Eq. (4.10)].

5. KINETIC THEORY FOR THE STOCHASTIC LIOUVILLE EQUATION

The results of Sec. 3 will now be specialized to the stochastic Liouville equation, i.e., Eq. (2.5), with the specific realizations of $\theta f(\mathbf{x},\mathbf{p},t;\alpha)$ given in (2.8a)–(2.8c).

A. $V(\mathbf{x},t;\alpha) = 1/2kx^2 + \delta V(\mathbf{x},t;\alpha)$

The mean part of the operator L in (2.5) is given in (4.1). On the other hand, the fluctuating part of L assumes the following form,

$$\delta L=\frac{\partial}{\partial \mathbf{x}}\,\delta V(\mathbf{x},t;\alpha)\cdot\frac{\partial}{\partial \mathbf{p}}+O(\hbar^2). \tag{5.1}$$

On the basis of the first-order smoothing approximation only [cf. Eq. (3.2)], one has the kinetic equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}$$
$$=\frac{\partial}{\partial \mathbf{p}}\cdot\left[\int_0^t d\tau\, E\left\{\frac{\partial}{\partial \mathbf{x}}\,\delta V(\mathbf{x},t;\alpha)\frac{\partial}{\partial \mathbf{x}'}\,\delta V(\mathbf{x}',t-\tau;\alpha)\right\} \times\frac{\partial}{\partial \mathbf{p}'}E\{f(\mathbf{x}',\mathbf{p}',t-\tau;\alpha)\}\right], \tag{5.2}$$

where

$$\mathbf{x}'=\mathbf{x}\cos\omega_0\tau-(1/m\omega_0)\mathbf{p}\sin\omega_0\tau, \tag{5.3a}$$

$$\mathbf{p}'=\mathbf{p}\cos\omega_0\tau+m\omega_0\mathbf{x}\sin\omega_0\tau. \tag{5.3b}$$

By virtue of the homogeneity and stationarity of the random function $\delta V(\mathbf{x},t;\alpha)$ (cf. Sec. 4 A),

$$E\left\{\frac{\partial}{\partial \mathbf{x}}\delta V(\mathbf{x},t;\alpha)\frac{\partial}{\partial \mathbf{x}'}\,\delta V(\mathbf{x}',t-\tau;\alpha)\right\}=-\frac{\partial^2}{\partial \mathbf{y}\,\partial \mathbf{y}}\,\Gamma(\mathbf{y},\tau), \tag{5.4}$$

where

$$\mathbf{y}=\mathbf{x}-\mathbf{x}'=\mathbf{x}(1-\cos\omega_0\tau)+(1/m\omega_0)\mathbf{p}\sin\omega_0\tau. \tag{5.5}$$

Finally, Eq. (5.2) can be written as follows:

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)$$
$$=\frac{\partial}{\partial \mathbf{p}}\cdot\left[\int_0^\infty d\tau\int_{R^3}d\mathbf{y}\,\delta\left(\mathbf{y}-\mathbf{x}(1-\cos\omega_0\tau)+\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\right) \times\left(-\frac{\partial}{\partial \mathbf{y}\,\partial \mathbf{y}}\,\Gamma(\mathbf{y},\tau)\right)\cdot\left(\frac{\sin\omega_0\tau}{m\omega_0}\,\frac{\partial}{\partial \mathbf{x}}\cos\omega_0\tau\,\frac{\partial}{\partial \mathbf{p}}\right) \times E\left\{f\left(\mathbf{x}\cos\omega_0\tau-\frac{1}{m\omega}\mathbf{p}\sin\omega_0\tau,\mathbf{p}\cos\omega_0\tau +m\omega_0\mathbf{x}\sin\omega_0\tau,t-\tau;\alpha\right)\right\}\right]. \tag{5.6}$$

For random fluctuations which are statistically homogeneous, wide-sense stationary, and δ correlated in time $[\Gamma(\mathbf{y},\tau)=\gamma(\mathbf{y})\delta(\tau)]$, the time integration on the right-hand side of (5.6) can be carried out explicitly, with the result

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial \mathbf{p}}\cdot\left[\mathbf{D}\cdot\frac{\partial}{\partial \mathbf{p}}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}\right], \tag{5.7a}$$

$$\mathbf{D}=-\tfrac{1}{2}\lim_{\mathbf{y}\to 0}\left(\frac{\partial^2}{\partial \mathbf{y}\,\partial \mathbf{y}}\,\gamma(\mathbf{y})\right). \tag{5.7b}$$

The right-hand side of this transport equation is identical to that in Eq. (6.2) of Paper I, which was obtained under the assumption that $\omega_0=0$. This shows that there is no interaction between the deterministic potential field profile and the random variations under the presently specified statistical properties. It should also be noted that if, in addition to the prescribed properties, $\delta V(\mathbf{x},t;\alpha)$ is a Gaussian process, the kinetic equation (5.7) is the *exact* statistical equation for the mean Wigner distribution function within the stochastic Liouville approximation. (The proof of an analogous statement can be found in the second part of the Appendix in Paper I.)

Equation (5.7) is a variant of the *equation of Kramers*.[29] A fundamental solution for it can be found by a method introduced by Wang and Uhlenbeck.[30] Equation (5.7) can be also obtained from (4.19) or, equivalently, from the three-dimensional analog of (C1a) (cf. Appendix C). If, in the latter, the term $\gamma(\hbar\mathbf{u})$ is expanded to order $\hbar^2$, and an inverse Fourier transform is performed with respect to variables $\mathbf{u}$ and $\mathbf{q}$ [cf. Eq. (C2)], the ensuing transport equation is identical to (5.7). As a result, the expressions for the first- and second-order averaged observables listed in Appendix C remain unchanged. However, third- and higher-order observables calculated on the basis of (C1) will contain terms of at least first order in $\hbar$, which will be absent in the corresponding Liouville approximation.

In the long-time Markovian approximation, (5.2) simplifies to

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\frac{\partial}{\partial \mathbf{p}}\cdot\left(\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial \mathbf{p}}+\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial \mathbf{x}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.8}$$

This is a Fokker–Planck equation in phase space. The space- and momentum-dependent dyadic diffusion coefficients are given by

$$\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})=\int_0^\infty d\tau\int_{R^3}d\mathbf{y}\,\delta\left(\mathbf{y}-\mathbf{x}(1-\cos\omega_0\tau)-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\right) \times\left(-\frac{\partial^2}{\partial \mathbf{y}\,\partial \mathbf{y}}\,\Gamma(\mathbf{y},\tau)\right)\cos\omega_0\tau, \tag{5.9a}$$

$\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p}) =$

$$\int_0^\infty d\tau \int_{R^3} d\mathbf{y}\, \delta\Big(\mathbf{y} - \mathbf{x}(1-\cos\omega_0\tau) - \frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\Big) \times \Big(-\frac{\partial^2}{\partial\mathbf{y}\,\partial\mathbf{y}}\Gamma(\mathbf{y},\tau)\Big)\frac{\sin\omega_0\tau}{m\omega_0}. \qquad (5.9b)$$

Equation (5.8) can be derived by applying the long-time Markovian approximation directly to the stochastic Liouville equation. Alternatively, it can be derived from the transport equation corresponding to the long-time Markovian approximation of the stochastic Wigner equation [cf. Eq. (4.8)] under the restriction that $\delta V(\mathbf{x},t;\alpha)$ varies slowly in space. This can be done by following the method used by Landau to derive the Fokker–Planck equation for a plasma from a Boltzmann equation (cf. Paper I and Ref. 31).

B. $V(\mathbf{x},t;\alpha) = 1/2\,kx^2\,[1+\delta G(t;\alpha)]$

The exact stochastic Wigner equation assumes in this case the form

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big) f(\mathbf{x},\mathbf{p},t;\alpha) = k\delta G(t;\alpha)\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}} f(\mathbf{x},\mathbf{p},t;\alpha). \qquad (5.10)$$

One has, then, in the first-order smoothing approximation,

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = k^2\frac{\partial}{\partial\mathbf{p}}\cdot\Big[\int_0^t d\tau\,\Gamma(\tau)\mathbf{x}\mathbf{x}'\cdot\frac{\partial}{\partial\mathbf{p}'}E\{f(\mathbf{x}',\mathbf{p}',t-\tau;\alpha)\}\Big], \qquad (5.11)$$

where $\Gamma(\tau) = E\{\delta G(t;\alpha)\delta G(t-\tau;\alpha)\}$, and $\mathbf{x}',\mathbf{p}'$ are given in (5.3). The kinetic equation (5.11) can be rewritten as follows:

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = k^2\frac{\partial}{\partial\mathbf{p}}\cdot\int_0^t d\tau\,\Gamma(\tau)\mathbf{x}\Big[\mathbf{x}\cos\omega_0\tau - \frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau\Big] \cdot\Big(\frac{\sin\omega_0\tau}{m\omega_0}\frac{\partial}{\partial\mathbf{x}} + \cos\omega_0\tau\frac{\partial}{\partial\mathbf{p}}\Big)E\Big\{f\Big(\mathbf{x}\cos\omega_0\tau - \frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau,\ \mathbf{p}\cos\omega_0\tau + m\omega_0\mathbf{x}\sin\omega_0\tau,\ t-\tau;\alpha\Big)\Big\}. \qquad (5.12)$$

For a harmonic oscillator whose frequency is modulated by a wide-sense stationary, δ-correlated random process, viz., $\Gamma(\tau) = D\delta(\tau)$, where D is a constant, the integration on the right-hand side of (5.11) can be performed explicitly, yielding[32]

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = k^2 D\Big(\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big)^2 E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \qquad (5.13)$$

The one-dimensional version of this equation was derived previously by Mollow (cf. Ref. 4).

Equation (5.13) corresponds to a Fokker–Planck equation in phase space, with a quadratic diffusion coefficient. The latter is due entirely to the presence of random fluctuations. No exact fundamental solution for (5.13) seems to be possible to the general case. However, closed systems of equations for moments of any order can be obtained. For example, since

$$E\{\mathbf{x}(t;\alpha)\} = \int_{R^3} d\mathbf{x}\int_{R^3} d\mathbf{p}\,\mathbf{x}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \qquad (5.14a)$$

$$E\{\mathbf{p}(t;\alpha)\} = \int_{R^3} d\mathbf{x}\int_{R^3} d\mathbf{p}\,\mathbf{p}E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}, \qquad (5.14b)$$

one derives from (5.13) the following equations of motion:

$$\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\} = \frac{1}{m}E\{\mathbf{p}(t;\alpha)\}, \qquad (5.15a)$$

$$\frac{d}{dt}E\{\mathbf{p}(t;\alpha)\} = -kE\{\mathbf{x}(t;\alpha)\}. \qquad (5.15b)$$

The initial conditions required for their solution are obtainable from (5.14), viz.,

$$E\{\mathbf{x}(0;\alpha)\} = \int_{R^3} d\mathbf{x}\int_{R^3} d\mathbf{p}\,\mathbf{x}E\{f(\mathbf{x},\mathbf{p},0;\alpha)\} \equiv \mathbf{x}_0, \qquad (5.16a)$$

$$E\{\mathbf{p}(0;\alpha)\} = \int_{R^3} d\mathbf{x}\int_{R^3} d\mathbf{p}\,\mathbf{p}E\{f(\mathbf{x},\mathbf{p},0;\alpha)\} \equiv \mathbf{p}_0. \qquad (5.16b)$$

It then readily follows that

$$E\{\mathbf{x}(t;\alpha)\} = \mathbf{x}_0\cos\omega_0 t + (k/m)^{-1/2}\mathbf{p}_0\sin\omega_0 t, \qquad (5.17a)$$

$$E\{\mathbf{p}(t;\alpha)\} = \mathbf{p}_0\cos\omega_0 t - (k/m)^{1/2}\mathbf{x}_0\sin\omega_0 t, \qquad (5.17b)$$

where $\omega_0 = (k/m)^{1/2}$. We next note the following: (1) The random perturbation $\delta G(t;\alpha)$ in this case has no effect whatsoever at the level of the first two moments. (This, of course, is not the case for higher moments); (2) Equation (5.17) gives the expressions for the position and momentum of a classical harmonic oscillator characterized by a frequency ω_0. This is due to the fact that the stochastic Liouville equation (5.10) is identical to the equation governing the classical distribution function $f_c(\mathbf{x},\mathbf{p},t;\alpha) = \delta[\mathbf{x}-\mathbf{x}(t;\alpha)]\delta[\mathbf{p}-\mathbf{p}(t;\alpha)]$, $f_c(\mathbf{x},\mathbf{p},0;\alpha) = \delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$, where $(d/dt)\mathbf{x}(t;\alpha) = (1/m)\mathbf{p}(t;\alpha)$, $(d/dt)\mathbf{p}(t;\alpha) = -k[1+\delta G(t;\alpha)]\mathbf{x}(t;\alpha)$, and $\mathbf{x}(0;\alpha) = \mathbf{x}_0$, $\mathbf{p}(0;\alpha) = \mathbf{p}_0$.

In the long-time Markovian approximation, (5.11) simplifies to the Fokker–Planck equation

$$\Big(\frac{\partial}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}} - k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\Big) E\{f(\mathbf{x},\mathbf{p},t;\alpha)\} = \Big(\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{p}} + \frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{x}}\Big) \times E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \qquad (5.18)$$

The two dyadic diffusion coefficients are given as follows:

$$\mathbf{D}^{(1)}(\mathbf{x},\mathbf{p}) = \Big[k^2\int_0^\infty d\tau\,\Gamma(\tau)\cos^2\omega_0\tau\Big]\mathbf{x}\mathbf{x} - \Big(\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin 2\omega_0\tau\Big)\mathbf{x}\mathbf{p}, \qquad (5.19a)$$

$$\mathbf{D}^{(2)}(\mathbf{x},\mathbf{p}) = -\Big[\Big(\frac{k}{m\omega_0}\Big)^2\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau\Big]\mathbf{x}\mathbf{p} + \Big(\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau\Big)\mathbf{x}\mathbf{x}. \qquad (5.19b)$$

In general, no exact solution to (5.18) seems to be possible. Nevertheless, closed systems of equations for moments of any order can be obtained by taking appropriate phase-space moments. For example, using the definitions of the average position and momentum [cf. Eq. (5.14)], the following equations of motion can be readily derived from (5.18):

$$\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\}=\frac{1}{m}E\{\mathbf{p}(t;\alpha)\}, \tag{5.20a}$$

$$\frac{d}{dt}E\{\mathbf{p}(t;\alpha)\}=-kE\{\mathbf{x}(t;\alpha)\}+c_1E\{\mathbf{x}(t;\alpha)\}-c_2E\{\mathbf{p}(t;\alpha)\}, \tag{5.20b}$$

with the constant coefficients c_1 and c_2 given by

$$c_1=\frac{k^2}{2m\omega_0}\int_0^\infty d\tau\,\Gamma(\tau)\sin 2\omega_0\tau, \tag{5.21}$$

$$c_2=\left(\frac{k}{m\omega_0}\right)^2\int_0^\infty d\tau\,\Gamma(\tau)\sin^2\omega_0\tau. \tag{5.22}$$

The latter one may be expressed in terms of the spectrum $\hat{\Gamma}(u)$ as follows,

$$c_2=\left(\frac{k}{m\omega_0}\right)^2\frac{\pi}{2}[\hat{\Gamma}(0)-\hat{\Gamma}(2\omega_0)]. \tag{5.23}$$

On the other hand, the former one may be written as

$$c_1=\frac{k^2}{2m\omega_0}\,\frac{\pi}{2}\,\hat{\Gamma}_H(2\omega_0), \tag{5.24}$$

where

$$\hat{\Gamma}_H(2\omega_0)=\frac{2}{\pi}\int_0^\infty d\tau\,\Gamma(\tau)\sin 2\omega_0\tau=\frac{1}{\pi}P\int_{-\infty}^\infty\frac{\hat{\Gamma}(u)}{u-2\omega_0}du \tag{5.25}$$

is the Hilbert transform of $\hat{\Gamma}(u)$.

Eliminating $E\{\mathbf{p}(t;\alpha)\}$ between (5.20a) and (5.20b), we obtain the second-order equation

$$\frac{d^2}{dt^2}E\{\mathbf{x}(t;\boldsymbol{\alpha})\}+c_2\frac{d}{dt}E\{\mathbf{x}(t;\alpha)\}+\omega_0^2\left(1-\frac{c_1}{\omega_0^2}\right)E\{\mathbf{x}(t;\alpha)\}=0 \tag{5.26}$$

for the mean position vector. It is clear from this expression that the presence of random fluctuations has a significant effect, even at the level of the first statistical moment. The average position is damped by an amount proportional to c_2. According to (5.23), this damping may be negative when the fluctuations are particularly strong at twice the unperturbed frequency. Furthermore, a shift in the oscillator frequency arises, which is determined by the Hilbert transform of the spectrum of the correlation function $\Gamma(\tau)$. Identical results have been reported recently by Van Kampen (cf. Ref. 20) who applied the long-time Markovian approximation directly to the equations of motion of one-dimensional classical harmonic oscillator. The coincidence of his results with ours is not surprising at all since the mean trajectory of the quantum mechanical oscillator is exactly the same with the path traversed by a classical harmonic oscillator. [More generally, this statement is valid whenever the potential field $V(\mathbf{x},t;\alpha)$ in (1.1) is such that the exact Wigner equation is of the form of a Liouville equation.]

C. $V(\mathbf{x},t;\alpha)=1/2\,k[\mathbf{x}-\mathbf{a}\delta H(t;\alpha)]^2$

The Wigner distribution function is governed in this case exactly by the stochastic Liouville equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)f(\mathbf{x},\mathbf{p},t;\alpha)=k\delta H(t;\alpha)\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{p}}f(\mathbf{x},\mathbf{p},t;\alpha). \tag{5.27}$$

The corresponding kinetic equation for the mean Wigner distribution function in the first-order smoothing approximation has the form

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=k^2\frac{\partial}{\partial\mathbf{p}}\cdot\left[\int_0^t d\tau\,\Gamma(\tau)\mathbf{a}\mathbf{a}\cdot\left(\frac{\sin\omega_0\tau}{m\omega_0}\frac{\partial}{\partial\mathbf{x}}+\cos\omega_0\tau\frac{\partial}{\partial\mathbf{p}}\right)\times E\left\{f\left(\mathbf{x}\cos\omega_0\tau-\frac{1}{m\omega_0}\mathbf{p}\sin\omega_0\tau,\mathbf{p}\cos\omega_0\tau+m\omega_0\mathbf{x}\sin\omega_0\tau,t-\tau;\alpha\right)\right\}\right], \tag{5.28}$$

where $\Gamma(\tau)=E\{\delta H(t;\alpha)\delta H(t-\tau;\alpha)\}$.

For a random process $\delta H(t;\boldsymbol{\alpha})$ which is wide-sense stationary and δ correlated in time, viz., $\Gamma(\tau)=D\delta(\tau)$, where D is a constant, the time integration in (5.28) can be carried out explicitly. The resulting transport equation is

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=k^2D\left(\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{p}}\right)^2E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.29}$$

If, in addition to the above assumptions $\delta H(t;\alpha)$ is a Gaussian random process, (5.29) is the *exact* statistical equation for $E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}$.

The stochastic Liouville equation (5.27) is identical to the equation governing the classical distribution function $f_c(\mathbf{x},\mathbf{p},t;\alpha)=\delta[\mathbf{x}-\mathbf{x}(t;\alpha)]\delta[\mathbf{p}-\mathbf{p}(t;\alpha)]$, $f_c(\mathbf{x},\mathbf{p},0;\alpha)=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{p}-\mathbf{p}_0)$ associated with the Brownian motion of a simple, classical, harmonic oscillator, viz., $(d/dt)\mathbf{x}(t;\alpha)=(1/m)\mathbf{p}(t;\alpha)$, $(d/dt)\mathbf{p}(t;\alpha)=-k\mathbf{x}(t;\alpha)+\mathbf{a}\delta H(t;\boldsymbol{\alpha})$, with $\mathbf{x}(0;\alpha)=\mathbf{x}_0$, $\mathbf{p}(0;\alpha)=\mathbf{p}_0$. Equation (5.29) has an exact fundamental solution since, except for the initial condition, it is identical to the equation satisfied by $E\{f_c(\mathbf{x},\mathbf{p},t;\alpha)\}$, and the latter has been studied extensively (cf. Ref. 30).

In the long-time Markovian approximation, (5.28) reduces to the simpler transport equation

$$\left(\frac{\partial}{\partial t}+\frac{1}{m}\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{x}}-k\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{p}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}=\left(\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(1)}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{D}^{(2)}\cdot\frac{\partial}{\partial\mathbf{x}}\right)E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}. \tag{5.30}$$

The dyadic diffusion coefficients are given by the expressions

$$\mathbf{D}^{(1)} = (k^2 \int_0^\infty d\tau\, \Gamma(\tau) \cos\omega_0\tau)\mathbf{a}\,\mathbf{a}, \tag{5.31a}$$

$$\mathbf{D}^{(2)} = \left(\frac{k^2}{m\omega_0} \int_0^\infty d\tau\, \Gamma(\tau) \sin\omega_0\tau\right)\mathbf{a}\,\mathbf{a}. \tag{5.31b}$$

They can be easily written in terms of the spectrum $\hat{\Gamma}(u)$ and its Hilbert transform $\hat{\Gamma}_H(u)$ as follows:

$$\mathbf{D}^{(1)} = \pi k^2 \mathbf{a}\,\mathbf{a}\hat{\Gamma}(\omega_0)/2, \tag{5.32a}$$

$$\mathbf{D}^{(2)} = \pi k^2 \mathbf{a}\,\mathbf{a}\hat{\Gamma}_H(\omega_0)/(2m\omega_0). \tag{5.32b}$$

Since both $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are constant, it is possible to determine a general fundamental solution for the Fokker–Planck equation (5.30).

APPENDIX A: THE UNCERTAINTY PRINCIPLE IN PHASE SPACE

On the basis of the Schwartz inequality,

$$|f(\mathbf{x},\mathbf{p},t;\alpha)|^2 \le (2\pi\hbar)^{-6}[\int_{R^3} d\mathbf{y}\, |\psi^*(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)|^2] \times [\int_{R^3} d\mathbf{y}\, |\psi(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha)|^2]. \tag{A1}$$

Consider the integral

$$I_+ \equiv \int_{R^3} d\mathbf{y}\, |\psi^*(\mathbf{x}+\tfrac{1}{2}\mathbf{y},t;\alpha)|^2 = 6 \int_{R^3} d\mathbf{x}\, \psi\, |(\mathbf{x},t;\alpha)|^2. \tag{A2}$$

The total action, however, is conserved for every realization $\alpha \in A$, and is assumed to be normalized to unity (cf. Sec. 2). Therefore, $I_+ = 6$. Similarly,

$$I_- \equiv \int_{R^3} d\mathbf{y}\, |\psi(\mathbf{x}-\tfrac{1}{2}\mathbf{y},t;\alpha)|^2 = 6. \tag{A3}$$

Using these results in (A1) we obtain, finally,

$$|f(\mathbf{x},\mathbf{p},t;\alpha)| \le (\hbar\pi)^{-3}, \quad \forall \alpha \in A. \tag{A4}$$

APPENDIX B: DEGREE OF COHERENCE

Given a wavefunction $\psi(\mathbf{x},t;\alpha)$, the *degree of coherence*, $D(t)$, is defined as follows:

$$D^2(t) = (2\pi\hbar)^3 \int_{R^3} d\mathbf{x} \int_{R^3} d\mathbf{p}[E\{f(\mathbf{x},\mathbf{p},t;\alpha)\}]^2 = \int_{R^3} d\mathbf{x}_2 \int_{R^3} d\mathbf{x}_1\, |E\{\psi^*(\mathbf{x}_2,t;\alpha)\psi(\mathbf{x}_1,t;\alpha)\}|^2. \tag{B1}$$

This quantity is intimately linked with the irreversible loss of information (coherence) due to the statistical fluctuations.

The degree of coherence is characterized by the property

$$D^2(t) \le 1, \tag{B2}$$

the equality holding for the case of a purely coherent state. To show this we note that in the absence of random fluctuations (B1) reduces to

$$D^2(t) = \int_{R^3} d\mathbf{x}_2 \int_{R^3} d\mathbf{x}_1\, |\psi^*(\mathbf{x}_2,t)\psi(\mathbf{x}_1,t)|^2 = [\int_{R^3} d\mathbf{x}_2\, |\psi(\mathbf{x}_2,t)|^2][\int_{R^3} d\mathbf{x}_1\, |\psi(\mathbf{x}_1,t)|^2] = 1, \tag{B3}$$

the final equality following because of the conservation of the total action.

To prove the inequality $D^2(t) < 1$, which holds for a partially coherent (mixed) state, we use the Cauchy–Schwartz inequality,[33] viz.,

$$|E\{\psi^*(\mathbf{x},t;\alpha)\psi(\mathbf{x}_1,t;\alpha)\}|^2 \le E\{|\psi(\mathbf{x}_2,t;\alpha)|^2\}E\{|\psi(\mathbf{x}_1,t;\alpha)|^2\}, \tag{B4}$$

in conjunction with (B1). We then have

$$D^2(t) \le [\int_{R^3} d\mathbf{x}_2\, E\{|\psi(\mathbf{x}_2,t;\alpha)|^2\}][\int_{R^3} d\mathbf{x}_1\, E\{|\psi(\mathbf{x}_1,t;\alpha)|^2\}] = 1, \tag{B5}$$

the last equality following from the fact that the total mean action is conserved and is normalized to unity.

APPENDIX C: INTEGRATION OF THE KINETIC EQUATION (4.19)

We shall integrate here the transport equation (4.19) and use the result to determine several averaged observables. For simplicity, we shall restrict the discussion to the one-dimensional case.

Taking a double Fourier transform of (4.19), we obtain the initial value problem

$$\left\{\frac{\partial}{\partial t} - \frac{1}{m} q \frac{\partial}{\partial u} - ku\frac{\partial}{\partial q} + \frac{1}{\hbar^2}[\gamma(0) - \gamma(\hbar u)]\right\} E\{\hat{f}(q,u,t;\alpha)\} = 0, \tag{C1a}$$

$$E\{\hat{f}(q,u,0;\alpha)\} = \hat{f}_0(q,u), \tag{C1b}$$

where

$$E\{\hat{f}(q,u,t;\alpha)\} = (2\pi)^{-2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, \exp[-i(qx+up)]E\{f(x,p,t;\alpha)\}. \tag{C2}$$

We next introduce a new function

$$g(q,u,t) = \exp(\nu t)E\{\hat{f}(q,u,t;\alpha)\}, \tag{C3}$$

together with a new set of variables (r,ϕ), defined by the relations

$$u = (mk)^{-1/2} r \cos\phi, \tag{C4a}$$

$$q = r\sin\phi. \tag{C4b}$$

The equation for the time evolution of $\tilde{g}(\phi,t) = g[r\sin\phi, (mk)^{-1/2} r\cos\phi, t]$ now takes the following form,

$$\left(\frac{\partial}{\partial t} + \omega_0 \frac{\partial}{\partial \phi} - \frac{1}{\hbar^2}\tilde{\gamma}(\phi)\right)\tilde{g}(\phi,t) = 0, \tag{C5a}$$

$$\tilde{g}(\phi,0) = \tilde{g}_0(\phi), \tag{C5b}$$

where $\tilde{\gamma}(\phi) = \gamma[(mk)^{-1/2}\hbar r\cos\phi]$.

The solution of (C5) can be found by the method of characteristics. It is given by

$$\tilde{g}(\phi,t) = \left\{\exp\frac{1}{\hbar^2}\int_0^t d\tau\, \tilde{\gamma}(\phi - \omega_0\tau)\right\}\tilde{g}_0(\phi - \omega_0 t). \tag{C6}$$

Returning to the original variables, we finally have

$$E\{\hat{f}(q,u,t;\alpha)\} = \exp\left[-\nu t + \frac{1}{\hbar^2}\int_0^t d\tau\, \gamma\left(\hbar u\cos\omega_0\tau + \frac{\hbar q}{m\omega_0}\sin\omega_0\tau\right)\right] \times \hat{f}_0\left(q\cos\omega_0 t - m\omega_0 u\sin\omega_0 t, u\cos\omega_0 t + \frac{q}{m\omega_0}\sin\omega_0 t\right). \tag{C7}$$

Many important averaged physical observables can be found directly from (C7), making use of the fact that the moments of $E\{f(x,p,t;\alpha)\}$ can be expressed in terms of derivatives of $E\{\hat{f}(q,u,t;\alpha)\}$. For example, the averaged total energy of the system is given by the formula

$$E\{E(t)\}=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\left(\frac{p^2}{2m}+\tfrac{1}{2}kx^2\right)E\{f(x,p,t;\alpha)\}$$
$$=-(2\pi)^2\left(\frac{1}{2m}\frac{\partial^2}{\partial u^2}E\{\hat{f}(q,u,t;\alpha)\}+\tfrac{1}{2}k\frac{\partial^2}{\partial q^2}E\{\hat{f}(q,u,t;\alpha)\}\right)_{q=u=0}. \tag{C8}$$

Substituting (C7) into the above expression, we obtain

$$E\{E(t)\}=E\{E(0)\}-\gamma''(0)t/2m. \tag{C9}$$

Since $\gamma''(0)<0$, we can see immediately that this model predicts amplification of the energy of the particle due the stochastic variations of the potential field. The formula (C9) is also valid for the case of free propagation ($\omega_0\to 0$). For the three-dimensional case, (C9) is replaced by

$$E\{E(t)\}=E\{E(0)\}-3\gamma''(0)t/2m. \tag{C10}$$

Expressions for other physical averaged observables are listed below:

(i) *Mean centroid of a wavepacket*:

$$E\{x_c(t)\}=E\{x_c(0)\}\cos\omega_0 t+\frac{1}{m\omega_0}E\{p_c(0)\}\sin\omega_0 t; \tag{C11a}$$

(ii) *Mean momentum*:

$$E\{p_c(t)\}=E\{p_c(0)\}\cos\omega_0 t-m\omega_0 E\{x_c(0)\}\sin\omega_0 t; \tag{C11b}$$

(iii) *Spatial spread of a wavepacket*:

$$E\{\sigma_x^2(t)\}=E\{\sigma_x^2(0)\}\cos^2\omega_0 t+(m\omega_0)^{-2}E\{\sigma^2(0)\}\sin^2\omega_0 t\times\frac{1}{m\omega_0}E\{\sigma_{xp}^2(0)\}\sin 2\omega_0 t-\frac{\gamma''(0)}{(m\omega_0)^2}\times\left(\frac{t}{2}-\frac{\sin 2\omega_0 t}{4\omega_0}\right); \tag{C11c}$$

(iv) *Momentum spread of a wavepacket*:

$$E\{\sigma_p^2(t)\}=E\{\sigma_p^2(0)\}\cos^2\omega_0 t+(m\omega_0)^2E\{\sigma_x^2(0)\}\times\sin^2\omega_0 t-m\omega_0E\{\sigma_{xp}^2(0)\}\sin 2\omega_0 t-\gamma''(0)\left(\frac{t}{2}+\frac{\sin 2\omega_0 t}{4\omega_0}\right). \tag{C11d}$$

In the limit as $\omega_0\to 0$ (free propagation), these results simplify as follows:

(i) $$E\{x_c(t)\}=E\{x_c(0)\}+\frac{1}{m}E\{p\ (0)\}t; \tag{C12a}$$

(ii) $$E\{p_c(t)\}=E\{p_c(0)\}; \tag{C12b}$$

(iii) $$E\{\sigma_x^2(t)\}=E\{\sigma_x^2(0)\}+\frac{1}{m^2}E\{\sigma_p^2(0)\}t^2-\frac{2}{m}E\{\sigma_{xp}^2(0)\}t-\frac{\gamma''(0)}{3m^2}t^3; \tag{C12c}$$

(iv) $$E\{\sigma_p^2(t)\}=E\{\sigma_p^2(0)\}-\gamma''(0)t. \tag{C12d}$$

It is interesting to note that the average spread of a wavepacket grows with time due to the presence of stochastic fluctuations. The growth is proportional to the first power of time for a particle in the field of an elastic force, and to the third power of time for a freely propagating particle. On the other hand, the spread of momentum grows linearly with time in both cases.

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Lorentz transformations as space–time reflections. II. Timelike reflections

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Continuing previous work, some features of the covariant factorization of Lorentz transformations, into complementary space–time reflections, are further discussed in terms of timelike reflections. Several properties of timelike reflections are shown, which bear interesting relations with the Lorentz tensor performing active isometric transformations between two inertial observers, while their geometric meaning is also briefly examined. Some product rules for timelike reflections appear, as a background enabling the discussion of the group multiplication laws for Lorentz isometry tensors. This brings into the fore the realization of the restricted Lorentz group attained in the present formalism. The instances of two multiplication laws of the Lorentz tensors are examined. Next, the same problem (i.e., factorization of an ordinary rotation into two complementary reflections) is readily solved for the Euclidean 3-space. Both formalisms (Euclidean and Minkowskian) are essentially the same. Indeed, factorization of an isometric transformation into two complementary reflections is a general property of flat Riemannian geometry. A few concluding remarks are presented.

1. INTRODUCTION

Continuing previous work,[1] in this note we further analyze Lorentz transformations in a manifestly Lorentz covariant manner.

It is well understood today that the best (and certainly the most elegant) approach to relativity theory obtains by considering it as the geometry of absolute space—time.[2] The very program of every relativity theory requires us to emphasize this remark as strongly as possible. Thus, for instance, the Einsteinian principle of special relativity demands that the laws of physics have to conform to this geometric approach in a manifestly Lorentz covariant fashion.

It is clear that Lorentz transformations themselves, considered as the fundamental laws of free motion, should not evade this normative rule of geometric covariance. In other words, one should represents Lorentz transformations by means of a (rank two) space—time *tensor*, able to perform active isometric transformations between inertial observers characterized by their 4-velocities.[3] The problem, however, seems to have been neglected in the literature.[4] As was shown in Paper I, its solution casts some new light on the structure of Lorentz transformations, by showing the rather simple (and important) role played by space—time reflections.[5]

The present paper dwells only on timelike reflections, as a background for handling restricted Lorentz transformations by means of a sprightly enough absolute formalism. In Sec. 2 we review some algebraic features of timelike relfections, and we also present their relations with the Lorentz tensor introduced in Paper I. Several product rules for timelike reflections are shown, while their rather intuitive geometric meaning is neatly stated. Next, in Sec. 3 we discuss two group multiplication laws for the Lorentz tensors, which bring into the fore the realization of the restricted Lorentz group attained in the present formalism. Again, the space—time geometry involved in these algebraic manipulations appears quite intuitively, and is briefly discussed. In Sec. 4, the same problem is succinctly solved for Euclidean three-dimensional space (i.e., factorization of an ordinary proper rotation into two complementary reflections). The great similitude between the Euclidean and the Minkowskian factorization formalisms is stressed, for they are essentially the same. Finally, in Sec. 5 we end up with a few general remarks.

2. TIMELIKE REFLECTIONS

With the aim of studying those properties of timelike reflections which play an outstanding role in the performance of proper orthochronous Lorentz transformations, let us first briefly recall some results already presented in Paper I. For the sake of handiness, in the present note we omit tensor indices throughout; instead, we use matrix notation to denote 4-vectors and rank two 4-tensors.[6] Accordingly, for the Lorentz tensor $L^{\mu}{}_{\nu}$ presented in Paper I, characterizing an active restricted transformation from an old v-frame into a new u-frame, we write[7]

$$L(v,u)=I-(u-v)\circ u^{+}-(1+v^{+}\cdot u)^{-1}(u+v)\circ v^{+}\circ(I-u\circ u^{+}), \tag{2.1}$$

where I stands for the 4×4 identity and where v and u denote the 4-velocities of two inertial observers. It was shown in Paper I that if we transform space—time events by means of this $L(v,u)$ tensor, the familiar features for an active Lorentz transformation obtain. As we shall see presently, it is useful to write, instead of Eq. (2.1), the more compact equivalent expression:

$$L(v,u)=I-(1+v^{+}\circ u)^{-1}(v+u)\circ(v+u)^{+}+2v\circ u^{+}, \tag{2.2}$$

at which we arrive after some straightforward steps. In this form one immediately observes that the Minkowski adjoint L^{+} merely interchanges the role of the 4-velocities v and u; i.e.,

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$$L^{+}(v,u)=L(u,v), \tag{2.3}$$

and also that

$$L(v,v)=I, \tag{2.4}$$

as it should be. Moreover, the following transformations hold:

$$L(v,u)\cdot u=v, \tag{2.5}$$

$$L(v,u)\circ v=2(v^{+}\cdot u)\,v-u, \tag{2.6}$$

which allow us to find the expected inversion law for a Lorentz matrix. Indeed, we get

$$L(v,u)\circ L(u,v)=I. \tag{2.7}$$

We next recall some notions concerning timelike reflections. As we know, if u is a unit timelike vector, the rank two tensor defined by

$$R(u)=I-2u\circ u^{+} \tag{2.8}$$

is a space—time operator reflecting every 4-vector by an hyperplane with timelike normal u. Simple immediate properties of timelike reflections (needed presently) follow:

$$R(u)=R^{+}(u)=R(-u), \tag{2.9}$$

$$R(u)\cdot R(u)=I, \tag{2.10}$$

$$L(v,u)\cdot R(u)\circ L(u,v)=R(v), \tag{2.11}$$

$$R(v)\cdot u=-L(v,u)\circ v, \tag{2.12}$$

$$R(u)\circ n\circ R^{T}(u)=n. \tag{2.13}$$

Of course, space—time reflections are Lorentz transformations by themselves [i.e., Eq. (2.13) above]. They do not form a subgroup of the Lorentz group, however, for clearly the identity does not belong to the set, nor does the product of two reflections correspond to a reflection [cf. Eq. (2.24) below].

In order to further clarify the role played by timelike reflections in restricted Lorentz transformations, we observe that the Minkowski self-adjoint matrix given by

$$R(v+u)=I-(1+v^{+}\cdot u)^{-1}(v+u)\circ(v+u)^{+}, \tag{2.14}$$

which figures in Eq. (2.2), corresponds precisely to a space—time reflection along a unit timelike vector $v\,\hat{+}\,u$ (say) defined as follows:

$$v\,\hat{+}\,u=[2(1+v^{+}\cdot u)]^{-1/2}(v+u), \tag{2.15}$$

i.e., the "mean 4-velocity" of both inertial observers, duly normalized. We call this vector the *normalized sum* of v and u. We have, indeed,[8]

$$R(v+u)=I-2(v\,\hat{+}\,u)\circ(v\,\hat{+}\,u)^{+}. \tag{2.16}$$

Hence, the Lorentz tensor presented in Eq. (2.2) corresponds to

$$L(v,u)=R(v+u)+2v\circ u^{+}. \tag{2.17}$$

In this manner, since

$$R(v+u)\circ u=-v, \tag{2.18}$$

we easily prove the fundamental result:

$$L(v,u)=R(v)\circ R(v+u)=R(v+u)\cdot R(u). \tag{2.19}$$

Thus we have two equivalent factorizations of $L(v,u)$ into two "complementary" timelike reflections. The first factorization which appears in Eq. (2.19), i.e., $R(v)\cdot R(v+u)$, was already presented in Paper I.[9] Also, the fact that the essential features of the $L(v,u)$ transformation of events (namely, Fitzgerald contraction and time dilation) are entirely due to the $R(v+u)$ reflection factor has been remarked in that paper. The second factorization $R(v+u)\circ R(u)$ is new, and interesting. Indeed, Eq. (2.19) gives us a kind of "quasicommutation" rule for timelike reflections which, *sensu stricto*, do not commute. The result stated in Eq. (2.19) has an important meaning in what follows because it governs all those nontrivial facts of timelike reflection geometry, which we now proceed to review.

To further simplify our notation, let us also define the *normalized difference* between two timelike unit vectors, u and v, as the timelike vector

$$u\,\hat{-}\,v=-R(u)\circ v=2(u^{+}\circ v)u-v. \tag{2.20}$$

Clearly, this normalized difference is such that

$$(u\,\hat{-}\,v)\,\hat{+}\,v=u. \tag{2.21}$$

It must be observed that both $u\,\hat{+}\,v$ and $u\,\hat{-}\,v$ are timelike future-pointing unit vectors (as u and v are). We also remark that

$$v\,\hat{+}\,v=v\,\hat{-}\,v=v \tag{2.22}$$

holds for all 4-velocity v. Figure 1 is a space-time diagram representing the construction of the $u\,\hat{-}\,v$ vector in the (u,v)-flat. We observe, quite generally, that $u\,\hat{-}\,v$ is not the same as $-(v\,\hat{-}\,u)$ (while, of course, $v\,\hat{+}\,u=u\,\hat{+}\,v$, always). Finally, the following useful relation,

$$v\,\hat{-}\,(v\,\hat{+}\,u)=v\,\hat{+}\,(v\,\hat{-}\,u), \tag{2.23}$$

can be proved, and shall be used presently.

After this preliminaries, let us next write $v\,\hat{+}\,u=w$ in Eq. (2.19); that is, we set $v=w\,\hat{-}\,u$ and $u=w\,\hat{-}\,v$. The following *product law* for timelike reflections obtains[8]:

$$R(v)\cdot R(u)=L(v,u-v)=L(v-u,u), \tag{2.24}$$

quite generally. These equivalent Lorentz tensors correspond to transformations from the v-frame into the $(u\,\hat{-}\,v)$-frame, and from the $(v\,\hat{-}\,u)$-frame into the u-frame, respectively. We represent these relations in Fig. 2, which helps clarifying the intuitive meaning of

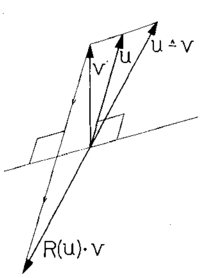

FIG. 1. Space—time diagram representing the construction of the $u\,\hat{-}\,v$ normalized timelike vector.

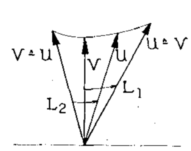

FIG. 2. Space—time representation of the identity $L_1=L_2$, with $L_1=L(v,u-v)$ and $L_2=L(v-u,u)$.

Eq. (2.24). Indeed, the vectors $v \hat{-} u$, v, u, and $u \hat{-} v$, all belong in the same 2-flat and, moreover, it is clear that the "rotation" from $v \hat{-} u$ into u is the same as the "rotation" from v to $u \hat{-} v$. Furthermore, by the same token, for any Lorentz tensor $L(v,u)$ we get the identities

$$L(v,u) = L[v+u, u+(v-u)] = L[v+(v-u), v+u], \tag{2.25}$$

which geometric meaning is easy to grasp. In the same manner, using the quasicommutation law stated in Eq. (2.19), we immediately get

$$R(v)\cdot R(u)\circ R(v) = R(v-u), \tag{2.26}$$

as the *reflection law for timelike reflections*. Hence, a new *law of inversion* for the Lorentz tensor obtains; namely,[10]

$$L^{-1}(v,u) = L(v, v-u). \tag{2.27}$$

We readily interprete this law, since v is the $L(v,u)$ transformed vector of u[cf. Eq. (2.5)], while $v \hat{-} u$ is the image of v upon $L(v,u)$, and therefore the transformation from u to v is the same as the transformation from v to $v \hat{-} u$ [cf. Eq. (2.6)]. Furthermore, we may tickle the argument and get a new geometric identity:

$$L(v,u) = L(u, u-v) = L(v-u, v). \tag{2.28}$$

Finally, we also note the useful relation

$$R(u+v)\circ R(v-u)\cdot R(u+v) = R(u-v), \tag{2.29}$$

which can be readily proved.

This completes our review of the essentials of timelike reflection geometry. Similar features can be obtained for spacelike reflections, while changing some minor details.

3. SOME GROUP MULTIPLICATION RULES

In the present section we study two group multiplication laws for the $L(v,u)$ tensors; that is, we tackle the problem raised by the Lorentz tensor realization of the restricted Lorentz group.

As was shown in Paper I, each L tensor of the form (2.2), with v and u arbitrarily given, corresponds to a well-defined proper orthochronous Lorentz matrix with six independent parameters. Notwithstanding the previously found identities (2.25) and (2.28), it should be clear that this association is an isomorphism. Indeed, the shown identities have a purely formal character.[10] Hence the six parameters are identified, without ado, by means of the six independent components of the 3-velocities v and u describing the motion of the inertial observers relative to any Galilean working frame we may choose. The explicit structure of the $L(v,u)$ tensor relative to a general w-frame [i.e., $v \neq w$, $u \neq w$, and $w^T = (1,0)$] is quite involved. As a matter of fact, however, the simplest interpretation of the $L(v,u)$ tensor (as an active Lorentz transformation operator) arrives while choosing the v-frame as our working frame,[11] i.e., by setting $v^T = (1,0)$, which standpoint clearly left the L tensor with only three independent parameters, namely, the components of the velocity u with respect to the working scaffold. Therefore, if we adopt this point of view (as we do in what follows), $L(v,u)$ corresponds to a restricted Lorentz transformation without rotation.

There are, of course, several multiplication laws for the L tensors, which bear some interest for each admits a different geometric meaning, while they all bring the restricted Lorentz group into the fore. When looking for group multiplication rules, for the sake of simplicity, we will only pay attention to those combinations of two L's which contain three inertial observers (v, u, and w, say, with v for the 4-velocity of our working frame). In effect, these are the most simple combinations of L tensors contrived to give us a six-parameter outcoming tensor product.

The first multiplication rule attains quite directly if we use Eq. (2.24). Indeed, we immediately obtain

$$L(v, u-v)\circ L(u, w-u) = L(v, w-v), \tag{3.1}$$

where the three parameters contained in u have been eliminated in the outcoming result. This group multiplication law of the Lorentz tensors in interesting in that it neatly shows that the set of L tensors in the form $L(v, u-v)$ fulfills the restricted Lorentz group in a manifestly Lorentz covariant fashion. We have

$$L(v, v-v) = I, \tag{3.2}$$

$$L^{-1}(v, u-v) = L(u, v-u), \tag{3.3}$$

$$\begin{aligned} L(v,u-v)\cdot[L(u,w-u)\circ L(w,z-w)] \\ = [L(v,u-v)\circ L(u,w-u)]\circ L(w,z-w) \\ = L(v,z-v). \end{aligned} \tag{3.4}$$

It must be borne in mind that these $L(v,u-v)$ tensors are not the same as the $L(v,v-u)$ tensors obtained in Eq. (2.27).[12] Incidentally, we readily observe that

$$L(v,u)\circ L(v,u) = L(v, u-v) \tag{3.5}$$

is a general rule for "squaring" a Lorentz tensor, and, thus, by the same token, the formula

$$L(v,u) = L(v, v+u)\cdot L(v, v+u) \tag{3.6}$$

immediately solves the problem of finding the "square root" of a given proper orthochronous Lorentz matrix. We wish to remark, once again, the manifest covariance of the whole procedure leading to these results.

As our second example, let us briefly discuss the outcome of first transforming the space—time points x from the working frame (v-frame) into the u-frame, and next from the u-frame into a new w-frame. Let us then compare this product with the direct transformation from the initial v-frame into the final w-frame. Thus, we set

$$x' = L(v,u)\cdot x, \tag{3.7}$$

and also

$$x'' = L(v, w')\circ x', \tag{3.8}$$

where clearly

$$w' = L(v,u)\cdot w \tag{3.9}$$

is the 4-velocity of the w-observer as seen from the u-frame. Of course,

$$L(v,w')=L(v,u)\circ L(u,w)\cdot L(u,v) \quad (3.10)$$

is the L tensor used by the u-observer in its own frame while performing the transformation $u\to w$; i.e., $v\to w'$ [since $v=L(v,u)\cdot u$, as we know]. Moreover, Eq. (3.10) states precisely the transformation of the $L(u,w)$ mixed tensor from the v-frame into the u-frame. Therefore, the product of these transformations corresponds to

$$x''=L(v,u)\cdot L(u,w)\cdot x. \quad (3.11)$$

As is well known, transformation (3.11) does not in general coincide with the direct transformation from v into the w-frame, say,

$$x'''=L(v,w)\cdot x. \quad (3.12)$$

Indeed, if we adopt the v-frame as our working frame, then Eq. (3.8) requires the w'-Cartesian-base to be parallel with the v-Cartesian-base, while in Eq. (3.7) parallelism among the v and u 3-space bases is required. However, because of the relativistic effect of rotation of the Cartesian bases used by three inertial observers, parallelism is not quite generally a transitive property in relativistic geometry. Hence, parallelism among the 3-bases used in the w-frame and in the v-frame [as required in Eq. (3.12)] is not a general consequence of Eqs. (3.7) and (3.8); i.e., x'' and x''' may differ by a space rotation.

In Eq. (3.11) we have the following multiplication law:

$$L(v,u)\circ L(u,w)=R(v+u)\circ R(u+w)$$
$$=L[v+u,(w+u)-(u+v)], \quad (3.13)$$

where the intermediary parameters in u have *not* been eliminated. On the other hand, in order to relate this tensor product with the $L(v,w)$ tensor we observe that

$$v\circ w^{+}=-L(v,u)\cdot u\cdot u^{+}\circ R(u+w)=-v\circ v^{+}\cdot R(v+w), \quad (3.14)$$

and thus, after some manipulations, we get

$$L(v,u)\cdot L(u,w)=L(v,w)-R(v+w)$$
$$+R(v+u)\cdot R(u)\circ R(u+w). \quad (3.15)$$

Hence, we may write the following result:

$$x''=[R(v+u)\cdot R(u+w)\cdot R(w+v)+2v\cdot v^{+}]\cdot x'''. \quad (3.16)$$

Now, since we set $v^T=(1,0)$, from Eq. (3.16) we immediately get the time components of the events x'' and x''' relative to the working frame; i.e., we have

$$t''=v^{+}\circ x''=v^{+}\circ x'''=t''', \quad (3.17)$$

as it should be. In the same manner, for the space components of x'' and x''' (relative to the v-frame) we use the orthogonal projector $I-v\cdot v^{+}$. We thus obtain

$$x''_T=R(v,u,w)\cdot x''_T, \quad (3.18)$$

where

$$x''_T=(I-v\cdot v^{+})\circ x'',\quad x''_T=(I-v\cdot v^{+})\cdot x'' \quad (3.19)$$

and where

$$R(v,u,w)=R(v+u)\cdot R(u+w)\cdot R(w+v) \quad (3.20)$$

is the product of three timelike reflections.

In order to properly interprete Eq. (3.18), let us explicitly write the orthogonal projector $I-v\circ v^{+}$ with respect to the v-frame. It corresponds to the 4×4 arrangement

$$I-v\circ v^{+}=\begin{bmatrix}0 & 0^T\\ 0 & \mathrm{I}\end{bmatrix}, \quad (3.21)$$

where, clearly, 0 denotes the null 3-column-vector, and **I** is the 3×3 identity. In the same manner, let

$$R(v,u,w)=\begin{bmatrix}A & \mathrm{B}^T\\ \mathrm{C} & \mathrm{R}\end{bmatrix} \quad (3.22)$$

(say) be the matrix arrangement of the components of $R(v,u,w)$, relative to the working frame. Since

$$R^{+}(v,u,w)\cdot R(v,u,w)=I \quad (3.23)$$

and

$$R(v,u,w)\circ R(v)=R(v)\cdot R(v,u,w), \quad (3.24)$$

we have

$$A=\pm 1,\quad \mathrm{B}=\mathrm{C}=0; \quad (3.25)$$

moreover, we also get

$$\mathrm{R}^T\circ\mathrm{R}=\mathrm{I}. \quad (3.26)$$

Thus, relative to the working frame, we have found that

$$R(v,u,w)=\begin{bmatrix}\pm 1 & 0^T\\ 0 & \mathrm{R}\end{bmatrix}, \quad (3.27)$$

where **R** is an orthogonal 3×3 matrix, as it should be.

Hence, we have found that the image-events x'' and x''', corresponding to the same object event x, have equal time components relative to the v-frame, while their space components in this frame are related by means of a (proper or improper) 3-rotation. This completes our discussion of transformations (3.11) and (3.12).

4. ROTATIONS AND REFLECTIONS IN EUCLIDEAN 3-SPACE

In this section we briefly discuss the factorization method for an ordinary proper rotation into two complementary reflections for Euclidean three-dimensional space. The formalism in this case turns out to be so simple that we do not claim originality for it. Anyhow, it seems important to observe that we can cast ordinary proper rotations into two factorized reflections forms, which bear a great similarity to the Minkowskian factorization formalism previously discussed, and which (although elementary) is generally unrecognized in the current literature of mathematical physics. Again, the tool afforded by this approach allows us to work out rotations in ordinary space by means of reflections, while these are somehow simpler to handle. In particular, we wish to mention here that a formalism obtains which is specially adapted for stuyding Pauli spinors,[13] even if we do not touch on applications in the present note.

As we did in Paper I,[14] let us consider in ordinary 3-space two orthonormal right-handed Cartesian triads, $\{i,j,k\}$ and $\{i',j',k'\}$, say, such that[15]

$$j = -[1-(k^T \circ k')^2]^{-1/2}(I - k\circ k^T)\cdot k', \tag{4.1}$$

$$j' = [1-(k^T \circ k')^2]^{-1/2}(I - k'\circ k'^T)\circ k, \tag{4.2}$$

$$i' = i. \tag{4.3}$$

These triads satisfy the orthgonality conditions, as well as the relations of completeness. Therefore (as is well known, indeed), these orthonormal Cartesian bases are related by means of the proper rotation

$$R(k,k') = i\circ i'^T + j\cdot j'^T + k\cdot k'^T. \tag{4.4}$$

As can be proved, after some straightforward steps, the following expression obtains:

$$R(k,k') = I - (1+k^T\cdot k')^{-1}(k+k')\circ(k+k')^T + 2k\circ k'^T \tag{4.5}$$

[cf. Eq. (2.2)]. Therefore, while introducing the normalized sum

$$k \hat{+} k' = [2(1+k^T\circ k')]^{-1/2}(k+k'), \tag{4.6}$$

we get [cf. Eq. (2.17)]

$$R(k,k') = R(k+k') + 2k\cdot k'^T, \tag{4.7}$$

where $R(k+k')$ performs the reflections by the plane orthogonal to $k\hat{+}k'$; namely,

$$R(k+k') = I - 2(k\hat{+}k')\cdot(k\hat{+}k')^T. \tag{4.8}$$

However, once we arrive at this point, it is an easy matter to further prove that [cf. Eq. (2.19)]

$$R(k,k') = R(k+k')\cdot R(k') = R(k)\cdot R(k+k'). \tag{4.9}$$

where $R(k)$ and $R(k')$ are reflection operators by the planes orthogonal to k and k', respectively. Equation (4.9) states the expected factorizations of the proper rotation (which brings k' into k) in terms of two elementary reflections.

5. CONCLUDING REMARKS

We conclude this note with some few remarks. It seems that the reflection factorization approach affords an interesting tool for handling "rotations" in flat geometry. Indeed, the great similarity between the Euclidean and the Minkowskian factorization formalism should be stressed, for they are essentially the same. It is clear that factorization of isometric transformations (i.e., "rotations") into two complementary reflections appears as a general property of flat Riemannian spaces, since the method is resting exclusively on the most general properties of the flat metric.

It is also interesting to remark that this formalism seems to probe the structure of Lie groups operating as isometric groups in flat spaces, without recourse to the infinitesimal transformations and the associated Lie algebra, as is usually done. Indeed, the possibility arises of having enough further information on the behavior of general space—time reflections, for instance, as to end up with a complete understanding of the structure of the Lorentz group. In this sense, the compactness of the reflection method for handling Dirac spinors[13] seems to reveal a very fundamental relation between space—time reflections and the structure of the Lorentz group. On these grounds, although reflections do not form a group by themselves, further research would be desirable as to group theoretic possibilities of reflection geometry.

[1]J. Krause, "Lorentz transformations as space—time reflections," J. Math. Phys. 18, 889 (1977), hereafter referred to as Paper I.

[2]Cf. J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1965), 2nd ed., p. 34.

[3]"Subluminal" Lorentz transformations are here alluded, to be sure; "superluminal" transformations [cf. E. Recami and R. Mignani, Nuovo Cimento 4, 209, 398 (1974)] are not included in the present formalism.

[4]Nevertheless, there are some exceptions; cf., e.g., S. L. Basanski, J. Math. Phys. 6, 1201 (1965), on which we already comment in Paper I.

[5]By the same token, a new tool obtains for handling geometric problems in space—time; e.g., see Paper I, Appendix B. Other applications of physical interest will be published elsewhere.

[6]Tensors of higher rank than two are not considered in this paper. A contravariant vector v^μ, $\mu = 0,1,2,3$, is simply denoted as v, and it corresponds (by definition) to a 4-column, while the 4-row v^T is written for the transposed matrix. The Minkowski metric is represented by the matrix $n = \mathrm{diag}(+---)$. We thus introduce the *Minkowski adjoint* (say) v^+, of the vector column v, as $v^+ = v^T\cdot n$. ("Dots" indicate matrix multiplication.) Hence, $v^+\cdot v = v_\mu v^\mu$. In the same way, for a rank two mixed tensor $S^\mu\cdot{}_\nu$, we simply write S, while the associated Minkowski adjoint matrix $S^+ = n\cdot S^T\cdot n$ corresponds to the tensor $S_\mu^{\cdot\cdot\nu}$. Rank two tensors in the forms $S^{\mu\nu}$ and $S_{\mu\nu}$ are avoided in this work (with the only exception of n). Translation to the usual tensor notation is easy. We set $c=1$, throughout.

[7]Cf. Paper I, Eq. (1.1) of that article.

[8]It should be understood that all "sums" and "differences" which appear in the arguments of our tensor operators are duly normalized, according to Eqs. (2.15) and (2.20). Thus, we write $R(v+u)$ instead of $R(v\hat{+}u)$, and $L(v,u-v)$ to denote $L(v,u\hat{-}v)$, etc.

[9]Cf. Paper I, Eq. (3.5).

[10]It is clear that if v' and u' are unit timelike vectors belonging in the 2-flat (v,u), and such that $v' = L(v,u)\cdot u'$, then $L(v',u') = L(v,u)$, quite generally. Hence, the shown identities are special (and interesting) instances of this theorem.

[11]Cf. Paper I, Eqs. (1.2) and (1.3).

[12]The detailed properties of this group, which we denote by $\mathcal{L}^2$ [cf. Eq. (3.5)], will be discussed elsewhere.

[13]Cf. Paper I, Appendix B, where an application to Dirac spinors is presented.

[14]Cf. Paper I, Appendix A.

[15]See Paper I, Eqs. (A5)—(A8).

Strings and gauge invariance[a)]

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Within the context of the nonrelativistic theory of dyons, we study a number of interrelated issues concerning the quantum formulation of magnetic charge. We begin by solving the two-body Schrödinger equation for an arbitrarily oriented singularity line (string) in terms of the known solutions with the string on the z axis. Charge quantization conditions emerge by requiring that the wavefunctions be single valued. The general solutions express the necessary gauge covariance of the wavefunctions. These results provide a basis for the reconsideration of the phase factor in the dyon–dyon scattering amplitude. Finally, the connection between the formulations in terms of vector potentials and in terms of intrinsic spin is investigated. This approach leads to a rederivation of the gauge transformation properties of the theory.

I. INTRODUCTION

The recent revival of interest[1-3] in the subject of magnetic charge has focused much attention on the fundamental features of charge quantization and the associated "string". Misconceptions continue to persist concerning the Lorentz invariance of the theory and the observability of the string. Our object here is to reinvestigate this subject by examining the "gauge transformations" which relate possible vector potentials associated with different singularity lines (strings) and their connection to charge quantization conditions.

The context of our discussion is nonrelativistic dyon—dyon scattering (dyons are particles carrying both electric and magnetic charge), which was considered at length in a recent paper.[1] There, we assumed that the electromagnetic field was described by a single vector potential. Depending on whether or not there was explicit symmetry under rotations in the electromagnetic plane ($E\to E\cos\theta+H\sin\theta$, $H\to H\cos\theta-E\sin\theta$, and similarly for charges, currents, and potentials), the corresponding string was required to be infinite (symmetric case) or semi-infinite (unsymmetric case) and the charge quantization condition was found to be[4]

$$m'\equiv-(e_1g_2-e_2g_1)=\begin{cases} n, & \text{symmetric},\\ \tfrac{1}{2}n, & \text{unsymmetric},\end{cases} \tag{1.1}$$

where n is an integer. (The semi-infinite string gives the least restrictive quantization condition, but it is not forced upon us by the theory nor would it necessarily be realized by actual, physical dyons.) We here wish to explore to what extent we are free to choose the vector potentials that occur, how such choices are related by gauge transformations, and what types of quantization conditions emerge. Also of interest is the formulation of this problem in terms of an intrinsic spin, from which the charge quantization conditions and gauge transformation properties may be derived directly from the properties of angular momentum.

Section II deals with the vector potential description. In order that the Hamiltonian for the two-dyon system may be separated into center-of-mass and relative motion terms, of the four possible vector potentials (and associated strings), at most two can be independent. If the vector potentials have the same structure, we obtain quantization conditions of the form (1.1); otherwise, we obtain quantization conditions on the individual products

$$e_ag_b=\begin{cases} n, & \text{infinite},\\ \tfrac{1}{2}n, & \text{semi-infinite}.\end{cases} \tag{1.2}$$

Integer quantization occurs whenever an infinite string [see (2.12)] is used. The term "symmetry" however, does not apply here, since the charges do not occur in the invariant combination of (1.1).

All these relations between strings and quantization conditions arise from the requirement that the wavefunction describing the two-dyon system be single-valued. In the process, we find the wavefunction when the string is arbitrarily oriented. This immediately gives us the gauge transformation of the wavefunction when the orientation of the string is changed. For integer values of the charge combination [(1.1) or (1.2)], gauge transformations between infinite and semi-infinite strings, and vice versa, are allowable, in contrast to integer plus one-half values, where only a semi-infinite string is permissible.

As an application of the above transformations, we consider, in Sec. III, the scattering problem in which both the direction of the string and the direction of propagation of the incident particle are arbitrary. We find the general wavefunction, and thereby the scattering amplitude, for a state having a particular value of the projection of the total angular momentum along the incident direction, exhibiting explicitly the physical unobservability of the string.

In Sec. IV we examine another formulation that leads to the two-dyon problem, in which the angular momentum of the system is regarded as composed of two parts, orbital angular momentum and an intrinsic spin common to the system as a whole.[5] The interaction of the particles may be expressed solely in terms of this spin. A particular dyon system may be realized by diagonalizing the spin variable through a suitable unitary transformation. Different quantization conditions may be achieved by different diagonalizations. These unitary transformations are not gauge transformations, but a sequence of such transformations, which serves to reorient the direction of the string, is equivalent to the gauge transformation considered in Sec. II.

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In the Appendix we remark upon the singular nature of the gauge transformations which occur in Secs. II and IV. Consequently, when acting on the space of physical states, the angular momentum operator satisfies canonical commutation relations everywhere, and the field strength, which occurs, for example, in magnetic moment couplings, is string independent.

II. STRINGS

Throughout, we will discuss the nonrelativistic, quantum scattering of two dyons, with electric and magnetic charges e_1, g_1 and e_2, g_2, respectively. The Hamiltonian for the system is[2,4,6]

$$\mathcal{H} = \tfrac{1}{2}m_1 v_1{}^2 + \tfrac{1}{2}m_2 v_2{}^2 + \frac{e_1e_2 + g_1g_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{2.1}$$

where, in terms of the canonical momenta, the velocities are given by

$$\begin{aligned} m_1\mathbf{v}_1 &= \mathbf{P}_1 - e_1\mathbf{A}_{e2}(\mathbf{r}_1, t) - g_1\mathbf{A}_{m2}(\mathbf{r}_1, t), \\ m_2\mathbf{v}_2 &= \mathbf{P}_2 - e_2\mathbf{A}_{e1}(\mathbf{r}_2, t) - g_2\mathbf{A}_{m1}(\mathbf{r}_2, t). \end{aligned} \tag{2.2}$$

The electric (e) and magnetic (m) vector potentials are[1]

$$4\pi\mathbf{A}_e(\mathbf{r}, t) = 4\pi\nabla\lambda_e(\mathbf{r}, t) - \int (d\mathbf{r}')\mathbf{f}(\mathbf{r} - \mathbf{r}')\times\mathbf{H}(\mathbf{r}, t) \tag{2.3}$$

and

$$4\pi\mathbf{A}_m(\mathbf{r}, t) = 4\pi\nabla\lambda_m(\mathbf{r}, t) + \int (d\mathbf{r}')\,{}^*\mathbf{f}(\mathbf{r} - \mathbf{r}')\times\mathbf{E}(\mathbf{r}', t), \tag{2.4}$$

with

$$\begin{aligned} \lambda_e(\mathbf{r}, t) &= \int (d\mathbf{r}')\,\mathbf{f}(\mathbf{r} - \mathbf{r}')\cdot\mathbf{A}_e(\mathbf{r}', t), \\ \lambda_m(\mathbf{r}, t) &= \int (d\mathbf{r}')\,{}^*\mathbf{f}(\mathbf{r} - \mathbf{r}')\cdot\mathbf{A}_m(\mathbf{r}', t). \end{aligned} \tag{2.5}$$

Here, the functions f and *f represent the strings and must satisfy

$$\nabla\cdot{}^{(*)}\mathbf{f}(\mathbf{r} - \mathbf{r}') = 4\pi\delta(\mathbf{r} - \mathbf{r}'). \tag{2.6}$$

A priori, f and *f need not be related and could be different for each source. So, for the case of dyon—dyon scattering, it would seem that four independent strings are possible.

The first condition we impose on the Schrödinger equation,

$$\mathcal{H}\Psi = E\Psi, \tag{2.7}$$

is that it separates when center-of-mass and relative coordinates are employed, which implies

$$\begin{aligned} e_1\mathbf{A}_{e2}(\mathbf{r}_1, t) &= -g_2\mathbf{A}_{m1}(\mathbf{r}_2, t) \equiv e_1g_2\mathcal{A}(\mathbf{r}), \\ e_2\mathbf{A}_{e1}(\mathbf{r}_2, t) &= -g_1\mathbf{A}_{m2}(\mathbf{r}_1, t) \equiv e_2g_1\mathcal{A}'(\mathbf{r}), \end{aligned} \tag{2.8}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Correspondingly, there are relations between the various f functions,

$${}^*\mathbf{f}_1(\mathbf{x}) = -\mathbf{f}_2(-\mathbf{x}), \quad {}^*\mathbf{f}_2(\mathbf{x}) = -\mathbf{f}_1(-\mathbf{x}), \tag{2.9}$$

leaving only two independent ones. The Hamiltonian for the relative coordinates now reads

$$\mathcal{H} = \frac{1}{2\mu}[\mathbf{p} - e_1g_2\mathcal{A}(\mathbf{r}) + e_2g_1\mathcal{A}'(\mathbf{r})]^2 + \frac{e_1e_2 + g_1g_2}{r}, \tag{2.10}$$

where μ is the reduced mass.

If we further require that only one vector potential be present, $\mathcal{A} = \mathcal{A}'$, so that the combination appearing in (1.1) occurs in the Hamiltonian, one more relation is obtained between the two f functions,

$$\mathbf{f}_2(\mathbf{x}) = -\mathbf{f}_1(-\mathbf{x}). \tag{2.11}$$

Notice that (2.11) possesses two types of solutions. (1) There is a single string, necessarily infinite, satisfying

$$\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x}). \tag{2.12}$$

As shown in Appendix A of Ref. 1, the vector potentials transform in the same way as the charges and currents under E, H rotations whenever this condition is satisfied. This is the so-called symmetric case. (2) There are two strings, necessarily semi-infinite, which are negative reflections of each other. If identical semi-infinite strings are employed, so that $\mathcal{A} \neq \mathcal{A}'$, the individual charge products in (1.2) occur in the dynamics. The singularities of $\mathcal{A}$ and $\mathcal{A}'$ lie on lines parallel and antiparallel to the strings, respectively. We will see the consequences for the charge quantization condition of these different choices in the following.

We now return to the general situation embodied in (2.10). For simplicity, we choose the string associated with $\mathcal{A}$ to be a straight line lying along the direction n,

$$\mathcal{A} = \begin{cases} -\dfrac{1}{r}\dfrac{\mathbf{n}\times\mathbf{r}}{r - (\mathbf{n}\cdot\mathbf{r})}, & \text{semi-infinite,} \quad (2.13\text{a}) \\[2ex] -\dfrac{1}{r}\dfrac{1}{2}\left(\dfrac{\mathbf{n}\times\mathbf{r}}{r - (\mathbf{n}\cdot\mathbf{r})} - \dfrac{\mathbf{n}\times\mathbf{r}}{r + (\mathbf{n}\cdot\mathbf{r})}\right), & \text{infinite.} \quad (2.13\text{b}) \end{cases}$$

This result is valid in the gauge in which $\lambda_{e(m)}$ [Eq. (2.5)] is equal to zero.[1] Without loss of generality, we will take $\mathcal{A}'$ to be given by (2.13) with

$$\mathbf{n} \to \hat{z}.$$

(This corresponds to taking the string associated with $\mathcal{A}'$ to point along the $-z$ axis, $\mathbf{f}_1 \propto -\hat{z}$.)

We now wish to convert the resulting Hamiltonian, $\mathcal{H}$, into a form, $\mathcal{H}'$, in which all the singularities lie along the z axis. The Schrödinger equation in the latter case has been solved, for example, in Ref. 1, yielding the quantization condition (1.1). This conversion is effected by a unitary transformation[7] (essentially a gauge transformation),

$$\mathcal{H}' = e^{i\Lambda}\mathcal{H}e^{-i\Lambda}. \tag{2.14}$$

The differential equation determining Λ is

$$\nabla\Lambda = e_1g_2[\mathcal{A}'(\mathbf{r}) - \mathcal{A}(\mathbf{r})]. \tag{2.15}$$

We take n to be

$$\mathbf{n} = \sin\chi\cos\psi\hat{x} + \sin\chi\sin\psi\,\hat{y} + \cos\chi\hat{z}, \tag{2.16}$$

and use spherical coordinates, to find

$$\Lambda = -e_1g_2\beta(\mathbf{n}, \mathbf{r}) \tag{2.17}$$

where, for the semi-infinite string (Dirac)

$$\beta_D = \phi - \psi + (\cos\theta - \cos\chi)F_-(\theta, \phi - \psi, \chi) - 2\pi\eta(\chi - \theta), \tag{2.18a}$$

and for the infinite string (Schwinger)

$$\beta_S = \tfrac{1}{2}[(\cos\theta - \cos\chi)F_-(\theta, \varphi - \psi, \chi) - 2\pi\eta(\chi - \theta) + (\cos\theta + \cos\chi)F_+(\theta, \phi - \psi, \chi)]. \tag{2.18b}$$

The functions occurring here are

$$F_\pm(\theta,\alpha,\chi) = \int_0^\alpha \frac{d\phi'}{1 \pm \cos\chi\cos\theta \pm \sin\chi\sin\theta\cos\phi} = \frac{2}{|\cos\theta \pm \cos\chi|}\,\epsilon(\alpha) \times \tan^{-1}\left[\left(\frac{1\pm\cos(\chi+\theta)}{1\pm\cos(\chi-\theta)}\right)^{1/2} \tan\frac{|\alpha|}{2}\right], \tag{2.19}$$

where the arctangent is not defined on the principal branch but is such that $F_\pm(\theta,\alpha,\chi)$ is a monotone increasing function of α. The step functions occurring here are defined by

$$\eta(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi < 0, \end{cases} \tag{2.20}$$

$$\epsilon(\xi) = \begin{cases} 1, & \xi > 0, \\ -1, & \xi < 0. \end{cases} \tag{2.21}$$

The phases, β_D and β_S, satisfy the appropriate differential equation, (2.15), for $\theta \neq \chi$ (as well as $\theta \neq \pi - \chi$ for β_S) and are determined up to constants. The step functions are introduced here in order to make $e^{i\Lambda}$ continuous at $\theta = \chi$ and $\pi - \chi$, as will be explained below. We now observe that

$$F_\pm(\theta, 2\pi + \alpha, \chi) - F_\pm(\theta,\alpha,\chi) = \frac{2\pi}{|\cos\theta \pm \cos\chi|}, \tag{2.22}$$

so that the wavefunction,

$$\Psi = e^{-i\Lambda}\Psi' \tag{2.23}$$

(where Ψ' is the solution[1] to the problem with the singularity on the z axis) is single-valued under $\phi \to \phi + 2\pi$ when the quantization condition (1.2) is satisfied.

Notice that integer quantization follows when an infinite string is used while a semi-infinite string leads to half-integer quantization, since β_S changes by a multiple of 2π when $\phi \to \phi + 2\pi$, while β_D changes by a multiple of 4π. Notice that β_D possesses a discontinuity, which is a multiple of 4π, at $\theta = \chi$, while β_S possesses discontinuities, which are mutiples of 2π, at $\theta = \chi$, $\pi - \chi$. In virtue of the above derived quantization conditions, $e^{i\Lambda}$ is continuous everywhere. Correspondingly, the unitary operator $e^{i\Lambda}$, which relates solutions of differential equations with different vector potentials, is alternatively viewed as a gauge transformation relating physically equivalent descriptions of the same system, since it converts one string into another. [Identical arguments applied to the case when only one vector potential is present leads to the conditions (1.1) for infinite and semi-infinite strings, respectively.]

It is now a simple application of the above results to transform a system characterized by a single vector potential with an infinite string along the direction n into one in which the singularity line is semi-infinite and lies along the z axis. This can be done in a variety of ways; particularly easy is to break the string at the origin and transform the singularities to the z axis.

Making use of (2.17) with $-e_1 g_2 \to \frac{1}{2}m'$ and (2.18a) for n and $-$n, we find

$$\Lambda = m'\beta_S'(\mathbf{n}, \mathbf{r}) \tag{2.24}$$

with

$$\beta_S' = \phi - \psi + \beta_S. \tag{2.25}$$

In particular, we can relate the wavefunctions for infinite and semi-infinite singularity lines on the z axis by setting $\chi = 0$ in (2.25),

$$\beta_S' = \phi - \psi,$$

so

$$\Psi(\text{infinite}) = e^{-im'(\phi-\psi)}\Psi(\text{semi-infinite}) \tag{2.26}$$

which, in its ϕ dependence, is in agreement with the result found in Ref. 1 [see Eq. (3.24) there]. Note that (2.25) or (2.26) reiterates that an infinite string requires integer quantization.

III. SCATTERING

In the above, we related the wavefunction when the string lies along the direction n with that when the string lies along the z axis. When there is only a single vector potential (which, for simplicity, we will assume throughout the following), this relation is

$$\Psi_{\mathbf{n}} = e^{-im'\beta(\mathbf{n},\mathbf{r})}\,\Psi', \tag{3.1}$$

where β is given by (2.18a), (2.18b), or (2.25), for the various cases. For concreteness, if we take Ψ' to be a state corresponding to a semi-infinite singularity line along the $+z$ axis, then β is either β_D [(2.18a)] or β_S' [(2.25)] depending on whether the singularity characterized by n is semi-infinite or infinite. By means of (3.1), we can easily build up the relation between solutions corresponding to two arbitrarily oriented strings, with n and n$'$ say,

$$\Psi_{\mathbf{n}'} = e^{-im'[\beta(\mathbf{n}',\mathbf{r}) - \beta(\mathbf{n},\mathbf{r})]}\Psi_{\mathbf{n}}, \tag{3.2}$$

which expresses the gauge covariance properties of the wavefunctions.

For scattering, we require a solution that consists of an incoming plane wave and an outgoing spherical wave. We will consider an eigenstate of $\mathbf{J}\cdot\hat{k}$ where J is the total angular momentum (see Sec. IV),

$$\mathbf{J} = \mathbf{r}\times(\mathbf{p} + m'\mathcal{A}_{\mathbf{n}}) + m'\mathbf{r}, \tag{3.3}$$

and $\hat{k}$ is the unit vector in the direction of propagation of the incoming wave (not necessarily the z axis). This state cannot be an eigenstate of $\hat{k}\cdot(\mathbf{r}\times\mathbf{p})$, since this operator does not commute with the Hamiltonian. However, since

$$e^{i\Lambda}\hat{k}\cdot\mathbf{J}e^{-i\Lambda} = \hat{k}\cdot(\mathbf{r}\times\mathbf{p}) - m', \tag{3.4}$$

with

$$\Lambda = m'[\beta(\mathbf{n},\mathbf{r}) - \beta_D(\hat{k},\mathbf{r})], \tag{3.5}$$

the incoming state with eigenvalue[8,9]

$$(\hat{k}\cdot\mathbf{J})' = -m' \tag{3.6}$$

is simply related to an ordinary modified plane wave [η is defined in (3.11), below],

$$\Psi_{\text{in}} = e^{-i\Lambda} \exp\{i[\mathbf{k}\cdot\mathbf{r} + \eta \ln(kr - \mathbf{k}\cdot\mathbf{r})]\}. \quad (3.7)$$

This state exhibits the proper gauge covariance under reorientations of the string. In (3.5), the appropriate β's are given by (2.18a) or (2.25).

The asymptotic form of the wavefunction is

$$\Psi \sim e^{-im'\beta(\mathbf{n},\mathbf{r})} \sum_{j\bar{m}} A_{kj\bar{m}} \mathcal{Y}^{m'}_{j\bar{m}}(\hat{r}) e^{im'\phi} \times \frac{1}{kr} \sin\left(kr - \eta \ln 2kr - \frac{\pi}{2} L + \delta_L\right). \quad (3.8)$$

The summation in (3.8) is the general form of the solution when the singularity line is semi-infinite, extending along the $+z$ axis, as shown in Ref. 1. In particular, $\mathcal{Y}^{m'}_{jm}$ is a generalized spherical harmonic $[\hat{r} = (\theta, \phi)]$,

$$\langle jm' | \exp(i\psi J_3)\exp(i\theta J_2)\exp(i\varphi J_3) | jm\rangle = \exp(im'\psi)(2j+1)^{-1/2} \mathcal{Y}^{m'}_{jm}(\hat{r}), \quad (3.9)$$

δ_L is the Coulomb phase shift for noninteger L,

$$\delta_L = \arg\Gamma(L + 1 + i\eta), \quad (3.10)$$

and

$$L + \tfrac{1}{2} = [(j+\tfrac{1}{2})^2 - m'^2]^{1/2}, \quad \eta = \frac{\mu}{k}(e_1 e_2 + g_1 g_2). \quad (3.11)$$

Upon defining Ψ_{out} by[8]

$$\Psi \sim e^{-i\Lambda}[\exp\{i[\mathbf{k}\cdot\mathbf{r} + \eta \ln(kr - \mathbf{k}\cdot\mathbf{r})]\} + \Psi_{\text{out}}], \quad (3.12)$$

where Λ is given by (3.5), we find

$$\Psi_{\text{out}} = \frac{1}{r} e^{i(kr - \eta\ln 2kr)} e^{im'\gamma} f(\bar{\theta}). \quad (3.13)$$

In terms of the scattering angle, $\bar{\theta}$, which is the angle between $\mathbf{k}$ and $\mathbf{r}$, the scattering amplitude is[1]

$$2ikf(\bar{\theta}) = \sum_{j=|m'|}^{\infty} \sqrt{2j+1}\, \mathcal{Y}^{m'}_{jm'}(\pi - \bar{\theta}, 0)\exp[-i(\pi L - 2\delta_L)]. \quad (3.14)$$

The extra phase factor in (3.13) is given by (where $\hat{k}$ is characterized by θ', ϕ' and $-\hat{k}$ by $\pi - \theta'$, $\phi' \pm \pi$)

$$\gamma = \beta_D(\hat{k}, -\hat{k}) + \phi - \phi' \mp \pi - \beta_D(\hat{k}, \mathbf{r}) + \bar{\phi}, \quad (3.15)$$

where[1,10]

$$\tan^{-1}\tfrac{1}{2}\bar{\phi} = \cos\left(\frac{\theta + \pi - \theta'}{2}\right)\sin\left(\frac{\phi - \phi' \mp \pi}{2}\right) \times \left[\cos\left(\frac{\theta - \pi + \theta'}{2}\right)\cos\left(\frac{\phi - \phi' \mp \pi}{2}\right)\right]^{-1}. \quad (3.16)$$

Straightforward evaluation shows that

$$\tfrac{1}{2}\gamma = 0 \pmod{2\pi}, \quad (3.17)$$

so that there is no additional phase factor in the outgoing wave. Thus the phase factor found in Ref. 1 multiplying the outgoing wave is, in fact, an overall phase factor multiplying the entire scattering wavefunction. Of course this clarification in no way changes either the scattering amplitude or cross sections calculated there.

IV. SPIN

Classically, the electromagnetic field due to two dyons at rest carries angular momentum [see (1.1) for m']

$$\mathbf{S}_{\text{classical}} = m'\hat{r}. \quad (4.1)$$

A quantum mechanical transcription of this fact allows us to replace the nonrelativistic description explored above, in which the interaction is through vector potentials (apart from the Coulomb term), by one in which the particles interact with an intrinsic spin. The derivation of the magnetic charge problem from this point of view seems first to have been carried out by Goldhaber[5] in a simplified context, but it has recently been revived for 't Hooft monopoles[3,8] (where the spin is called "isospin").

Before introducing the notion of spin, we first consider the angular momentum of the actual dyon problem. For simplicity we will describe the interaction between two dyons in terms of a single vector potential $\mathcal{A}$, and an infinite string satisfying (2.12). (The other cases are simple variations on what we do here and the consequences for charge quantization are the same as found in Sec. II.) Then the relative momentum of the system is

$$\mathbf{p} = \mu\mathbf{v} - m'\mathcal{A}. \quad (4.2)$$

Since

$$\nabla \times \mathcal{A} = \mathbf{r}/r^3 - \mathbf{f}(\mathbf{r}), \quad (4.3)$$

we have the following commutation property valid everywhere,

$$\mu\mathbf{v} \times \mu\mathbf{v} = -im'[\mathbf{r}/r^3 - \mathbf{f}(\mathbf{r})]. \quad (4.4)$$

Motivated by the classical situation, we assert that the total angular momentum operator is

$$\mathbf{J} = \mathbf{r} \times \mu\mathbf{v} + m'\hat{r}. \quad (4.5)$$

This is confirmed[11] by noting that, almost everywhere, $\mathbf{J}$ is the generator of rotations:

$$\frac{1}{i}[\mathbf{r}, \mathbf{J}\cdot\delta\boldsymbol{\omega}] = \delta\boldsymbol{\omega} \times \mathbf{r}, \quad (4.6)$$

$$\frac{1}{i}[\mu\mathbf{v}, \mathbf{J}\cdot\delta\boldsymbol{\omega}] = \delta\boldsymbol{\omega} \times \mu\mathbf{v} - m'\mathbf{f}(\mathbf{r}) \times (\delta\boldsymbol{\omega} \times \mathbf{r}), \quad (4.7)$$

where $\delta\boldsymbol{\omega}$ stands for an infinitesimal rotation. The presence of the extra term in (4.7) is consistent only because of the quantization condition.[2] For example, consider the effect of a rotation on the time evolution operator,

$$e^{-i\mathbf{J}\cdot\delta\boldsymbol{\omega}}\exp[-i\int dt\, H]\, e^{i\mathbf{J}\cdot\delta\boldsymbol{\omega}} = \exp[-i\int dt(H + \delta H)] \quad (4.8)$$

where

$$\delta H = i[H, \mathbf{J}\cdot\delta\boldsymbol{\omega}] = m'\mathbf{v}\cdot[\mathbf{f}(\mathbf{r}) \times \delta\mathbf{r}], \quad (4.9)$$

and

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}. \quad (4.10)$$

Using the representation

$$\mathbf{f}(\mathbf{r}) = 4\pi\int_C d\mathbf{x}\,\tfrac{1}{2}[\delta(\mathbf{r} - \mathbf{x}) - \delta(\mathbf{r} + \mathbf{x})], \quad (4.11)$$

where C is any contour starting at the origin and extending to infinity, and the notation

$$dt\mathbf{v} = d\mathbf{r},$$

we have

$$-i\int dt\,\delta H = -im'4\pi\int d\mathbf{r}\cdot(d\mathbf{x}\times\delta\mathbf{r})\tfrac{1}{2}[\delta(\mathbf{r}-\mathbf{x})-\delta(\mathbf{r}+\mathbf{x})]. \tag{4.12}$$

Since the possible values of the integral are 0, $\pm\frac{1}{2}$, ± 1, the unitary time development operator is unaltered by a rotation only if m' is an integer. (Evidently, half-integer quantization results from use of a semi-infinite singularity line.)

Effectively, then, J satisfies the canonical angular momentum commutation relations (also see the Appendix)

$$\frac{1}{i}\,\mathbf{J}\times\mathbf{J}=\mathbf{J}, \tag{4.13}$$

and is a constant of the motion

$$\frac{d}{dt}\,\mathbf{J}=\frac{1}{i}\,[H,\mathbf{J}]=0. \tag{4.14}$$

And, corresponding to the classical field angular momentum (4.1), the component of J along the line connecting the two dyons, m', should be an integer.

The identification of m' as an angular momentum component invites us to introduce an independent spin operator S. We do this by first writing[11]

$$m'=\mathbf{S}\cdot\hat{\mathbf{r}} \tag{4.15}$$

and

$$\mu\mathbf{v}=\mathbf{p}+\frac{\mathbf{S}\times\mathbf{r}}{r^2}, \tag{4.16}$$

which substituted into (4.5) yields

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{S}. \tag{4.17}$$

We now ascribe independent canonical commutation relations to S, and regard (4.15) as an eigenvalue statement. The consistency of this assignment is verified by noting that the commutation property for $\mu\mathbf{v}$ [(4.4) without the f term] still holds true, and that $\mathbf{S}\cdot\hat{\mathbf{r}}$ is a constant of the motion

$$[\mathbf{S}\cdot\hat{\mathbf{r}},\,\mu\mathbf{v}]=0. \tag{4.18}$$

In this angular momentum description, the Hamiltonian, (2.10), can be written in the form

$$H=\frac{1}{2\mu}\left[p^2+\frac{2\mathbf{S}\cdot\mathbf{L}}{r^2}+\frac{S^2-(\mathbf{S}\cdot\hat{\mathbf{r}})^2}{r^2}\right]+\frac{e_1e_2+g_1g_2}{r}, \tag{4.19}$$

where the orbital angular momentum occurs,

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}. \tag{4.20}$$

The total angular momentum, J, appears when the operator,

$$p_r^2=\frac{1}{r^2}\left((\mathbf{r}\cdot\mathbf{p})^2+\frac{1}{i}\,\mathbf{r}\cdot\mathbf{p}\right), \tag{4.21}$$

is introduced into the Hamiltonian

$$H=\frac{1}{2\mu}\left(p_r^2+\frac{J^2-(\mathbf{J}\cdot\hat{\mathbf{r}})^2}{r^2}\right)+\frac{e_1e_2+g_1g_2}{r}. \tag{4.22}$$

In an eigenstate of J^2 and $\mathbf{J}\cdot\hat{\mathbf{r}}$,

$$(J^2)'=j(j+1),\quad (\mathbf{J}\cdot\hat{\mathbf{r}})'=m', \tag{4.23}$$

(4.22) yields the radial Schrödinger equation solved in Ref. 1. This modified formulation, only formally equivalent to our starting point, makes no reference to a vector potential or string.

We now proceed to diagonalize the S dependence of the Hamiltonian, (4.19) or (4.22), subject to the eigenvalue constraint

$$(\mathbf{S}\cdot\hat{\mathbf{r}})'=m'. \tag{4.24}$$

This is most easily done by diagonalizing the angular momentum operator[8] (4.17). In order to operate in a framework sufficiently general to include our original symmetrical starting point, we first write S as the sum of two independent spins,

$$\mathbf{S}=\mathbf{S}_a+\mathbf{S}_b. \tag{4.25}$$

We then subject J to a suitable[5] unitary transformation

$$\mathbf{J}'=U\mathbf{J}U^{-1}, \tag{4.26}$$

where

$$U=\exp[i(\mathbf{S}_a\cdot\hat{\boldsymbol{\phi}})\theta]\exp[i(\mathbf{S}_b\cdot\hat{\boldsymbol{\phi}})(\theta-\pi)], \tag{4.27}$$

which rotates $\mathbf{S}_{a,b}\cdot\hat{\mathbf{r}}$ into $\pm(S_{a,b})_3$. This transformation is easily carried out by making use of the representation in terms of Euler angles,

$$\exp(i\mathbf{S}\cdot\hat{\boldsymbol{\phi}}\,\theta)=\exp(-i\phi S_3)\exp(i\theta S_2)\exp(i\phi S_3). \tag{4.28}$$

The general form of the transformed angular momentum,

$$\mathbf{J}'=\mathbf{r}\times\left[\mathbf{p}+\frac{\hat{\boldsymbol{\phi}}}{r}\sin\theta\left(\frac{S_{a3}}{1+\cos\theta}+\frac{S_{b3}}{1-\cos\theta}\right)\right]+\hat{\mathbf{r}}(S_a-S_b)_3, \tag{4.29}$$

is subject, *a priori*, only to the constraint (4.24), or

$$(S_a-S_b)_3'=m'. \tag{4.30}$$

We recover the unsymmetrical and symmetrical formulations by imposing the following supplementary eigenvalue conditions:

$$(1)\ S_{a3}'=0, \tag{4.31a}$$

$$(2)\ (S_a+S_b)_3'=0. \tag{4.31b}$$

These yield the angular momentum in the form (4.5), the vector potential appearing there being, respectively,

$$(1)\ \mathbf{A}=-\frac{\hat{\boldsymbol{\phi}}}{r}\cot\frac{\theta}{2}, \tag{4.32a}$$

$$(2)\ \mathbf{A}=-\frac{\hat{\boldsymbol{\phi}}}{r}\cot\theta, \tag{4.32b}$$

which are (2.13) with $\mathbf{n}=\hat{\mathbf{z}}$.

The effect of this transformation on the Hamiltonian is most easily seen from the form (4.22),

$$U\left[p_r^2+\frac{J^2-(\mathbf{J}\cdot\hat{\mathbf{r}})^2}{r^2}\right]U^{-1}=p_r^2+\frac{1}{r^2}(\mathbf{r}\times\mu\mathbf{v})^2=(\mu v)^2, \tag{4.33}$$

making use of (4.21), or

$$H'=UHU^{-1}=\tfrac{1}{2}\mu v^2+\frac{e_1e_1+g_1g_2}{r}. \tag{4.34}$$

So by means of the transformation given in (4.27) we have derived the explicit magnetic charge problem, expressed in terms of J′ and H', from the implicit formulation in terms of the spin. These transformations are not really gauge transformations, because the physical dyon theory is defined only after the eigenvalue conditions (4.30) and (4.31) are imposed. The unsymmetrical condition (1) [(4.31a)] gives rise to the Dirac formulation of magnetic charge, with a semi-infinite singularity line, and, from (4.30), m' either integer or half-integer. The symmetrical condition (2) [(4.31b)] gives the Schwinger formulation: an infinite singularity line [with (2.12) holding], and integer quantization of m'. These correlations, which follow directly from the commutation properties of angular momentum (the group structure), are precisely the conditions required for the consistency of the magnetic charge theory, as we have seen in Sec. II.

Even though the individual unitary operators U are not gauge transformations, a sequence of them, which serves to reorient the string direction, is equivalent to such a transformation. For example, if we formally set $S_a=0$ in (4.27),

$$U_{(1)}=\exp[i\mathbf{S}\cdot\hat{\phi}(\theta-\pi)], \tag{4.35}$$

we have the transformation which generates a vector potential with singularity along the positive z axis, (4.32a), while

$$U_{(2)}=\exp[i\mathbf{S}\cdot\mathbf{u}_2(\Theta-\pi)] \tag{4.36}$$

generates a vector potential with singularity along n, (2.13a), where Θ is the angle between n and r,

$$\cos\Theta=\cos\theta\cos\chi+\sin\theta\sin\chi\cos(\phi-\psi) \tag{4.37}$$

[the coordinates of n are given by (2.16)], and

$$\mathbf{u}_2=\frac{\mathbf{n}\times\mathbf{r}}{|\mathbf{n}\times\mathbf{r}|}. \tag{4.38}$$

The transformation which carries (4.32a) into (2.13a) is

$$U_{(12)}=U_{(2)}U_{(1)}^{-1}. \tag{4.39}$$

Since $U_{(12)}$ reorients the string from the direction $\hat{z}$ to the direction n, it must have the form

$$U_{(12)}=\exp(i\mathbf{S}\cdot\mathbf{n}\Phi)\exp(-i\mathbf{S}\cdot\hat{\psi}\chi). \tag{4.40}$$

The angle of rotation about the n axis, Φ, is most easily determined by considering the case $\mathbf{S}=\frac{1}{2}\sigma$, and introducing a right-handed basis,

$$\mathbf{u}_1=\mathbf{n},\ \mathbf{u}_2=\frac{\mathbf{n}\times\mathbf{r}}{|\mathbf{n}\times\mathbf{r}|},\ \mathbf{u}_3=\mathbf{n}\times\mathbf{u}_2. \tag{4.41}$$

Then, straightforward algebra yields

$$\cos\tfrac{1}{2}\Phi=\frac{\sin\frac{1}{2}\theta\cos\frac{1}{2}\chi-\cos\frac{1}{2}\theta\sin\frac{1}{2}\chi\cos(\phi-\psi)}{\sin\frac{1}{2}\Theta} \tag{4.42a}$$

and

$$\sin\tfrac{1}{2}\Phi=\frac{-\cos\frac{1}{2}\theta\sin\frac{1}{2}\chi\sin(\phi-\psi)}{\sin\frac{1}{2}\Theta}. \tag{4.42b}$$

The corresponding transformation carrying the vector potential with singularities along the negative z axis [(4.32a) with $\theta\to\theta-\pi$], into the vector potential with singularities along the direction of $-\mathbf{n}$ [(2.13a) with $\mathbf{n}\to-\mathbf{n}$], are obtained from (4.40) and (4.42) [see also (4.35) and (4.36)] by the substitutions

$$\theta\to\theta+\pi,\quad \Theta\to\Theta+\pi. \tag{4.43}$$

The combination of these two cases gives the transformation of the infinite string, for which (4.27) is the prototype.

Since the effect of $\exp(-i\mathbf{S}\cdot\hat{\psi}\chi)$ is completely given by

$$\exp(-i\mathbf{S}\cdot\hat{\psi}\chi)S_3\exp(i\mathbf{S}\cdot\hat{\psi}\chi)=\mathbf{S}\cdot\mathbf{n}, \tag{4.44}$$

that is, for the transformation (4.40),

$$\begin{aligned}U_{(12)}&\left[\mathbf{r}\times\left(\mathbf{p}+\frac{\hat{\phi}}{r}\cot\frac{\theta}{2}S_3\right)-\hat{r}S_3\right]U_{(12)}^{-1}\\&=\exp(i\mathbf{S}\cdot\mathbf{n}\Phi)\left[\mathbf{r}\times\left(\mathbf{p}+\frac{\hat{\phi}}{r}\cot\frac{\theta}{2}\mathbf{S}\cdot\mathbf{n}\right)-\hat{r}\mathbf{S}\cdot\mathbf{n}\right]\\&\times\exp(-i\mathbf{S}\cdot\mathbf{n}\Phi),\end{aligned} \tag{4.45}$$

in a state where $\mathbf{S}\cdot\mathbf{n}$ has a definite eigenvalue, $-m'$, $U_{(12)}$ is effectively just the gauge transformation which reorients the string from the z axis to the direction n. And, indeed, in this case,

$$\tfrac{1}{2}\Phi=\tfrac{1}{2}\beta_D \pmod{2\pi}, \tag{4.46}$$

where β_D is given by (2.18a) as determined by the differential equation method.

V. CONCLUSIONS

There is no classical Hamiltonian theory of magnetic charge, since, without introducing an arbitrary unit of action,[12] unphysical elements (strings) are observable. In the quantum theory, however, there is a unit of action, $\hbar$, and since it is not the action W which is observable, but $\exp(iW/\hbar)$, a well-defined theory exists provided charge quantization conditions of the form (1.1) or (1.2) are satisfied. The precise form of the quantization condition depends on the nature of the strings, which define the vector potentials. It may be worth noting that the situation which first comes to mind, namely, a single vector potential with a single string, implies Schwinger's symmetrical formulation with integer quantization.[2]

We have seen in the nonrelativistic treatment of the two-dyon system that the charge quantization condition is essential for all aspects of the self-consistency of the theory. Amongst these we list the nonobservability of the string, the single valuedness and gauge covariance of the wavefunctions and the compatability with the commutation relations of angular momentum. In fact, all these properties become evident when it is recognized that the theory may be derived from an angular momentum formulation.[13,14]

APPENDIX: SINGULAR GAUGE TRANSFORMATIONS

We here wish to show that it is precisely the singular nature of the gauge transformations (2.14) and (4.33) which is required for the consistency of the theory, that is, the nonobservability of the string. To illustrate this, we will consider a simpler context than that of the text, that is, an electron moving in the field of a static magnetic charge of strength g, which produces a magnetic field

$$\mathbf{H}=g\frac{\hat{r}}{r^2}. \tag{A1}$$

The string appears in the relation of H to the vector potential,

$$H = \nabla \times A + gf(r), \tag{A2}$$

where the string function f satisfies (2.6). Reorienting the string consequently changes A,

$$A \to A', \tag{A3}$$

which induces a phase change in the wavefunction,

$$\psi \to \psi' = \exp(i\Lambda)\psi. \tag{A4}$$

The equation determining Λ is (2.15),

$$\nabla\Lambda = e(A' - A), \tag{A5}$$

which makes manifest that this is a gauge transformation of a singular type, since

$$\nabla \times \nabla\Lambda \neq 0. \tag{A6}$$

Recognition of this fact is essential in understanding the commutation properties of the mechanical momentum,

$$\pi = p - eA, \tag{A7}$$

since

$$\pi \times \pi = -\nabla \times \nabla + ie(\nabla \times A). \tag{A8}$$

(Here, the parentheses indicate that ∇ acts only on A, and not on anything else to the right.) Consider the action of the operator (A8) on an energy eigenstate ψ. Certainly $\nabla \times \nabla\psi = 0$ away from the string; on the string, we isolate the singular term by making a gauge transformation reorienting the string,

$$\psi = \exp(-i\Lambda)\psi', \tag{A9}$$

where ψ' is regular on the string associated with A. Hence

$$-\nabla \times \nabla\psi = \begin{cases} 0 & \text{off string}, \\ i(\nabla \times \nabla\Lambda)\psi & \text{on string}, \end{cases} \tag{A10}$$

so by (A5) and (A2),

$$-\nabla \times \nabla\psi(r) = iegf(r)\psi(r). \tag{A11}$$

Thus, when acting on an energy eigenstate [which transforms like (A4) under a string reorientation], (A8) becomes

$$\pi \times \pi \to ie[(\nabla \times A) + gf(r)] = ieH. \tag{A12}$$

This means that, under these conditions, the commutation properties of the angular momentum operator (4.5),

$$J = r \times \pi - eg\hat{r}, \tag{A13}$$

are precisely the canonical ones

$$\frac{1}{i}[r, J \cdot \delta\omega] \to \delta\omega \times r, \tag{A14a}$$

$$\frac{1}{i}[\pi, J \cdot \delta\omega] \to \delta\omega \times \pi. \tag{A14b}$$

In Sec. IV, we considered the operator properties of J on the class of states for which $\nabla \times \nabla = 0$, so an additional string term appears in the commutator (4.7). Nevertheless, in this space, **J** is consistently recognized as the angular momentum, because the time evolution operator is invariant under the rotations generated by J. Here, we have considered the complementary space, which includes the energy eigenstates, in which case the angular momentum attribution of **J** is immediate, from (A14).

Incidentally, note that the replacement (A12) is necessary to correctly reduce the Dirac equation describing an electron moving in the presence of a static magnetic charge,

$$(\gamma\pi + m)\psi = 0, \tag{A15a}$$

to nonrelativistic form, since the second order version of (A15a) is

$$(\pi^2 + m^2 - e\sigma \cdot H)\psi = 0, \tag{A15b}$$

where H is the fully gauge invariant, string independent, field strength (A1), rather that $(\nabla \times A)$, as might be naively anticipated.[15] (This form validates the consideration of the dipole moment interaction of Ref. 1, where the nonrelativistic scattering, including anomalous magnetic moment contributions, was analyzed numerically.)

Similar remarks apply to the non-Abelian, spin, formulation of the theory, given by (4.19). If we define the non-Abelian vector potential by

$$eA = -\frac{S \times r}{r^2}, \tag{A16}$$

the mechanical momentum of a point charge moving in this field is

$$\pi = p - eA, \tag{A17}$$

and the magnetic field strength is determined, analogously to (A12), by

$$eH = \frac{1}{i}\,\pi \times \pi = (\nabla \times eA) - ieA \times eA = -S \cdot \hat{r}\,\frac{\hat{r}}{r^2}. \tag{A18}$$

This reduces to the Abelian field strength (A1) in an eigenstate of $S \cdot \hat{r}$,

$$(S \cdot \hat{r})' = -eg, \tag{A19}$$

which is a possible state, since $S \cdot \hat{r}$ is a constant of the motion,

$$[S \cdot \hat{r}, \pi] = 0. \tag{A20}$$

The Abelian description is recovered from this one by means of the unitary transformation (4.28),

$$U = \exp(-i\phi S_3)\exp(i\theta S_2)\exp(i\phi S_3). \tag{A21}$$

Under this transformation, the mechanical momentum, (A17), takes on the Abelian form,

$$U\pi U^{-1} = p + \hat{\phi}\,\frac{S_3}{r}\tan\frac{\theta}{2}, \tag{A22}$$

where we see the appearance of the Abelian potential

$$eA = -S_3\,\frac{\hat{\phi}}{r}\,\tan\frac{\theta}{2}, \tag{A23}$$

corresponding to a string along the $-z$ axis. In an eigenstate of S_3,

$$S_3' = (US \cdot \hat{r}\,U^{-1})' = -eg, \tag{A24}$$

this is the Dirac vector potential. To find the relation between this vector potential and the field strength, we apply the unitary transformation (A21) to the operator

$$e\mathbf{H} = \nabla \times e\mathbf{A} + e\mathbf{A} \times \nabla - ie\mathbf{A} \times e\mathbf{A} \tag{A25}$$

to obtain[16]

$$Ue\mathbf{H}U^{-1} = (\nabla \times e\mathbf{A}) - iU\nabla \times \nabla U^{-1} = (\nabla \times e\mathbf{A}) - S_3\mathbf{f}(\mathbf{r}), \tag{A26}$$

where $\mathbf{f}$ is the particular string function

$$\mathbf{f}(\mathbf{r}) = -4\pi\hat{k}\eta(-z)\delta(x)\delta(y), \tag{A27}$$

η being the unit step function. In this way the result (A2) is recovered.

[1]J. Schwinger, K.A. Milton, W.-y. Tsai, L.L. DeRaad, Jr., and D.C. Clark, Ann. Phys. (N.Y.) **101**, 451 (1976).

[2]J. Schwinger, Phys. Rev. D **12**, 3105 (1975).

[3]T.T. Wu and C.N. Yang, Phys. Rev. D **12**, 3845 (1975); C.N. Yang, "Gauge Fields," in *Proceedings of the Sixth Hawaii Topical Conference in Particle Physics*, 1975 (University of Hawaii, Honolulu, 1976), p. 489 ff; Y. Nambu, Phys. Rev. D **10**, 4262 (1974); G. 't Hooft, Nucl. Phys. B **79**, 276 (1974); A.M. Polyakov, JETP Lett. **20**, 194 (1974).

[4]We use Gaussian units, as well as $\hbar = c = 1$.

[5]A.S. Goldhaber, Phys. Rev. **140**, B1407 (1965).

[6]The action W associated with this Hamiltonian gives the correct Lorentz equations of motion even when the trajectory crosses the string, because of the charge quantization condition. It suffices to note that, quantum mechanically, the physically significant object is $\exp(iW/\hbar)$; crossing the string results in a phase change of 2π.

[7]An earlier discussion of gauge transformations and the charge quantization condition for monopoles can be found in B. Zumino, in *Strong and Weak Interactions—Present Problems*, edited by A. Zichichi (Academic, New York, 1966), p. 711.

[8]D.G. Boulware, L.S. Brown, R.N. Cahn, S.D. Ellis, and C. Lee, Phys. Rev. D **14**, 2708 (1976).

[9]If a different eigenvalue of $\hat{k}\cdot\mathbf{J}$ is used, a phase factor, referring to the azimuthal angle about the $\hat{k}$ direction, appears in (3.7).

[10]The signs of the numerator and denominator, which determine the sine and the cosine, respectively, imply in which quadrant the angle lies.

[11]J. Schwinger, Science **165**, 757 (1969).

[12]The requirement for such a unit of action is shown by T.M. Yan, Phys. Rev. **160**, 1182 (1967). This unit of action also appears in the recent work of R.A. Brandt and J.R. Primack [Phys. Rev. D **15**, 1798 (1977)] as the charge product e_0g_0. In both cases such an arbitrary element is necessary in order to obtain the correct equations of motion on the string.

[13]Another approach to charge quantization and rotations of the string direction, based on angular momentum considerations, can be found in C.A. Hurst, Ann. Phys. (N.Y.) **50**, 51 (1968). See also H.J. Lipkin, W.I. Weisberger, and M. Peshkin, Ann. Phys. (N.Y.) **53**, 203 (1969).

[14]After the completion of this work a related treatment of some of these same questions appeared in R.A. Brandt and J.R. Primack, Phys. Rev. D **15**, 1175 (1977).

[15]This connection between the singular nature of the gauge transformations and the correct magnetic moment coupling can also be found in Ref. 14.

[16]The second equality in (A26) can be most easily obtained by applying Stoke's theorem to a vanishingly small southern polar cap.

Subcriticality and supercriticality of energy dependent neutron transport in slab geometry

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The energy dependent transport system in an anisotropic medium in slab geometry subjecting possible internal source q and incoming fluxes ψ_0, ψ_1 is discussed. It has been shown in an earlier paper that under certain conditions on the average number of secondary neutrons per collision c, the scattering cross section σ, and the optical slab length $2a$, this system has a unique nonnegative solution for all inputs q, ψ_0, ψ_1. The aim of this paper is to establish analogous conditions on c, σ, a so that the system has no nonnegative solution when there is either internal source or incoming fluxes (or both), and it only has the trivial solution when neither internal source nor incoming fluxes are present in the system. This conclusion together with the earlier results yield explicit conditions for insuring the supercriticality and the subcriticality of the energy dependent system and therefore lead to analytical upper and lower bounds for the critical value c^* in terms of σ and a.

I. INTRODUCTION

In a previous paper the author[1] discussed the criticality and subcriticality question for the energy dependent neutron transport in an anisotropic homogeneous medium in slab geometry. The main result in that paper is the establishment of some explicit conditions on the constant c, the average number of secondary neutrons per collision, in terms of the scattering (including fission) cross section σ and the optical thickness $2a$ of the slab so that the energy dependent system is subcritical; that is, the system has a unique nonnegative solution for every internal source q and incoming fluxes ψ_0, ψ_1. The purpose of this paper is to establish some analogous conditions on c in terms of σ, a so that the energy-dependent system has no nonnegative solution for any internal source and incoming fluxes and it only has the trivial solution when there is neither source nor incoming fluxes. The nonexistence of a solution for every inhomogeneous boundary value problem and the existence of only the trivial solution for the homogeneous problem insure that the system is supercritical (see the definition in Sec. 2). These results extend those obtained in Ref. 2 for the case of the monoenergetic system. Since the condition for supercriticality given in this paper and that established in Ref. 1 for subcriticality are both explicit they can be readily used to determine whether and when it is possible or not possible to find a physically meaningful solution. As an example, our results show that if $\int \sigma(\cdot, E')dE' \geq 1$ and $c > 2[1 - E_2(2a)]^{-1}$, where E' is the energy variable, and $E_2(z)$ is the second order exponential integral [see Eq. (2.18)], then it is impossible to find a solution by any existing method when q, ψ_0, ψ_1 are not all identically zero. However, there always exists a unique solution when $\int \sigma(\cdot, E')dE' \leq 1$ and $c < [1 - E_2(a)]^{-1}$ (cf. Ref. 1). More specific conditions on c, σ, a for the nonexistence problem are given in the following section. These conditions together with those obtained in Ref. 1 lead to some explicit upper and lower bounds for the critical value c^* of the energy dependent system.

2. SUBCRITICALITY AND SUPERCRITICALITY

Consider the energy-dependent neutron transport in an anisotropic homogeneous medium in slab geometry with its faces located at $x = -a$ and $x = a$. If the system is subjected to an internal source q and incoming fluxes ψ_0, ψ_1 at the slab faces, then the neutron density $N \equiv N(x, \mu, E)$ is governed by the equation

$$\mu \frac{\partial N}{\partial x} + N = \frac{c}{2} \int_{E_0}^{E_1} \int_{-1}^{1} \sigma(\mu, E; \mu', E') \times N(x, \mu', E')\, d\mu' \, dE' + q(x, \mu, E) \tag{2.1}$$

$$(-a \leq x \leq a), \quad -1 \leq \mu \leq 1, \quad E_0 \leq E \leq E_1),$$

and the boundary condition

$$N(-a, \mu, E) = \psi_0(\mu, E) \quad (0 < \mu \leq 1, \quad E_0 \leq E \leq E_1),$$
$$N(a, \mu, E) = \psi_1(\mu, E) \quad (-1 \leq \mu < 0, \quad E_0 \leq E \leq E_1), \tag{2.2}$$

where c is the average number of secondary neutrons per collision, μ is the direction cosine relative to the x axis, (E_0, E_1) is the energy interval, and σ is the scattering (including fission) cross section which satisfies the condition

$$\tfrac{1}{2} \int_{E_0}^{E_1} \int_{-1}^{1} \sigma(\mu, E; \mu', E')\, d\mu' \, dE' \leq 1.$$

As in Ref. 1 we have taken the total cross section as one so that $2a$ should be considered as the optical thickness of the slab. For physical reasons we assume that σ, q, ψ_0, ψ_1 are all nonnegative continuous (or piecewise continuous) functions of their respective arguments. We also assume that $\sigma(\mu, E; \mu', E') = \sigma(\mu', E'; \mu, E)$.

It is shown in Ref. 1 that the boundary value problem (2.1), (2.2) can be reduced to the integral equation

$$N(x, \mu, E) = (F(N))(x, \mu, E) \quad [(x, \mu, E) \in D], \tag{2.3}$$

where $D = [-a, a] \times [-1, 1] \times [E_0, E_1]$ and

$(F(N))(x, \mu, E)$

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$$
\begin{cases}
\exp[-(x+a)/\mu]\psi_0(\mu,E)+\int_0^{(x+a)/\mu}\exp(-\tau)(f(N)) \\
\times(x-\tau\mu,\mu,E)\,d\tau \quad (0<\mu\le 1), \\
(f(N))(x,0,E) \quad (\mu=0) \\
\exp[(a-x)/\mu]\psi_1(\mu,E)+\int_0^{(a-x)/(-\mu)}\exp(-\tau)(f(N)) \\
\times(x-\tau\mu,\mu,E)\,d\tau \quad (-1\le\mu<0)
\end{cases}
\tag{2.4}
$$

$(-a\le x\le a, E_0\le E\le E_1)$.

In the above relation the function $f(N)$ is given by

$$(f(N))(x,\mu,E)\equiv\frac{c}{2}\int_{E_0}^{E_1}\int_{-1}^{1}\sigma(\mu,E;\mu',E')\times N(x,\mu',E')\,d\mu'\,dE'+q(x,\mu,E). \tag{2.5}$$

Our investigation for the subcriticality and the supercriticality of (2.1), (2.2) is based on the integral form (2.3).

We say that the system (2.3) is subcritical if for every internal source q and incoming fluxes ψ_0, ψ_1 this system has a unique nonnegative solution; it is said to be critical if the corresponding homogeneous system (i.e., $q=\psi_0=\psi_1=0$) has a nontrivial nonnegative solution; and it is called supercritical if (2.3) has no nonnegative solution for any q, ψ_0, ψ_1, not all identically zero, and it only has the trivial solution when q, ψ_0, ψ_1 are all identically zero.

For notational convenience, we set

$$
\begin{aligned}
&\bar{\sigma}(\mu',E')=\sup\{\sigma(\mu,E;\mu',E');\ -1\le\mu\le1,\ E_0\le E\le E_1\},\\
&\underline{\sigma}(\mu',E')=\inf\{\sigma(\mu,E;\mu',E');\ -1\le\mu\le1,\ E_0\le E\le E_1\},\\
&\bar{\sigma}_M(\mu')=\max\{\textstyle\int_{E_0}^{E_1}\bar{\sigma}(\mu',E')\,dE',\ \int_{E_0}^{E_1}\bar{\sigma}(-\mu',E')\,dE'\},\\
&\underline{\sigma}_m(\mu')=\min\{\textstyle\int_{E_0}^{E_1}\underline{\sigma}(\mu',E')\,dE',\ \int_{E_0}^{E_1}\underline{\sigma}(-\mu',E')\,dE'\}.
\end{aligned}
\tag{2.6}
$$

Notice from the hypothesis $\sigma(\mu,E;\mu',E')=\sigma(\mu',E';\mu,E)$ that $\bar{\sigma}(\mu',E')=\bar{\sigma}(\mu,E)$ and $\underline{\sigma}(\mu',E')=\underline{\sigma}(\mu,E)$. Before discussing the supercriticality of the system we state the following theorem from Ref. 1.

Theorem 2.1: Assume that $\bar{\sigma}(\mu',E')>0$ and that

$$c\int_0^1\bar{\sigma}_M(\mu')(1-\exp(-a/\mu'))\,d\mu'<1. \tag{2.7}$$

Then for any nonnegative inputs q, ψ_0, ψ_1 the integral equation (2.3) has a unique nonnegative solution. In particular, the homogeneous system (i.e., $q=\psi_0=\psi_1=0$) has only the trivial solution $N=0$.

The result in Theorem 2.1 implies that under the condition (2.7) the transport system is subcritical. As a direct consequence of Theorem 2.1 we also have

Corollary 1: Assume that $\bar{\sigma}(\mu',E')>0$, $\bar{\sigma}_M(\mu')\le1$ and $c<[1-E_2(a)]^{-1}$. Then the system (2.3) is subcritical.

Proof: This follows from Theorem 2.1 using the relation

$$c\int_0^1\bar{\sigma}_M(\mu')(1-\exp(-a/\mu'))\,d\mu' \le c\int_0^1(1-\exp(-a/\mu'))\,d\mu' = c(1-E_2(a))<1.$$

In order to obtain a sufficient condition for the supercriticality of the system we first prepare the following lemma.

Lemma 2.1: Let $N(x,\mu,E)$ be a nonnegative solution of (2.3) and let $\underline{\sigma}(\mu',E')>0$ on $(-1,1)\times(E_0,E_1)$. Set

$$g(x)\equiv\int_{E_0}^{E_1}\int_{-1}^{1}\underline{\sigma}(\mu,E)N(x,\mu,E)\,d\mu\,dE \quad (-a\le x\le a). \tag{2.8}$$

If the functions N, ψ_0, ψ_1 are not all identically zero, then $g(x)>0$ for every $-a\le x\le a$.

Proof: Multiplication of (2.3) by $\underline{\sigma}(\mu,E)$ and integration over $(-1,1)\times(E_0,E_1)$ yield

$$
\begin{aligned}
g(x)=&\int_{E_0}^{E_1}\int_0^1\underline{\sigma}(\mu,E)\exp(-(x+a)/\mu)\psi_0(\mu,E)\,d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_0^1\int_0^{(x+a)/\mu}\exp(-\tau)\underline{\sigma}(\mu,E)f(N)(x-\tau\mu,\mu,E)\times d\tau\,d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_{-1}^0\underline{\sigma}(\mu,E)\exp((a-x)/\mu)\psi_1(\mu,E)\,d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_0^1\int_0^{(a-x)/(-\mu)}\exp(-\tau)\underline{\sigma}(\mu,E)f(N)\times(x-\tau\mu,\mu,E)\,d\tau\,d\mu\,dE.
\end{aligned}
\tag{2.9}
$$

Replacing μ by $(-\mu)$ in the last two integrals and using the relation (2.5) for $f(N)$ we obtain

$$
\begin{aligned}
g(x)\ge&\int_{E_0}^{E_1}\int_0^1[\underline{\sigma}(\mu,E)\exp(-(x+a)/\mu)\psi_0(\mu,E)+\underline{\sigma}(-\mu,E)\\
&\times\exp((a-x)/(-\mu))\psi_1(-\mu,E)]\,d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_0^1\Big[\int_0^{(x+a)/\mu}\exp(-\tau)\underline{\sigma}(\mu,E)q(x-\tau\mu,\mu,E)\,d\tau\\
&+\int_0^{(a-x)/\mu}\exp(-\tau)\underline{\sigma}(-\mu,E)q(x+\tau\mu,-\mu,E)\,d\tau\Big]d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_0^1\int_0^{(x+a)/\mu}\exp(-\tau)\underline{\sigma}(\mu,E)\Big(\frac{c}{2}\int_{E_0}^{E_1}\int_{-1}^1\underline{\sigma}(\mu',E')\\
&\times N(x-\tau\mu,\mu',E')\,d\mu'\,dE'\Big)\,d\tau\,d\mu\,dE\\
&+\int_{E_0}^{E_1}\int_0^1\int_0^{(a-x)/\mu}\exp(-\tau)\underline{\sigma}(-\mu,E)\Big(\frac{c}{2}\int_{E_0}^{E_1}\int_{-1}^1\underline{\sigma}(\mu',E')\\
&\times N(x+\tau\mu,\mu',E')\,d\mu'\,dE'\Big)\,d\tau\,d\mu\,dE\\
\equiv&\, I_1(x)+I_2(x)+I_3(x)+I_4(x),
\end{aligned}
\tag{2.10}
$$

where $I_i(x)$, $i=1,\ldots,4$, represent the four integrals in (2.10). Let x_0 be a point in $[-a,a]$ such that $g(x_0)=\inf\{g(x);-a\le x\le a\}$. Then we have: (i) $I_1(x_0)>0$ when either $\psi_0\not\equiv0$ or $\psi_1\not\equiv0$; (ii) $I_2(x_0)\ge0$ when $q\ge0$; and (iii) $I_3(x_0)+I_4(x_0)>0$ when $N\ne0$. The result in (ii) is obvious. To show (i) we observe that if $\psi_0\not\equiv0$ on $(0,1)\times(E_0,E_1)$ or $\psi_1\not\equiv0$ on $(-1,0)\times(E_0,E_1)$ [i.e., $\psi_1(-\mu,E)\not\equiv0$ on $(0,1)\times(E_0,E_1)$], then for every $x\in[-a,a]$ the integrand in the first integral in (2.10) is strictly positive in some subdomain in $(0,1)\times(E_0,E_1)$ and thus the whole integral is positive. This implies, in particular, that $I_1(x_0)>0$ which proves (i). Since for any $x\in[-a,a]$, $\mu\in[0,1]$,

$$-a \le x - \tau\mu \le x \quad \text{for } 0 \le \tau \le (x+a)/\mu,$$

$$x \le x + \tau\mu \le a \qquad \text{for } 0 \le \tau \le (a-x)/\mu. \tag{2.11}$$

We see that if $N \not\equiv 0$, then either $N(x_0 - \tau\mu, \mu, E) > 0$ for $0 \le \tau \le (x_0 + a)/\mu$ in some subdomain of $(-a, x_0) \times (-1, 1) \times (E_0, E_1)$ or $N(x_0 + \tau\mu, \mu, E) > 0$ for $0 \le \tau \le (a - x_0)/\mu$ in some subdomain of $(x_0, a) \times (-1, 1) \times (E_0, E_1)$. In any case, at least one of the integrals in the bracket in the third and fourth integrals [i.e., in $I_3(x)$ and $I_4(x)$] of (2.10) is strictly positive for the indicated intervals of τ. In view of (2.11) we have either $I_3(x_0) > 0$ or $I_4(x_0) > 0$ which leads to the result in (iii). It follows from (i)—(iii) that $g(x_0) > 0$ and therefore $g(x) > 0$ on $[-a, a]$. This proves the lemma.

Remark 2.1: Although the requirements of $N \not\equiv 0$ and $\underline{\sigma} > 0$ insure that $g(x) \not\equiv 0$, the positivity of $g(x)$ may not hold for every $x \in [-a, a]$ if N is not a solution of (2.3). In fact, if N is an arbitrary nonnegative nonzero function, then at any point x_1 where $N(x_1, \mu, E) = 0$ on $(-1, 1) \times (E_0, E_1)$ we have $g(x_1) = 0$. Thus the assumption of N being a solution of (2.3) is essential in the proof of Lemma 2.1. Notice that $I_1(x_0) > 0$ when either $\psi_0 \not\equiv 0$ or $\psi_1 \not\equiv 0$ but it is not always true that $I_2(x_0) > 0$ when $q \not\equiv 0$.

Using the result of Lemma 2.1 we now prove the following theorem.

Theorem 2.2: Assume that $\underline{\sigma}(\mu', E') > 0$ and

$$\frac{c}{2}\int_0^1 \underline{\sigma}_m(\mu')(1 - \exp(-2a/\mu'))\,d\mu' > 1. \tag{2.12}$$

Then given any inputs q, ψ_0, ψ_1 the system (2.3) has no nonnegative solution when q, ψ_0, ψ_1 are not all identically zero; and it only has the trivial solution when q, ψ_0, ψ_1 are all identically zero.

Proof: Assume that N is a nonnegative solution of (2.3) when q, ψ_0, ψ_1 are not all identically zero. Clearly $N \not\equiv 0$ and thus by Lemma 2.1, $g(x_0) = \inf g(x) > 0$ for some x_0 in $[-a, a]$. Since by (2.8).

$$g(x_0) \le \int_{E_0}^{E_1}\int_{-1}^{1} \underline{\sigma}(\mu', E') N(z, \mu', E')\,d\mu'\,dE'$$

for every $z \in [-a, a]$, and since the points $z \equiv x_0 - \tau\mu$ with $0 \le \tau \le (x_0 + a)/\mu$ and $z \equiv x_0 + \tau\mu$ with $0 \le \tau \le (a - x_0)/\mu$ are all in $[-a, a]$ we see from the definition of $I_3(x)$, $I_4(x)$ that

$$I_3(x_0) \ge \frac{c}{2} g(x_0) \int_{E_0}^{E_1}\int_0^1 \underline{\sigma}(\mu, E)(1 - \exp[-(x_0 + a)/\mu])\,d\mu\,dE$$

$$I_4(x_0) \ge \frac{c}{2} g(x_0) \int_{E_0}^{E_1}\int_0^1 \underline{\sigma}(-\mu, E)(1 - \exp[-(a - x_0)/\mu]) \times d\mu\,dE. \tag{2.13}$$

It follows from (2.10) with $x = x_0$ that

$$g(x_0) \ge I_1(x_0) + I_2(x_0) + \frac{c}{2} g(x_0) \int_0^1 \underline{\sigma}_m(\mu) \times [2 - \exp(-(x_0 + a)/\mu) - \exp(-(a - x_0)/\mu)]\,d\mu. \tag{2.14}$$

Since for each $\mu > 0$ the function $\rho(x) = 2 - \exp(-(x + a)/\mu) - \exp(-(a - x)/\mu)$ possesses the property that $\rho'' < 0$ for all x, the minimum of ρ on $[-a, a]$ occurs at $x = \pm a$ and is equal to $(1 - \exp(-2a/\mu))$. This implies that

$$g(x_0) \ge I_1(x_0) + I_2(x_0) + \frac{c}{2} g(x_0) \int_0^1 \underline{\sigma}_m(\mu)(1 - \exp(-2a/\mu))\,d\mu. \tag{2.15}$$

But by the hypothesis (2.12) and the fact that $I_1(x_0) + I_2(x_0) \ge 0$ and $g(x_0) > 0$ the above inequality is impossible. This contradiction shows that there exists no nonnegative solution when q, ψ_0, ψ_1 are not all identically zero. In the case of $q = \psi_0 = \psi_1 = 0$, then $I_1(x_0) = I_2(x_0) = 0$ and thus (2.15) is reduced to

$$g(x_0) \ge \frac{c}{2} g(x_0) \int_0^1 \underline{\sigma}_m(\mu)(1 - \exp(-2a/\mu))\,d\mu. \tag{2.16}$$

In view of (2.12) the above inequality is impossible unless $g(x_0) = 0$. However this insures that $N \equiv 0$, for otherwise Lemma 2.1 implies that $g(x_0) > 0$. Therefore, the only solution of (2.3) is the trivial solution $N \equiv 0$ when q, ψ_0, ψ_1 are all identically zero. This completes the proof of the theorem.

Remark 2.2: It is seen from (2.15) that if ψ_0, ψ_1 are not both identically zero then the system (2.3) has no nonnegative solution even when the left side of (2.12) is equal to one.

It is interesting to note that if $\underline{\sigma}_m(\mu')$ can be written as a polynomial of the form

$$\sigma_m(\mu') = a_0 + a_1\mu' + \cdots + a_m(\mu')^m,$$

then the condition (2.12) for supercriticality is reduced to

$$\frac{c}{2}\sum_{n=0}^{m} a_n[(n+1)^{-1} - E_{n+2}(2a)] > 1, \tag{2.17}$$

where $E_n(z)$ is the nth order exponential integral defined by

$$E_n(z) \equiv \int_1^\infty t^{-n}\exp(-zt)\,dt = \int_0^1 \mu^{n-2}\exp(-z/\mu)\,d\mu, \qquad n = 0, 1, 2, \cdots. \tag{2.18}$$

Similarly, if $\bar{a}_M(\mu') = b_0 + b_1\mu' + \cdots + b_m(\mu')^m$, then the condition (2.7) for subcriticality becomes

$$c\sum_{n=0}^{m} b_n[(n+1)^{-1} - E_{n+2}(a)] < 1. \tag{2.19}$$

Since the values of $E_n(z)$ have been tabulated in standard tables (e.g., see Ref. 3), numerical values for c in terms of a (or vice versa) can immediately be obtained from (2.17) and (2.19), respectively. In the special case of isotropic medium, $\sigma(\mu, E; \mu', E') \equiv \sigma(E, E')$ is independent of (μ, μ') and thus $\bar{\sigma}_M(\mu')$, $\underline{\sigma}_m(\mu')$ become the constants

$$\bar{\sigma}_M \equiv \int_{E_0}^{E_1} \bar{\sigma}(E')\,dE', \quad \underline{\sigma}_m = \int_{E_0}^{E_1} \underline{\sigma}(E')\,dE'. \tag{2.20}$$

In view of Theorem 2.1 and 2.2 we have the following conclusion.

Theorem 2.3: Assume that $\sigma \equiv \sigma(E, E')$ is independent of (μ, μ') and $\underline{\sigma}(E') > 0$. Then the system (2.3) is subcritical if

$$c\bar{\sigma}_M < [1 - E_2(a)]^{-1}; \tag{2.21}$$

while it is supercritical if

$$c\underline{\sigma}_m > 2[1 - E_2(2a)]^{-1}. \qquad (2.22)$$

Thus the critical value c^* of the system is bounded by

$$[\bar{\sigma}_M(1 - E_2(a))]^{-1} \leq c^* \leq 2[\underline{\sigma}_m(1 - E_2(2a))]^{-1}. \qquad (2.23)$$

When the transport system is energy independent the equations governing the density function $N \equiv N(x, \mu)$ are given by

$$\mu\frac{\partial N}{\partial x} + N = \frac{c}{2}\int_{-1}^{1} \sigma^*(\mu, \mu')N(x, \mu')\,d\mu' + q(x, \mu)$$

$$(-a \leq x \leq a, \ -1 \leq \mu \leq 1), \qquad (2.24)$$

$$N(-a, \mu) = \psi_0(\mu) \quad (0 < \mu \leq 1),$$
$$N(a, \mu) = \psi_1(\mu) \quad (-1 \leq \mu < 0). \qquad (2.25)$$

The above system can be deduced from (1.1), (1.2) by considering N, q, ψ_0, ψ_1 independent of E, $\sigma \equiv \sigma(\mu; \mu', E')$, and

$$\sigma^*(\mu, \mu') = \int_{E_0}^{E_1} \sigma(\mu; \mu', E')\,dE'.$$

Using the notation

$$\bar{\sigma}^*(\mu') = \sup\{\sigma^*(\mu, \mu'); -1 \leq \mu \leq 1\},$$
$$\underline{\sigma}^*(\mu') = \inf\{\sigma^*(\mu, \mu'); -1 \leq \mu \leq 1\},$$
$$\bar{\sigma}^*_M(\mu') = \max\{\bar{\sigma}^*(\mu'), \bar{\sigma}^*(-\mu')\},$$
$$\underline{\sigma}^*_m(\mu') = \min\{\underline{\sigma}^*(\mu'), \underline{\sigma}^*(-\mu')\}, \qquad (2.26)$$

and considering (2.24), (2.25) as a special case of (1.1), (1.2) we can deduce the following results from Theorems 2.1 and 2.2. These results have already been established in Refs. 1 and 2 by considering (2.24) and (2.25) directly.

Corollary 1: Assume that $\sigma^*(\mu, \mu') = \sigma^*(\mu', \mu)$ and $\underline{\sigma}^*_m(\mu') > 0$. Then the system (2.24), (2.25) is subcritical if

$$c\int_0^1 \bar{\sigma}^*_M(\mu')(1 - \exp(-a/\mu'))\,d\mu' < 1, \qquad (2.27)$$

and it is supercritical if

$$\frac{c}{2}\int_0^1 \underline{\sigma}^*_M(\mu')(1 - \exp(-2a/\mu'))\,d\mu' > 1. \qquad (2.28)$$

In the special case of monoenergetif, isotropic medium we may take $\sigma^* = 1$. In this situation we have the following simple criteria for the subcriticality and supercriticality of the system.

Corollary 2: For monoenergetic, isotropic medium the system (2.24), (2.25) is subcritical if

$$c < [1 - E_2(a)]^{-1}, \qquad (2.29)$$

and is supercritical if

$$c > 2[1 - E_2(2a)]^{-1}. \qquad (2.30)$$

Thus the critical value c^* is bounded by

$$[1 - E_2(a)]^{-1} \leq c^* \leq 2[1 - E_2(2a)]^{-1}. \qquad (2.31)$$

Remark 2.3: It is interesting to note that the difference between the upper and lower bounds of c^* is small for small values of slab thickness ($2a$) and grows larger as a increases. However, the upper bound is at most twice as much as the lower bound. For numerical examples, these bounds are given respectively, by (19.87, 23.02), (3.60, 4.70), and (1.17, 2.08) when $a = 0.01$, 0.10, and 1.00. The limiting case as $a \to \infty$ is $1 \leq c^* \leq 2$ (cf. Ref. 2).

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Interaction and stability of localized solutions in a classical nonlinear scalar field theory[a]

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The interaction between the localized solutions of a nonlinear scalar field theory are studied. We also study the stability of the above solutions under certain time-dependent perturbations.

I. INTERACTION BETWEEN LOCALIZED SOLUTIONS

We consider the solvable nonlinear scalar field[1] based on the Lagrangian density

$$L=\left(\frac{\partial\phi}{\partial t}\right)^2-(\nabla\phi)^2+g\phi^6+\frac{h}{r^2}\phi^2 \tag{1}$$

with $\phi=\phi(\mathbf{x},t)$ a real scalar field, g a positive constant and the constant $h\leq\frac{1}{4}$. The field equation associated with (1) is

$$-\phi_{tt}+\Delta\phi+3g\phi^5+(h/r^2)\phi=0, \tag{2}$$

which admits the spherically symmetric static solutions of the form

$$\phi=\frac{Zr^\alpha}{[(4/\beta^2)Z^4g+r^\beta]^{1/2}} \tag{3}$$

with the conditions $\beta=\pm 2(1-4h)^{1/2}$

$$\alpha=(\beta-2)/4 \tag{4}$$

and where Z is an arbitrary constant. The energy associated with the solution (3) is

$$E=\tfrac{1}{4}E_R\beta^2=E_R(1-4h), \tag{5}$$

where $E_R=\pi^2/2g^{1/2}$ is the energy obtained by Rosen in the case $\beta=2$.[2]

Let us now consider the interaction between the localized solutions (3):

(A) If ϕ_1 and ϕ_2 are two solutions with constants Z_1 and Z_2, then we get the solution

$$\phi(\phi_1,\phi_2)=\frac{F(Z_1,Z_2)r^\alpha}{[(4g/\beta^2)F^4(Z_1,Z_2)+r^\beta]^{1/2}}$$

where $F(Z_1,Z_2)$ is any function of Z_1 and Z_2. In particular if $F(Z_1,Z_2)=Z_1\pm Z_2$ then $\phi(\phi_1,\phi_2)=\pm\phi(\phi_2,\phi_1)$.

The above fact suggests the following rule for the superposition of two spherically symmetric solutions of (2):

$$\phi^2=2Z^2(Z_1^2/\phi_1^2+Z_2^2/\phi_2^2)^{-1},\quad Z=[(Z_1^4+Z_2^4)/2]^{1/4}. \tag{6}$$

Here ϕ describes a localized solution with the same energy as either ϕ_1 or ϕ_2. Thus we can interpret ϕ as a bound state of ϕ_1 and ϕ_2 such that the bound state energy is the mass of either ϕ_1 or ϕ_2. Assuming $Z_1\gg Z_2$, we have that $\phi\approx 2^{1/4}\phi_1$ when $r\to 0$ and $\phi\approx 2^{-1/4}\phi_1$ when $r\to\infty$.

The above results hold for n localized solutions; in particular, we get

$$\phi^2=nZ^2\left(\sum_{i=1}^n\frac{Z_i^2}{\phi_i^2}\right)^{-1},\quad Z=\left(\frac{1}{n}\sum_{i=1}^n Z_i^4\right)^{1/4}. \tag{7}$$

(B) Equation (2) also admits the solution

$$\phi(\mathbf{r}-\xi)=\frac{Z|\mathbf{r}-\xi|^\alpha}{[(4/\beta^2)Z^4g+|\mathbf{r}-\xi|^\beta]^{1/2}} \tag{8}$$

where ξ is a constant vector, which locates the center of the localized solution. For the superposition of localized solutions of the form (8) when $\beta=2$ ($\alpha=0$), we have the following rule:

$$\phi^2=nZ^2\left(\sum_{i=1}^n\frac{Z_i^2}{\phi_i^2}\right)^{-1} \tag{9}$$

$$Z=\left[\frac{1}{n}\sum_{i=1}^n Z_i^4+\sum_{i=1}^n \mathbf{r}_i^2-\frac{1}{n}\left(\sum_{i=1}^n\mathbf{r}_i\right)^2\right]^{1/4}$$

where $\mathbf{r}_i$ is the constant vector associated with ϕ_i.

(C) Let us find the static force between two localized solutions by the method of Rosen *et al.*[3] The method involves integration of the normal component of the energy–momentum tensor $T^{\mu\nu}$ over a surface enclosing the localized solution, which then yields the force on the "particle."

Consider two localized solutions centered at points $\mathbf{r}_1=(0,0,-R/2)$ and $\mathbf{r}_2=(0,0,R/2)$ at time $t=0$. The separation R is assumed to be much larger than the sizes $(4Z_i^4g/\beta^2)^{1/\beta}$ of the localized solutions. Suppose we have the initial conditions for Eq. (2)

$$\phi(\mathbf{r},0)=\phi_1(\mathbf{r}-\mathbf{r}_1)+\phi_2(\mathbf{r}-\mathbf{r}_2),\quad \left(\frac{\partial\phi}{\partial t}\right)_{t=0}=0. \tag{10}$$

To find the force on localized solution ϕ_2, we have to compute the surface integral of $T^{\mu\nu}$ over a surface enclosing the center of ϕ_2 but excluding the center of ϕ_1. Following Rosen, we take the unbounded surface prescribed by $Z=0$. Since in this case ϕ_i are with cylindrical symmetry about the Z axis, the force on ϕ_2 is

$$F^1=F^2=0,$$
$$F^3=\int_{Z=0}T^{33}\,dx\,dy, \tag{11a}$$

where

$$T^{33}=\phi_t^2-\phi_x^2-\phi_y^2+\phi_z^2+g\phi^6+h/r^2\phi^2. \tag{11b}$$

[a)]Research supported by a fellowship of the Program of Cultural Cooperation between the United States and Spain.
[b)]On leave from Departamento de Fisica Teorica, Universidad de Zaragoza, Zaragoza, Spain.

In the limit of large R, we obtain

$$F^3 = -\left(\frac{\pi}{2}2^{\beta/2}\frac{\beta+2}{\beta+6}\right)[16Z_1Z_2 - (\beta-2)(Z_1-Z_2)^2] \times \frac{1}{R^{(\beta+2)/2}} \qquad (12)$$

which corresponds to the potential energy

$$V = -\left(\frac{\pi}{2}2^{\beta/2}\frac{\beta+2}{\beta+6}\right)[16Z_1Z_2 - (\beta-2)(Z_1-Z_2)^2]\frac{2}{\beta}\frac{1}{R^{\beta/2}}. \qquad (13)$$

As we can see when $\beta > 2$ $(0<\beta<2)$ we have the anti-Coulomb interaction obtained by Rosen[4] plus a repulsive (attractive) interaction, which corresponds to the repulsive (attractive) scalar potential in Eq. (2).

When $\beta = 2$, with the help of (9) we get the localized solution χ as a superposition of ϕ_1 and ϕ_2 given in (10) as

$$\chi = \frac{Z}{(Z^4g + r^2)^{1/2}}, \qquad Z = \left(\frac{Z_1^4+Z_2^4}{2} + \frac{1}{2g}R^2\right)^{1/4}. \qquad (14)$$

Now if in (10) we consider $\phi(\mathbf{r},0)=\chi$ and $\phi_t(\mathbf{r},0)=0$, then $F^3 = 0$. This means that the state represented by χ is a static state of two localized solutions and there is no force between them. But the above state χ is unstable because, if $\phi_t(\mathbf{r},0)\neq 0$ it follows from (11) that a force F^3 appears between ϕ_1 and ϕ_2.

II. STABILITY OF THE LOCALIZED SOLUTIONS

Let us now consider the stability of the solutions (3) under scalar perturbations acting in a short time.

(A) We have the same Lagrangian (1) plus the interaction

$$L_I = [\varphi(t)/r^2]\phi^2 \qquad (15)$$

with $\varphi = \sin(\pi t/T)$ if $t\in[0,T]$ and $\varphi = 0$ otherwise. The field equation is

$$-\phi_{tt} + \Delta\phi + 3g\phi^5 + \{[h+\varphi(t)]/r^2\}\phi = 0. \qquad (16)$$

Numerically (the Appendix), we studied the Eq. (16) with the initial conditions:

$$\phi(\mathbf{r},0) = \frac{Zr^{\alpha}}{(4/\beta^2 Z^4 g + r^{\beta})^{1/2}}, \qquad (17)$$

$$\phi_t(\mathbf{r},0) = 0.$$

We considered the three cases $\beta = 10, 15, 20$ for which $\phi(\mathbf{r},0)$ is concentrated near $r=0$. Also we studied the perturbation for three values of $T \gtrless \tau_\beta$, where τ_β is the

TABLE I.

t	$\lvert\phi\rvert_{max}(r)$	E
0.00	0.708(1.)	3.93
0.04	0.709(1.)	3.78
0.08	0.711(1.)	3.83
0.12	0.713(1.)	3.93
0.16	0.718(1.)	3.93
0.205	0.728(1.)	3.93

TABLE II.

t	$\lvert\phi\rvert_{max}(r)$	E
0.000	0.708(1.)	3.937
0.004	0.989(1.)	6420.18
0.006	2.243(1.)	125637.21
0.008	∞	

lifetime of the corresponding state.[1] The behavior of the solution is the same in all cases. The energy associated with $\phi(r,t)$ changes in the interval $[0,T]$, remaining constant when $t\geq T$ and such that $E(0)=E(T)$. Also $\lvert\phi(r,t)\rvert_{max}$ increases continuously, even when $t>T$. Thus we can expect unbounded growth in ϕ for as $t\to\infty$. For the case

$$\beta = 20 \quad (\tau_\beta = 0.08), \quad Z=1, \quad g=100, \quad T=0.1 \qquad (18)$$

we represent in Table I the variation of $\lvert\phi\rvert_{max}$ and the energy E. The weakness of the perturbation represented by (15) is reflected in the fact that when it stops at $t=T$, there is a negligible variation in the shape of ϕ compared with its shape at $t=0$. Also at $t=T$ we have $\lvert\phi_t/\phi\rvert \leq 10^{-4}$.

(B) As above, we consider the Lagrangian (1) but with the interaction

$$L_I = \varphi(t)\phi^2 \qquad (19)$$

with $\varphi(t)$ the same as in (A). The field equation is

$$-\phi_{tt} + \Delta\phi + 3g\phi^5 + (h/r^2)\phi + \varphi(t)\phi = 0. \qquad (20)$$

We proceed as before, considering the same cases, and we find the same behavior in all the cases except

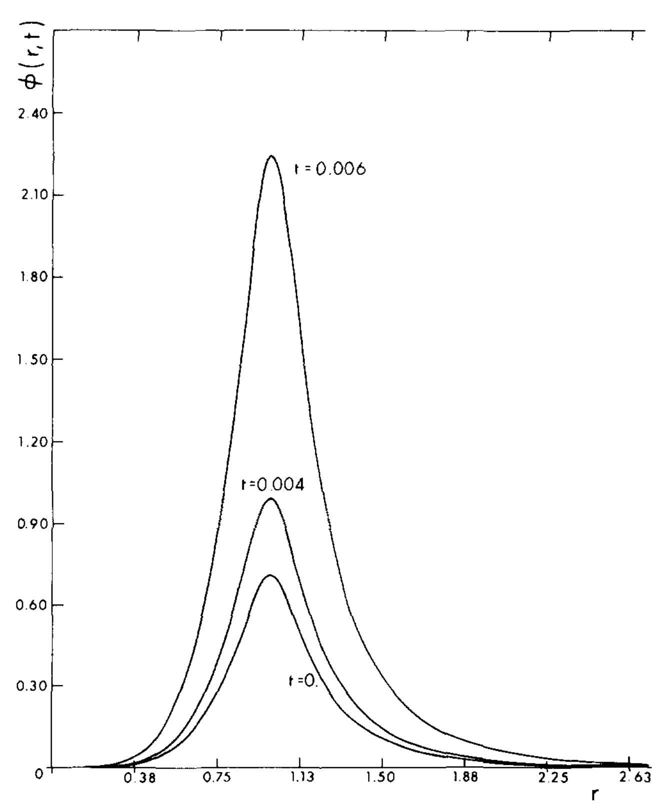

FIG. 1. Time development of the amplitude $\phi(r,t)$ corresponding to the Eq. (20) with initial conditions (17)–(18). The time and space intervals are $\Delta t = 0.002$ and $\Delta r = 0.002$.

for the behavior with the interaction given by (15), for which the amplitude ϕ increases without bound in a very short time. The effect of the interaction given by (19) must be stronger than that given by (15) in order to destroy the localized solution. We represent in Table II the variation of $|\phi|_{max}$ and the energy E, while in Fig. 1 we illustrate the time development of $\phi(r,t)$, for the initial conditions (17), (18).

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APPENDIX

Equations (16) and (20) are particular cases of the equation

$$U_{tt} - U_{rr} - 2U_r/r - 3gU^5 + q(r,t)u = 0. \tag{A1}$$

With the change $V = rU$, we get

$$V_{tt} - V_{rr} - \frac{3g}{r^4}V^5 + q(r,t)V = 0 \tag{A2}$$

with $V(0,t) = 0$. For numerical purposes we replace (A2) with[5]

$$\frac{V_j^{n+1} - 2V_j^n + V_j^{n-1}}{(\Delta t)^2} - \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{(\Delta r)^2} - \frac{g}{2(j\Delta r)^4}\frac{(V_j^{n+1})^6 - (V_j^{n-1})^6}{V_j^{n+1} - V_j^{n-1}} + q(j\Delta r, n\Delta t)\tfrac{1}{2}(V_j^{n+1} + V_j^{n-1}) = 0, \tag{A3}$$

where Δt and Δr are the time and space intervals. The function $V(r,t)$ is approximated by $V_j^n = V(j\Delta r, n\Delta t)$. When q is time-independent, the quantity which approximates the energy

$$\sum_j \Delta r\left\{\left(\frac{V_j^{n+1} - V_j^n}{\Delta t}\right)^2 + \left(\frac{V_{j+1}^{n+1} - V_j^{n+1}}{\Delta r}\right)\left(\frac{V_{j+1}^n - V_j^n}{\Delta r}\right) - \frac{g}{2(j\Delta r)^4}[(V_j^{n+1})^6 + (V_j^n)^6] + \tfrac{1}{2}q(j\Delta r)[(V_j^{n+1})^2 + (V_j^n)^2]\right\} \tag{A4}$$

is conserved.[5]

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On equivalence of parabolic and hyperbolic super-Hamiltonians[a)]

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Three types of super-Hamiltonians occur in generally covariant field theories: linear in the momenta (hypersurface kinematics), parabolic in the momenta (parametrized field theories on a given Riemannian background), and hyperbolic in the momenta (geometrodynamics). Three simple models are discussed in which the linear or parabolic super-Hamiltonian can be cast, essentially by a canonical transformation, into an equivalent hyperbolic form: (1) The scalar field propagating on a $(1+1)$-dimensional flat Minkowskian background, (2) hypersurface kinematics on a $(1+n)$-dimensional flat Minkowskian background, and (3) geometrodynamics of a $(1+2)$-dimensional vacuum spacetime. The implications for constraint quantization are mentioned.

1. INTRODUCTION

Parametrized Hamiltonian dynamics of tensor fields propagating on a given geometrical background[1] resembles in most respects the Hamiltonian geometrodynamics,[2] in which the fields are coupled to geometry by Einstein's law of gravitation. In both theories, the Hamiltonian is a linear combination of the constraint functions, called super-Hamiltonian and supermomentum. The coefficients of this linear combination enter the action as Lagrange multipliers and have the same geometrical interpretation in both schemes, suggested by their accepted names, the lapse and the shift functions. The variation of the Lagrange multipliers leads to the super-Hamiltonian and supermomentum constraints which limit the choice of the canonical field variables. The Poisson brackets between the constraint functions have the same universal structure.[3] The supermomenta are always linear functions of the field momenta.[4]

Inspecting the super-Hamiltonians, however, we come across the first fundamental difference between the two theories. The geometrodynamical super-Hamiltonian is a quadratic function of the field momenta, characterized by a hyperbolic "supermetric."[5] The super-Hamiltonians of standard tensor fields propagating on a given background are quadratic in the field momenta, but linear in the "kinematical" momenta canonically conjugate to the embedding variables. These super-Hamiltonians thus have a "parabolic" character.[6]

Viewing the super-Hamiltonian as the starting point of canonical quantization, one is led to a Schrödinger equation for the field propagating on a given background, but to a Klein—Gordon equation for quantum geometrodynamics. The latter situation is never encountered in the conventional Lorentz invariant quantum field theory; the single particle may obey the Klein—Gordon equation, but the evolution equation for the quantized field is always of the Schrödinger type. The Klein—Gordon field equation in quantum geometrodynamics leaves us with unsettling unresolved problems touching the very interpretation of the formalism.

Two general proposals have been made how to linearize the geometrodynamical super-Hamiltonian in at least some of the field momenta. From the ADM perturbation analysis[7] it is apparent that the direct information about time is carried by the geometrodynamical momentum, rather than by the hypersurface geometry in the role of the configuration coordinate. The author suggested[8] that the geometrodynamical super-Hamiltonian is to be cast from the hyperbolic to a parabolic form by a canonical transformation before the fields are quantized. The transition to such an "extrinsic time representation" was accomplished for specific minisuperspace models.[9] York[10] has developed a beautiful general scheme defining the extrinsic time and the conjugate energy density and casting the super-Hamiltonian into a parabolic form, the energy density entering linearly into the new super-Hamiltonian. A possible use of York's scheme in canonical quantization has been explored by Teitelboim.[11] The principal difficulty to be overcome is the implicit and highly nonlocal structure of the new super-Hamiltonian, which makes it extremely difficult to decide on a proper factor ordering.

The second proposal attempts to linearize the quadratic gravitational super-Hamiltonian similarly as the Dirac equation linearizes the ordinary Klein—Gordon equation.[12] The resulting scheme is equivalent to super-gravity.[13] The price to pay for the linearization is thus a supplementary $\frac{3}{2}$-spin field coupled to geometry.

In this paper, we want to discuss much simpler aspects of the linearization problem. We shall construct three elementary models in which a hyperbolic super-Hamiltonian constraint is cast into an equivalent parabolic (or linear) form by a suitable canonical transformation. In our models, both the old and the new constraints are local in the respective field variables, and no supplementary spinor fields are necessary. The canonical transformations used are in one case a linear transformation of the canonical

a)Partially supported by NSF Grant No. MPS-74-16311 to the University of Utah.

variables, in another case a transformation in the configuration space of the system complemented by adding a gradient to the field momentum. These are transformations which, unlike the general canonical transformation, are expected to preserve the quantization scheme.

Neither of the models yields easily to a generalization. We do not propose them as solutions to present difficulties, but rather as idealized models (similar to soluble models of ordinary quantum field theory) on which there is a hope of exhibiting explicitly the relationship between the Klein—Gordon and the Schrödinger field quantizations. Even so, the rigorous comparison of these two schemes does not seem feasible at the present moment, due to the unresolved difficulties how to define the densities, rather than the integrated constraints, as meaningful operators.

The first of our models shows how to reinterpret the massless real scalar field propagating on a (1+1)-dimensional flat Minkowskian background as a collection of three massless scalar fields, two with a positive, and the third with a negative energy densities, combined into a hyperbolic super-Hamiltonian. It also shows that the massive real scalar field leads to a model mimicking the supermetric of a curved superspace.

The second model shows that the linear (kinematical) part of the super-Hamiltonian constraint of a parametrized field theory on a flat Minkowskian background in $1+n$ dimensions can always be cast into an equivalent quadratic form. This model is interesting because it provides an example of the theory in which the Lagrange multipliers are only weakly equal to the lapse and shift functions, and the Poisson brackets between the super-Hamiltonians consequently lead to quadratic, rather than linear combinations of the constraint functions.

The third model analyzes the way in which the hyperbolic geometrodynamical super-Hamiltonian in 1+2 dimensions, necessarily generating the flat spacetime by the evolution of the 2-geometry, is equivalent to the linear kinematical hypersurface super-Hamiltonian, generating the same flat spacetime by the deformation of an initial embedding.

2. LINEAR, PARABOLIC AND HYPERBOLIC SUPER-HAMILTONIANS

To illustrate the three typical patterns in which the super-Hamiltonians depend on the canonical momenta, we briefly summarize the schemes of hypersurface kinematics,[9] of a parametrized field theory on a given Riemannian background,[1] and of geometrodynamics with tensor sources.[2]

A. Hypersurface kinematics

The position of a hypersurface in a given Riemannian spacetime $(\mathcal{M}, g)$ can be specified by the embedding

$$e: m \to \mathcal{M},\ x \in m \mapsto X \in \mathcal{M}.$$

In terms of the local coordinates x^a in the space m and the local coordinates X^α in the spacetime $\mathcal{M}$, the mapping e is described by the functions

$$X^\alpha = e^\alpha(x^a). \tag{2.1}$$

The Latin indices range from 1 to n, the Greek indices from 0 to n.

Differentiating Eq. (2.1) with respect to x^a, we get the tangent vectors $e^\alpha_a \equiv e^\alpha{}_{,a}$. The unit normal to the (spacelike) hypersurface is then determined by the conditions

$$n_\alpha e^\alpha_a = 0, \quad g^{\alpha\beta}[e] n_\alpha n_\beta = -1. \tag{2.2}$$

The Greek indices are raised and lowered by the spacetime metric tensor, the Latin indices by the space metric tensor

$$g_{ab}[e] = g_{\alpha\beta}[e] e^\alpha_a e^\beta_b. \tag{2.3}$$

A continuous deformation of the hypersurface through the spacetime is represented by a one-parameter family of hypersurfaces, i.e., by a curve $e(t)$ in the space of embeddings,

$$X^\alpha = e^\alpha(x^a, t). \tag{2.4}$$

The tangent vector to this curve at the embedding $e(t)$,

$$N^\alpha(x^a, t) = \frac{\partial e^\alpha(x^a, t)}{\partial t} \equiv \dot{e}^\alpha \tag{2.5}$$

is called the deformation vector. Its components along the hypersurface and normal to it are the shift and the lapse functions,

$$N^\alpha = N^a e^\alpha_a + N n^\alpha, \qquad N^a = N^\alpha e^a_\alpha, \quad N = -N^\alpha n_\alpha. \tag{2.6}$$

The kinematical process of deforming the hypersurface through a given Riemannian spacetime can be described in the canonical language. Start from the action functional

$$S[e^\alpha, p_\alpha; N^\alpha] \equiv \int dt \int_m d^n x (p_\alpha \dot{e}^\alpha - N^\alpha p_\alpha) \tag{2.7}$$

depending on the embedding variables $e^\alpha(x^a)$, their conjugate momenta $p_\alpha(x^a)$, and the Langrange multipliers $N^\alpha(x^a)$. Its variation with respect to p_α reproduces Eq. (2.5), informing us that the Lagrange multipliers N^α are to be interpreted as components of the deformation vector. Varying the action with respect to N^α, we learn that the momenta p_α are constrained to vanish,

$$p_\alpha = 0, \tag{2.8}$$

so that no physical or geometrical significance can be assigned to them. Finally, varying the action (2.7) with respect to the embedding variables, we see that the constraints (2.8) are preserved along the embedding curve $e(t)$,

$$\dot{p}_\alpha = 0. \tag{2.9}$$

The deformation vector N^α in the action (2.7) can be decomposed into the lapse and shift components according to Eq. (2.6) and the action cast into the form

$$S[e^\alpha, p_\alpha; N, N^a] = \int dt \int_m d^n x (p_\alpha \dot{e}^\alpha - NH - N^a H_a), \qquad (2.10)$$

with the super-Hamiltonian H and supermomentum H_a given as functionals of e^α and p_α,

$$H[e^\alpha, p_\alpha] = n^\alpha[e] p_\alpha \equiv -p_\perp, \qquad (2.11)$$

$$H_a[e^\alpha, p_\alpha] = e^\alpha_a p_\alpha \equiv p_a. \qquad (2.12)$$

The normal n^α is determined as a functional of the embedding by Eqs. (2.2). The super-Hamiltonian (2.11) is a linear function of the momenta p_α, reflecting an essentially trivial character of these variables.

The Poisson brackets among the constraint functions (2.11) and (2.12) close in the standard way[3],

$$[H(x), H(x')] = g^{ab}(x) H_a(x) \delta_{,b}(x, x') - (x \longleftrightarrow x'), \qquad (2.13)$$

$$[H_a(x), H(x')] = H(x) \delta_{,a}(x, x'), \qquad (2.14)$$

$$[H_a(x), H_b(x')] = H_b(x) \delta_{,a}(x, x') - (ax \longleftrightarrow bx'), \qquad (2.15)$$

ensuring the preservation of the $H = 0 = H_a$ constraints. The metric g^{ab} in Eq. (2.13) is considered as a functional of the embedding, Eq. (2.3).

All this shows that the hypersurface kinematics can be reproduced as a degenerate Hamiltonian dynamics of the e^α, p_α variables.

B. Parametrized field theory on a given Riemannian background

The hypersurface kinematics acquires a physical meaning when we follow the dynamics of a physical field along the embedding curve. We abstain from discussing the general formalism[1] and illustrate the situation by the simplest example of a field with nonderivative gravitational coupling: a real scalar field obeying the linear wave equation

$$\Box \phi - \mu^2 \phi = 0. \qquad (2.16)$$

The momentum $\pi_\phi(x)[e]$ conjugate to the field $\phi(x)[e] = \phi(e(x))$ on the hypersurface $e(x)$ is given in this case by the normal change of the field itself,

$$\pi_\phi(x)[e] = g^{1/2}(x)[e]\, n^\alpha(x)[e]\, \phi_{,\alpha}(e) = g^{1/2}(\dot{\phi} - N^a \phi_{,a}), \qquad (2.17)$$

and the dynamical evolution of the field is generated by the field super-Hamiltonian H^ϕ and supermomentum $H^\phi{}_a$ [14]:

$$H^\phi = \tfrac{1}{2} g^{-1/2} \pi_\phi{}^2 + \tfrac{1}{2} g^{1/2} (g^{ab} \phi_{,a} \phi_{,b} + \mu^2 \phi^2), \qquad (2.18)$$

$$H^\phi{}_a = \pi_\phi \phi_{,a}. \qquad (2.19)$$

The field super-Hamiltonian is quadratic (and positive definite) in the momentum π_ϕ. The evolution of the field is described by the action functional

$$S[e^\alpha, \phi, p_\alpha, \pi_\phi; N, N^a] = \int dt \int_m d^n x (p_\alpha \dot{e}^\alpha + \pi_\phi \dot{\phi} - NH - N^a H_a) \qquad (2.20)$$

with the super-Hamiltonian and supermomentum obtained by adding the field expressions (2.18), (2.19) to those describing the field kinematics, Eqs. (2.11), (2.12):

$$H[e^\alpha, p_\alpha, \phi, \pi_\phi] = -p_\perp + H^\phi, \qquad (2.21)$$

$$H_a[e^\alpha, p_\alpha, \phi, \pi_\phi] = p_a + H^\phi{}_a. \qquad (2.22)$$

The metric g_{ab} and its determinant g in the field super-Hamiltonian (2.18) are considered as functionals of the embedding through Eq. (2.3). The kinematical momenta p_α enter the super-Hamiltonian (2.21) linearly, while the field momentum π_ϕ occurs there quadratically, so that H has a "parabolic" structure in the momentum variables $\{p_\alpha, \pi_\phi\}$.

The variation of the action (2.20) in the field variables ϕ, π_ϕ yields the field equations equivalent to Eq. (2.16). By varying the kinematical momenta p_α, we learn that the Lagrange multipliers N, N^a are identical with the lapse and shift functions,

$$\dot{e}^\alpha = N n^\alpha + N^a e^\alpha_a. \qquad (2.23)$$

The variation of the multipliers leads to the constraints $H = 0 = H_a$ which endow the projections of the kinematical momentum with the physical meaning; using Eqs. (2.21), (2.22), we see that $p_\perp$ is equal to the energy density H^ϕ and $-p_a$ to the momentum density $H^\phi{}_a$ of the scalar field. The variation of e^α gives then the laws of conservation of energy and momentum.

The constraint functions (2.21), (2.22) satisfy the same Poisson bracket relations (2.13)–(2.15) as the kinematical constraint function (2.11)–(2.12).[3]

C. Geometrodynamics

The dynamics of the gravitational field in vacuum is described by the action functional[7,2]

$$S[g_{ab}, \pi^{ab}; N, N^a] = \int dt \int_m d^n x (\pi^{ab} \dot{g}_{ab} - NH - N^a H_a). \qquad (2.24)$$

The momentum π^{ab} canonically conjugate to the hypersurface metric g_{ab} is related to the extrinsic curvature K_{ab} by the formula

$$\pi^{ab} = g^{1/2}(K g^{ab} - K^{ab}). \qquad (2.25)$$

The gravitational super-Hamiltonian

$$H[g_{ab}, \pi^{ab}] = g^{-1/2} \left(\pi_{ab} \pi^{ab} - \frac{1}{n-1} \pi^2 \right) - g^{1/2} R \qquad (2.26)$$

is a quadratic function of the gravitational momenta π^{ab}. The coefficient $(n-1)^{-1}$ of π^2 depends on the dimension n of the space m.[15] The signature of the quadratic form (2.26) of the momenta is $(-,+,\ldots,+)$,[16] which shows the hyperbolic character of the super-Hamiltonian. The gravitational supermomentum

$$H_a[g_{cd},\pi^{cd}] = -2\pi^b_{a|b} \tag{2.27}$$

is linear in the momentum π^{ab}; the vertical stroke denotes the covariant derivative generated by the hypersurface metric g_{ab}.

The hyperbolic character of the super-Hamiltonian remains unchanged when we couple the gravitational field to a source. For a nonderivative coupling, this is achieved by adjoining the field super-Hamiltonian and supermomentum, like (2.18), (2.19), to the gravitational super-Hamiltonian (2.26) and supermomentum (2.27). For the scalar field, this adds just another square, $\frac{1}{2}g^{1/2}\pi_\phi^2$, to the gravitational form, adding an extra + to the signature of the extended supermetric.

The gravitational constraint functions (2.26), (2.27) again satisfy the universal Poisson bracket relations (2.13)—(2.15).[3]

We have thus exhibited the three different types of super-Hamiltonians we mentioned in the Introduction: the linear super-Hamiltonian of hypersurface kinematics, the parabolic super-Hamiltonian of a parametrized field theory on a given Riemannian background, and the hyperbolic super-Hamiltonian of geometrodynamics, with our without sources. In the following sections, we shall try to reconcile these three types of super-Hamiltonian by transforming them into each other for simple intuitive models.

3. REAL SCALAR FIELD ON A FLAT BACKGROUND IN 1 + 1 DIMENSIONS

As our first example of equivalence of the parabolic and hyperbolic constraints, take a real scalar field propagating on a flat (1 + 1)-dimensional Minkowskian background. If the privileged Minkowskian coordinates T, X are chosen to label the spacetime points, the embedding functions assume the form

$$e^\alpha(x) = \{T(x), X(x)\}; \tag{3.1}$$

x is an arbitrary curvilinear coordinate labeling the points of the one-dimensional hypersurface. The partial derivative with respect to x will be denoted by prime. There is only one vector tangent to the slice (3.1),

$$e^\alpha_1 = \{T', X'\}, \tag{3.2}$$

and the unit, future-pointing normal to this hypersurface is

$$n_\alpha = (X'^2 - T'^2)^{-1/2}\{-X', T'\}. \tag{3.3}$$

If we parametrize the hypersurface by the privileged Minkowskian coordinate X, $X=x$, the term $(X'^2 - T'^2)^{-1/2}$ becomes the Lorentz contraction factor for an observer moving perpendicular to the hypersurface with the 4-velocity n^α and the velocity dT/dX. The same combination of terms occurs in the hypersurface metric tensor

$$g_{11} = \eta_{\alpha\beta}e^\alpha_1 e^\beta_1 = X'^2 - T'^2 \tag{3.4}$$

and in the volume density

$$g^{1/2} = (X'^2 - T'^2)^{1/2}. \tag{3.5}$$

Let

$$p_\alpha = \{p_T, p_X\} \tag{3.6}$$

be the momenta conjugate to the embedding variables (3.1). The constraint functions (2.21) and (2.22) then assume the form[17]

$$H = (X'^2 - T'^2)^{-1/2}[X'p_T + T'p_X + \tfrac{1}{2}\pi_\phi{}^2 + \tfrac{1}{2}\phi'^2 + \mu^2(X'^2 - T'^2)\phi^2], \tag{3.7}$$

$$H_1 = T'p_T + X'p_X + \phi'\pi_\phi. \tag{3.8}$$

The supermomentum (3.8) has the structure appropriate for the collection of three space scalars, T, X, and ϕ.

The first glance at the super-Hamiltonian (3.7) shows that it is advantageous to rescale it by the factor (3.5),

$$\widetilde{H} \equiv g^{1/2}H = X'p_T + T'p_X + \tfrac{1}{2}\pi_\phi{}^2 + \tfrac{1}{2}\phi'^2 + \mu^2(X'^2 - T'^2)\phi^2. \tag{3.9}$$

If we want to preserve the form of the action (2.20), we must rescale the lapse function inversely to the super-Hamiltonian,

$$\widetilde{N} = g^{-1/2}N. \tag{3.10}$$

The new action, $S[e^\alpha, p_\alpha, \phi, \pi_\phi, \widetilde{N}, N^a]$, leads to an equivalent system of field equations.

The rescaling has an interesting effect on the Poisson brackets (2.13)—(2.15). Equation (2.14), expressing the fact that H is a scalar density of weight 1,[18] gets naturally replaced by the equation

$$[H_a(x), \widetilde{H}(x')] = 2\widetilde{H}(x)\delta_{,a}(x,x') + \widetilde{H}_{,a}(x)\delta(x,x'), \tag{3.11}$$

expressing the fact that $\widetilde{H}$ is a scalar density of weight 2. Equation (2.15) remains unaffected by the scaling, because it involves only the supermomenta. The real surprise awaits us, however, when we evaluate the Poisson bracket between the rescaled super-Hamiltonians,

$$\begin{aligned}[\widetilde{H}(x), \widetilde{H}(x')] &= [g^{1/2}(x)H(x), g^{1/2}(x')H(x')] \\ &= g^{1/2}(x)g^{1/2}(x')[H(x), H(x')] \\ &\quad - (x \longleftrightarrow x') \\ &\quad + [g^{1/2}(x), H(x')]g^{1/2}(x')H(x) \\ &\quad - (x \longleftrightarrow x'),\end{aligned}$$

which we are now going to do.

The last term vanishes after the interchange $(x \longleftrightarrow x')$ because it is proportional to the δ function,

$$[g^{1/2}(x), H(x')] = -g^{1/2}(x)K(x)\delta(x,x').$$

The first term can be simplified by using Eq. (2.13) and the identity

$$f(x)g(x')\delta_{,a}(x,x') = f(x)g(x)\delta_{,a}(x,x') + f(x)g_{,a}(x)\delta(x,x'). \qquad (3.12)$$

Therefore,

$$[\widetilde{H}(x), \widetilde{H}(x')] = g(x)g^{ab}(x)H_a(x)\delta_{,b}(x,x') - (x \longleftrightarrow x'). \qquad (3.13)$$

In a (1+1)-dimensional spacetime, $gg^{11}=1$, and Eq. (3.13) simplifies to

$$[\widetilde{H}(x), \widetilde{H}(x')] = H_1\delta'(x,x') - (x \longleftrightarrow x'), \qquad (3.14)$$

where $\delta'(x,x')$ is the δ function differentiated with respect to its first argument.

Note that the metric tensor disappears from Eq. (3.14). This may have quite important consequences. Indeed, the term g^{ab} on the right-hand side of the Poisson bracket relation (2.13) causes a number of related difficulties. Being a functional of the canonical coordinates e or, in geometrodynamics, of the canonical coordinates g_{ab}, it precludes the constraint functions $H_a(x)$, $H(x)$ to generate (an infinitely dimensional) Lie group.[19,20] In quantization, it is a source of yet unsurpassed factor ordering difficulties.[21] It is thus comforting to see the e-dependent terms disappear from the Poisson bracket (3.14) between the rescaled super-Hamiltonians. Let us emphasize that the proof of Eq. (3.14) is quite independent of our restriction to the scalar field ϕ and is thus applicable to an arbitrary parametrized field theory on a flat or curved (1+1)-dimensional Riemannian background. Unfortunately, the proof cannot be carried over to (1+1)-dimensional geometrodynamics, which is not a well-defined theory [the geometrodynamical super-Hamiltonian (2.26) makes no sense for $n=1$; see also the remarks at the beginning of Sec. 5]. Let us note in this context that Schwinger[22] tried to scale H into a weight 4 density in order to circumvent the factor ordering difficulties in (1+3)-dimensional geometrodynamics. In any case, the absence of the canonical coordinates e from the Poisson bracket relations (2.15), (3.11), (3.14) of an arbitrary parametrized field theory on a (1+1)-dimensional background deserves a careful exploration.

Having settled the question of the Poisson brackets, we return to the specific structure (3.8), (3.9) of the constraint functions. We see that both the super-Hamiltonian (3.9) and the supermomentum (3.8) are bilinear forms of the derivatives $\{T', X'\}$ of the embedding coordinates and the conjugate kinematical momenta $\{p_T, p_X\}$. We shall study the transformation

$$T, X; p_T, p_X \mapsto \xi, \eta; \pi_\xi, \pi_\eta \qquad (3.15)$$

given by the equations

$$\xi' = 2^{-1/2}(T' + p_X), \quad \eta' = 2^{-1/2}(p_X - T'),$$
$$\pi_\xi = 2^{-1/2}(X' + p_T), \quad \pi_\eta = 2^{-1/2}(X' - p_T), \qquad (3.16)$$

which have the inverse

$$X' = 2^{-1/2}(\pi_\xi + \pi_\eta), \quad T = 2^{-1/2}(\xi - \eta),$$
$$p_X = 2^{-1/2}(\xi' + \eta'), \quad p_T = 2^{-1/2}(\pi_\xi - \pi_\eta). \qquad (3.17)$$

This transformation has a number of important properties:

1. *It is canonical transformation*: This is easily checked by exhibiting its generating functional

$$F[X, p_T; \xi, \eta] = -2^{-1/2}\int_m dx((X' + p_T)\xi + (X' - p_T)\eta). \qquad (3.18)$$

The resulting equations

$$p_X = \frac{\delta F}{\delta X}, \quad T = -\frac{\delta F}{\delta p_T}, \quad \pi_\xi = -\frac{\delta F}{\delta \xi}, \quad \pi_\eta = -\frac{\delta F}{\delta \eta} \qquad (3.19)$$

coincide with the appropriate selection of Eqs. (3.16), (3.17).

2. *The transformation is nonlocal, but linear*: The linearity of the transformation is obvious. The nonlocality is equally obvious. For example, to get ξ and η, we should integrate p_X with respect to x, i.e., along the hypersurface. The canonical transformation (3.16) thus has the same general character as the canonical transformation from the local field coordinates and momenta to the normal coordinates and momenta, which is a standard tool of the quantum theory of free fields.

3. *It casts the kinematical part of* $\widetilde{H}$ *into a difference of squares*:

$$X'p_T + T'p_X = \tfrac{1}{2}(\pi_\xi^2 - \pi_\eta^2) + \tfrac{1}{2}(\xi'^2 - \eta'^2). \qquad (3.20)$$

4. *It leaves the kinematical part of* H_1 *unchanged*:

$$X'p_X + T'p_T = \xi'\pi_\xi + \eta'\pi_\eta. \qquad (3.21)$$

5. *The metric* g_{11} *becomes dependent on the new momenta, but it remains a quadratic form of the canonical variables*:

$$g_{11} = X'^2 - T'^2 = \tfrac{1}{2}(\pi_\xi + \pi_\eta)^2 - \tfrac{1}{2}(\xi' - \eta')^2. \qquad (3.22)$$

Let us remark that the transformation (3.16) is not the only one having the enumerated properties. E.g., the transformation

$$\xi' = 2^{-1/2}(X' + p_T), \quad \eta' = 2^{-1/2}(T' - p_X),$$
$$\pi_\xi = 2^{-1/2}(T' + p_X), \quad \pi_\eta = 2^{-1/2}(\pi_T - X') \qquad (3.23)$$

has the same desirable features. Equations (3.20) and (3.21) hold unchanged, while Eq. (3.22) gets replaced by

$$g_{11} = \tfrac{1}{2}(\xi' - \pi_\eta)^2 - \tfrac{1}{2}(\pi_\xi + \eta')^2. \qquad (3.24)$$

Let us first observe the effect of the canonical transformation (3.16) or (3.23) on the massless scalar field. Due to Eqs. (3.20), (3.21) we get

$$\widetilde{H} = -\tfrac{1}{2}(\pi_\eta^2 + \eta'^2) + \tfrac{1}{2}(\pi_\xi^2 + \xi'^2) + \tfrac{1}{2}(\pi_\phi^2 + \phi'^2), \qquad (3.25)$$

$$H_1 = \eta' \pi_\eta + \xi' \pi_\xi + \phi' \pi_\phi. \tag{3.26}$$

The super-Hamiltonian (3.25) and supermomentum (3.26) describe the collection of three noninteracting scalar fields, ϕ, ξ, and η, each of which satisfies the wave equation. The ϕ and ξ fields have the positive energy density, the η field has the negative energy density. Otherwise, all these fields enter the constraint functions in an entirely symmetrical fashion. There is no way to recognize from the form of the constraints (3.25), (3.26) that ϕ is a dynamical field, while ξ is a combination of kinematical variables; Eqs. (3.25), (3.26) are invariant with respect to the interchange

$$\xi,\ \pi_\xi \longleftrightarrow \phi, \pi_\phi. \tag{3.27}$$

The parabolic super-Hamiltonian (3.9) was cast by the canonical transformation into the hyperbolic form (3.25). The contravariant supermetric G^{AB} in the space of the canonical momenta

$$\pi_A \equiv \{\pi_\eta, \pi_\xi, \pi_\phi\} \tag{3.28}$$

has the diagonal form

$$G^{AB} = \tfrac{1}{2}\mathrm{diag}\{-1, 1, 1\}, \tag{3.29}$$

so that the superspace corresponding to the parameterized massless scalar field is flat.

Though the kinematical variable ξ cannot be recognized from the dynamical variable ϕ by inspecting the form of the constraint functions, there are inequalities limiting the range of the kinematical variables which remind us of the distinction between ξ and ϕ. These inequalities arise from the condition that the hypersurfaces must be spacelike,

$$g_{11} = X'^2 - T'^2 > 0. \tag{3.30}$$

In terms of the new canonical variables, Eq. (3.30) becomes

$$|\pi_\xi + \pi_\eta| > |\xi' - \eta'|. \tag{3.31}$$

We see that Eq. (3.31) mentions ξ and π_ξ, but not ϕ and π_ϕ.

The distinction between the kinematical and the dynamical variables becomes obvious even at the level of the super-Hamiltonian if we pass to the massive scalar field. Taking into account Eq. (3.22), we see that the canonical transformation (3.16) casts the parabolic super-Hamiltonian (3.9) into the hyperbolic form

$$\tilde{H} = -\tfrac{1}{2}(1-\mu^2\phi^2)\pi_\eta^{\,2} + \mu^2\phi^2\pi_\xi\pi_\eta + \tfrac{1}{2}(1+\mu^2\phi^2)\pi_\xi^{\,2} + \tfrac{1}{2}\pi_\phi^{\,2}$$
$$-\tfrac{1}{2}\eta'^2 + \tfrac{1}{2}\xi'^2 + \tfrac{1}{2}\phi'^2 - \tfrac{1}{2}\mu^2\phi^2(\xi' - \eta')^2. \tag{3.32}$$

Here, the kinematical and the dynamical variables are no longer on an equal footing, ϕ having clearly a distinguished position. The ξ, η fields are coupled to the ϕ field by the last (potential) term in Eq. (3.32). Moreover, the supermetric in the space (3.28) becomes dependent on the canonical coordinate ϕ,

$$G^{AB} = \tfrac{1}{2}\left\|\begin{matrix} -(1-\mu^2\phi^2), & \mu^2\phi^2, & 0 \\ \mu^2\phi^2, & (1+\mu^2\phi^2), & 0 \\ 0, & 0, & 1 \end{matrix}\right\|. \tag{3.33}$$

The Ricci tensor of the three-dimensional space $\{\eta, \xi, \phi\}$ with the contravariant metric (3.33) is

$$R_{AB} = \mu^2 \left\|\begin{matrix} 1, & -1, & 0 \\ -1, & 1, & 0 \\ 0, & 0, & 0 \end{matrix}\right\|. \tag{3.34}$$

The superspace $\{\eta(x), \xi(x), \phi(x)\}$ is thus obviously curved. This is surprising in face of the fact that the ϕ field obeys a linear equation in the original flat spacetime.

The super-Hamiltonian (3.32) is again hyperbolic, the metric (3.33) having the signature $(-,+,+)$ for an arbitrary ϕ [note that the determinant of (3.33) is always equal to -1].

The quantum field theory of a scalar field in flat spacetime is, of course, the simplest of the soluble theories. This makes the foregoing model, with all of its artifically introduced similarities to curved superspace, so attractive. One can hope to clarify the connection between the Schrödinger and Klein—Gordon quantizations along its simple outlines.

On the other hand, one should realize the limited applicability of the canonical transformations studied in this section. No immediate generalization offers itself to higher dimensional spacetimes ($n = 2, 3$). Even in a $(1+1)$-dimensional spacetime, the transformation looses its usefulness for other fields then the real and complex scalar fields. So, e.g., for the electromagnetic field in $1+1$ dimensions, g_{11} enters also into the electric energy part of the rescaled super-Hamiltonian $\tilde{H}$, making it of the fourth order in the new momenta. The ideas expressed here—the rescaling followed by a linear canonical transformation mixing the canonical coordinates with their conjugate momenta—thus work only in very special situations. The cylindrical gravitational waves are another simple model in which they find their application.[9]

4. HYPERSURFACE KINEMATICS GENERATED BY A HYPERBOLIC SUPER-HAMILTONIAN

The canonical transformation considered in the last section affected only the kinematical variables T, X, p_T, p_X. Its usefulness lies in its power to leave the kinematical supermomentum linear in the momenta, Eq. (3.21), while casting the linear kinematical super-Hamiltonian into a hyperbolic form, Eq. (3.20). The hypersurface kinematics in $1+1$ dimensions can be then generated by this hyperbolic super-Hamiltonian

Because the metric g_{11} enters the scalar super-Hamiltonian in a very special way, the rescaled field super-Hamiltonian is also quadratic in the momenta and the scalar field dynamics is again generated by a hyperbolic super-Hamiltonian, composed from the kinematical and the field parts. However, this property of the scalar field is rather fortuitous and the trick does not necessarily work for other fields. The only general theorem we have concerns the "hyperbolization" of hypersurface *kinematics* in a 1+1 dimensional flat spacetime.

We now explore a different method which allows us to generate the hypersurface kinematics in an arbitrarily dimensional flat spacetime by a hyperbolic super-Hamiltonian. First, we write down the equation governing the change of the embedding e^α along the embedding curve $e(t)$,

$$\dot{e}^\alpha = Nn^\alpha + N^a e^\alpha_a. \tag{4.1}$$

Of course, this is nothing else but the lapse-shift decomposition (2.6). We notice that e^α changes under the normal deformation ($N \neq 0, N^a = 0$) of the hypersurface by Nn^α, so that n^α is the normal velocity corresponding to the canonical coordinate e^α. In Sec. 2, we have seen that the density form of the normal velocity of the scalar field ϕ played the role of the momentum π_ϕ canonically conjugate to ϕ. It is thus natural to ask whether one can build the Hamiltonian formalism in which $g^{1/2}n_\alpha$ would be the momentum canonically conjugate to the embedding variable e^α.

Put therefore

$$\pi_\alpha \equiv g^{1/2} n_\alpha \tag{4.2}$$

and study its change along the embedding curve. Because

$$\dot{g}^{1/2} = -NK + (g^{1/2}N^a)_{,a}, \tag{4.3}$$

we get

$$\dot{\pi}_\alpha = -NKg^{1/2}n_\alpha + g^{1/2}e^a_\alpha N_{,a} + (\pi_\alpha N^a)_{,a}, \tag{4.4}$$

by differentiating the definition (2.3) of n_α. Can one find the Hamiltonian generating Eqs. (4.1), (4.4) as the canonical equations of motion?

For the shift part of Eqs. (4.1), (4.4) this is trivially accomplished by our old supermomentum (2.12), $H_a = e^\alpha_a \pi_\alpha$, contributing to the Hamiltonian by the term $\int d^n x N^a(x) H_a(x)$. Next, we find the potential $V[e]$ yielding the lapse terms in Eq. (4.4). Indeed,

$$V[e] = \int_m d^n x' g^{1/2}(x')[e] N(x') \tag{4.5}$$

is such a potential. Because

$$\delta g^{1/2}[e] = g^{1/2} g^{ab} e_{\alpha b} \delta e^\alpha_a, \tag{4.6}$$

we get

$$-\frac{\delta V[e]}{\delta e^\alpha(x)} = -\int_m d^n x' g^{1/2}(x') e^a_\alpha N(x') \delta_{,a'}(x',x)$$

$$= (g^{1/2}e^a_\alpha N)_{,a} = g^{1/2}e^a_\alpha N_{,a} + (g^{1/2}e^a_\alpha)_{|a} N.$$

The Gauss—Weingaarten equation

$$e^a_{\alpha|b} = -K^a_b\, n_\alpha \tag{4.7}$$

introduces the extrinsic curvature into the last term, and our expectation is fulfilled:

$$-NKg^{1/2}n_\alpha + g^{1/2}e^{a\alpha}N_{,a} = -\frac{\delta V[e]}{\delta e^\alpha}. \tag{4.8}$$

Next, we write down the kinetic term

$$T[e,\pi] \equiv \tfrac{1}{2}\int_m d^n x N g^{-1/2}[e] \eta^{\alpha\beta}\pi_\alpha \pi_\beta \tag{4.9}$$

which generates the lapse term in Eq. (4.1) by virtue of Eq. (4.2)

$$Ng^{1/2}n^\alpha = N\pi^\alpha = \frac{\delta T}{\delta \pi_\alpha}. \tag{4.10}$$

This leads us to the lapse part $T[e,\pi] + V[e]$ of the Hamiltonian. Unfortunately, the argument is spoiled by the presence of e in the kinetic term $T[e,\pi]$, which must be varied as well and which contributes to Eq. (4.4) by an unwanted term. The appearances are saved by recalling that we are building the formalism with the super-Hamiltonian *constraint*. All objectives are achieved by taking the lapse part of the Hamiltonian in the form $T[e,\pi] + \frac{1}{2}V[e]$, i.e., starting from the action

$$S[e^\alpha, \pi_\alpha, N, N^a] = \int_m d^n x(\pi_\alpha \dot{e}^\alpha - NH - N^a H_a), \tag{4.11}$$

with the super-Hamiltonian

$$H = \tfrac{1}{2}g^{-1/2}[e]\eta^{\alpha\beta}\pi_\alpha\pi_\beta + \tfrac{1}{2}g^{1/2}[e], \tag{4.12}$$

and the supermomentum

$$H_a = e^\alpha_a \pi_\alpha. \tag{4.13}$$

Indeed, the action (4.11)—(4.13) leads to the correct equations (4.1), (4.2), (4.4) for the embedding variables e^α and the conjugate momenta $\pi_\alpha = g^{1/2}n_\alpha$. Varying it with respect to π_α, we get

$$\dot{e}^\alpha = N^a e^\alpha_a + Ng^{-1/2}\pi^\alpha. \tag{4.14}$$

Next, the variation of the shift gives the supermomentum constraint

$$H_a \equiv e^\alpha_a \pi_\alpha = 0, \tag{4.15}$$

which implies that the momentum π_α must be parallel to the normal,

$$\pi_\alpha = \pi n_\alpha. \tag{4.16}$$

The variation of the lapse leads to the super-Hamiltonian constraint,

$$H = \tfrac{1}{2}(g^{-1/2}\eta^{\alpha\beta}\pi_\alpha\pi_\beta + g^{1/2}) = 0, \tag{4.17}$$

which fixes the proportionality factor π to $\pm g^{1/2}$. The solution for the momentum π_α is thus double-valued,

$$\pi_\alpha = \pm g^{1/2} n_\alpha. \tag{4.18}$$

Substituting (4.18) back into Eq. (4.14), we learn that N is (up to a possible sign) the lapse function, and N^a is the shift vector.

It is significant that N could be identified with the lapse function only modulo the constraints, i.e., weakly. As a consequence, the super-Hamiltonian also does not satisfy Eq. (2.13) strongly, but only weakly. Let us calculate

$[H(x),H(x')]$

$$= \tfrac{1}{4}[g^{-1/2}\eta^{\alpha\beta}\pi_\alpha\pi_\beta(x), g^{-1/2}\eta^{\gamma\delta}\pi_\gamma\pi_\delta(x')]$$

$$+ \{\tfrac{1}{4}[g^{-1/2}\eta^{\alpha\beta}\pi_\alpha\pi_\beta(x), g^{1/2}(x')] - (x \longleftrightarrow x')\}$$

directly. Using Eq. (4.6), we get the necessary variational derivatives

$$\frac{\delta g^{1/2}(x)}{\delta e^\alpha(x')} = g^{1/2}(x) e^a_\alpha(x)\delta_{,a}(x,x'),$$
$$\frac{\delta g^{-1/2}(x)}{\delta e^\alpha(x')} = -g^{-1/2}(x) e^a_\alpha(x)\delta_{,a}(x,x'), \tag{4.19}$$

while the identity (3.12) again polishes the final result,

$[H(x),H(x')]$

$$= (1 - g^{-1/2}(x)H(x))H^a(x)\delta_{,a}(x,x') - (x \longleftrightarrow x'). \tag{4.20}$$

We have thus arrived at an interesting example of generalized Hamiltonian theory in which the constraint functions close under the Poisson bracket operation not into a linear, but into a quadratic combination of the original constraint functions.

The action (4.11) with the hyperbolic super-Hamiltonian (4.12) is the "squared form" of the action

$S[e^\alpha, \pi_\alpha, N^\alpha]$

$$= \int dt \int_m d^nx\,(\pi_\alpha \dot{e}^\alpha - N^\alpha(\pi_\alpha - g^{1/2}n_\alpha[e])) \tag{4.21}$$

leading to the field equations

$$\dot{e}^\alpha = N^\alpha, \tag{4.22}$$

$$\dot{\pi}^\alpha = \frac{\delta}{\delta e(x)} \int_m d^nx'\, g^{1/2}(x') n_\beta(x') N^\beta(x') \tag{4.23}$$

and the constraint

$$H_\alpha \equiv \pi_\alpha - g^{1/2}n_\alpha[e] = 0. \tag{4.24}$$

Indeed, the variational derivative in Eq. (4.23) can be rearranged into

$$-\left[\frac{\delta}{\delta e^\alpha(x)} \int_m d^nx'\, g^{1/2}(x') N(x')\right]_{N(x')\ \mathrm{fixed}}$$

$$+ \int_m d^nx'\, g^{1/2}(x')\, \frac{\delta n_\beta(x')}{\delta e^\alpha(x)}.$$

The first term we have already evaluated, in Eqs. (4.5) and (4.8). The second term is equal to $(g^{1/2}n_\alpha N^a)_{,a}$, due to the formula

$$\frac{\delta n_\beta(x')}{\delta e^\alpha(x)} = -e^b_\beta(x') n_\alpha(x')\delta_{,b'}(x',x),$$

obtained by varying Eqs. (2.2). Therefore,

$$\frac{\delta}{\delta e^\alpha} \int_m d^nx'\, g^{1/2}(x') n_\beta(x') N^\beta(x')$$

$$= -NKg^{1/2}n_\alpha + g^{1/2}e^a_\alpha N_{,a} + (g^{1/2}n_\alpha N^a)_{,a}, \tag{4.25}$$

so that Eq. (4.23) coincides with our old equation (4.4). The actions (4.11), (4.12), and (4.21) both lead to the same correct equations of motion.

We pass from the action (4.21) to the action (4.11)–(4.13) in the following steps. We project the constraint (4.24) into the hypersurface and "square it,"

$$e^\alpha_a H_\alpha \equiv e^\alpha_a \pi_\alpha = 0,$$
$$H \equiv g^{-1/2}(\pi^\alpha + g^{1/2}n^\alpha)(\pi_\alpha - g^{1/2}n_\alpha) \tag{4.26}$$
$$= g^{-1/2}\eta^{\alpha\beta}\pi_\alpha\pi_\beta + g^{1/2} = 0.$$

The constraints (4.26) are equivalent to the original constraint (4.24), up to the ambiguity in sign introduced by the squaring operation. If we adjoin the new constraints (4.26) to the action by means of the new multipliers N^a and N, we arrive at the action (4.11), (4.12).

The last point which remains to be clarified is the connection between the linear action (4.21) and the standard linear action (2.7). Let us use the Greek epsilon to denote the embedding variable conjugate to the π_α. Our aim is to show that the transition from (2.7) to (4.21),

$$e^\alpha = \epsilon^\alpha, \quad p_\alpha = \pi_\alpha - g^{1/2}n_\alpha, \tag{4.27}$$

is a canonical transformation. Indeed, Eq. (4.27) follows from the generating functional

$$F[\epsilon^\alpha, p_\alpha] = -\int_m d^nx\,(p_\alpha\epsilon^\alpha + (n+1)^{-1}g^{1/2}[\epsilon]n_\alpha[\epsilon]\epsilon^\alpha) \tag{4.28}$$

by variational differentiation

$$e^\alpha = -\frac{\delta F}{\delta p_\alpha}, \quad \pi_\alpha = -\frac{\delta F}{\delta \epsilon^\alpha}. \tag{4.29}$$

The last variational derivative is evaluated with the help of Eq. (4.25) (for $N^\beta = \epsilon^\beta$) and the relations

$$n_{\alpha,b} = -K_{ab}e^a_\alpha, \quad e_{a\alpha|b} = -K_{ab}n_\alpha.$$

The term $g^{1/2}n_\alpha$ in Eq. (4.27) is thus shown to be the functional gradient of an integral given by the last term in Eq. (4.28).

We conclude that the hyperbolic super-Hamiltonian (4.12) follows from the standard linear super-Hamiltonian (2.11) by the canonical transformation (4.27) followed by the squaring operation (4.26). Either one of these super-Hamiltonians correctly generates the kinematics of hypersurface in a $(1+n)$-dimensional Minkowskian spacetime.

5. 1 + 2 DIMENSIONAL GEOMETRODYNAMICS

The metric field in $1+3$ dimensions possesses $2\,\infty^3$ dynamical degrees of freedom. While other fields (scalar, electromagnetic) remain dynamic in a $(1+2)$- or $(1+1)$-dimensional spacetimes, the metric field is an exception. The Einstein law of gravitation cannot be written down in $1+1$ dimensions, because the Einstein tensor identically vanishes (another way of seeing this is to realize that the Hilbert action in two dimensions becomes a topological invariant due to the Gauss—Bonnet theorem). Correspondingly, the super-Hamiltonian ceases to be well-defined for $n=1$.

In $1+2$ dimensions, the Einstein law makes sense. However, the vanishing of the Einstein tensor in the vacuum is equivalent to the vanishing of the full Riemann tensor. The matter thus curves spacetime only locally and the spacetime outside bodies is necessarily flat. The gravitational field in $1+2$ dimensions has no true dynamical degrees of freedom.

Still, we can generate the flat spacetime by the evolution of the 2-geometry $g_{ab}(x)[e]$ through the action functional (2.24) with the supermomentum (2.27) and the hyperbolic super-Hamiltonian (2.26),

$$H[g_{ab},\pi^{ab}]=g^{-1/2}(\pi_{ab}\pi^{ab}-\pi^2)-g^{1/2}R. \tag{5.1}$$

The quantization of this model was attempted by Leutwyler.[23]

On the other hand, we know that the kinematics of hypersurfaces in a flat spacetime is described by the action (2.10) with the standard linear super-Hamiltonian (2.11) and supermomentum (2.12). The geometry carried by the hypersurface e^α can then be defined by the equation

$$g_{ab}(x)[e]=\eta_{\alpha\beta}e^\alpha_a e^\beta_b. \tag{5.2}$$

We know that the geometrodynamical scheme (2.24), (2.27) and (5.1) and the hypersurface kinematics scheme (2.10)—(2.12) and (5.2) must be equivalent to each other, but we are confronted by the problem how to spell out this equivalence in purely Hamiltonian terms.

The obvious idea is to complete Eq. (5.2) into a canonical transformation

$$e^\alpha, p_\alpha \rightarrow g_{ab}, p^{ab}. \tag{5.3}$$

For $n=2$, the number of variables is just right: each of the spacetime quantities e^α, p_α has three components, and each of the symmetric space tensors g_{ab}, p^{ab} also has three independent components. Our first attempt to produce the geometrodynamical momentum (2.25) is, however, doomed to failure; we have thus denoted the momentum arrived at by the canonical transformation (5.3) by another symbol, namely, p^{ab}.

The natural generating functional of Eq. (5.2) is

$$F[e^\alpha,p^{ab}]=\int_m d^2x\, \eta_{\alpha\beta}e^\alpha_a e^\beta_b\, p^{ab}; \tag{5.4}$$

indeed,

$$g_{ab}=\frac{\delta F}{\delta p^{ab}}$$

yields Eq. (5.2), while

$$p_\alpha=\frac{\delta F}{\delta e^\alpha}=-2(p^{ab}e_{a\alpha})_{,b} \tag{5.5}$$

specifies the transformation of the momenta.

As usual, Eq. (5.5) is resolved neither with respect to the old nor the new momenta. Bypassing this problem, let us rather see what combinations of the new momenta we must know when calculating the quantities of interest, namely, the constraint functions.

Because the expression $p^{ab}e_{a\alpha}$ is a space vector density, we can replace the partial derivative $,b$ in Eq. (5.5) by the covariant derivative $|b$. Using the Gauss—Weingaarten equation (4.7), we get

$$p_\alpha=-2p^{ab}{}_{|b}e_{a\alpha}+2K_{ab}p^{ab}n_\alpha. \tag{5.6}$$

The projections of p_α onto the surface and perpendicular to it yield the supermomentum (2.12) and the super-Hamiltonian (2.11),

$$H_a=e^\alpha_a p_\alpha=-2p^b_{a|b}, \tag{5.7}$$

$$H=n^\alpha p_\alpha=-2K_{ab}p^{ab}. \tag{5.8}$$

In supermomentum, the resolution problem is completely solved, because the final expression in (5.7) is given entirely in terms of the new variables, g_{ab} and p^{ab}. The extrinsic curvature in the super-Hamiltonian, however, is determined from the Gauss—Weingaarten equation,

$$K_{ab}[e]=n_\alpha[e]e^\alpha_{a|b}, \tag{5.9}$$

and as such it is still a functional of the embedding rather than of the 2-geometry g_{ab}.

To obtain K_{ab} as a functional of geometry, it is best to proceed in an indirect way. We know that the spacetime is flat and that its flatness in $1+2$ dimensions is equivalent to the vanishing of the Einstein tensor $G_{\alpha\beta}$. We also know[24] that the $\perp\perp$ and $\perp a$ projections of this tensor contain only the metric and the extrinsic curvature,

$$-2G_{\perp\perp}\equiv-2n^\alpha G_{\alpha\beta}n^\beta=(K_{ab}K^{ab}-K^2-R)=0, \tag{5.10}$$

$$G_{a\perp}\equiv-e^\alpha_a G_{\alpha\beta}n^\beta=(K^b_a-K\delta^b_a)_{|b}=0, \tag{5.11}$$

while the ab projection contains also the normal change of the extrinsic curvature and is thus irrelevant for our purposes. Equations (5.1) and (5.11) could have been obtained, of course, also directly by manipulating Eq. (5.9).

An important fact is that the three equations (5.10) and (5.11) are just sufficient (with appropriate boundary conditions) to determine the three components of K_{ab} as functionals of the metric. An equally important fact is that we cannot write down an explicit form of this functional, because only Eq. (5.1) is algebraic, the second equation, Eq. (5.11), being a differential equation. The best we can do in this situation is to formulate our action principle as a variational principle with supplementary conditions:

The actual field extremizes the action

$$S[g_{ab},p^{ab},K_{ab},N,N^a]\equiv\int dt\int_m d^2x\,(p^{ab}\dot g_{ab}-NH-N^aH_a), \tag{5.12}$$

$$H=-2K_{ab}p^{ab},\quad H_a=-2p^b_{a|b}, \tag{5.13}$$

for all variations of the variables g_{ab}, p^{ab}, K_{ab}, N^a, N satisfying the supplementary algebraic-differential conditions (5.10) and (5.11).

This is a far cry from the geometrodynamical action principle (2.24), (2.27), and (5.1) which, first, is a free variational principle, and second, contains the super-Hamiltonian (5.1) quadratic in the momentum π^{ab}, while the super-Hamiltonian (5.13) is still linear in the momentum p^{ab}.

To resolve this discrepancy, we start from a slightly different canonical transformation than that produced by the generating functional (5.4). We put

$$F[e^\alpha, \pi^{ab}] = \int_m d^2x(\eta_{\alpha\beta}e^\alpha_a e^\beta_b \pi^{ab} - 2g^{1/2}[e]K[e]), \tag{5.14}$$

with the extrinsic curvature again given by Eq. (5.9). The additional term does not affect the metric (5.2), but it modifies the kinematical momentum,

$$\pi_\alpha = \frac{\delta F}{\delta e^\alpha} = -2(\pi^{ab}e_{a\alpha})_{|b} - 2\frac{\delta}{\delta e^\alpha}\int_m d^2x\, g^{1/2}[e]K[e]. \tag{5.15}$$

The last term is most easily evaluated from the projection formula for the spacetime curvature scalar,[25]

$$Ng^{1/2}\,{}^3R = -2(g^{1/2}K)^\cdot + Ng^{1/2}[K_{ab}K^{ab} - K^2) + R] + 2(g^{1/2}KN^a)_{,a} + 2(g^{1/2}g^{ab}N_{,b})_{,a}. \tag{5.16}$$

In flat spacetime, the left-hand side of Eq. (5.16) vanishes. The last two terms on the right-hand side of Eq. (5.16) are boundary terms, and $N = -n_\alpha \dot{e}^\alpha$. Therefore,

$$-2\frac{\delta}{\delta e^\alpha}\int_m d^2x\, g^{1/2}[e]K[e] = g^{1/2}\{(K_{ab}K^{ab} - K^2) + R\}n^\alpha. \tag{5.17}$$

The kinematical momentum (5.15) thus assumes the form

$$\pi_\alpha = -2\pi^{ab}{}_{|b}e_{a\alpha} + \{2\pi^{ab}K_{ab} + g^{1/2}(K_{ab}K^{ab} - K^2) + g^{1/2}R\}n_\alpha, \tag{5.18}$$

leading to the super-Hamiltonian

$$H = n^\alpha\pi_\alpha = -2K_{ab}\pi^{ab} - g^{1/2}(K_{ab}K^{ab} - K^2) - g^{1/2}R, \tag{5.19}$$

and supermomentum

$$H_a = e^\alpha_a \pi_\alpha = -2\pi^b_{a|b}. \tag{5.20}$$

Superficially, nothing is gained by making the new canonical transformation (5.2) and (5.18) instead of the old one. The new super-Hamiltonian (5.19) certainly looks more complicated than the old super-Hamiltonian (5.8), and the old troubles remain unresolved: The variational principle is still conditional, subject to the supplementary conditions (5.10) and (5.11), and the new super-Hamiltonian (5.19) is still linear in the new momentum π^{ab}. However, varying the new action

$$S[g_{ab}, \pi^{ab}, K_{ab}, N, N^a] \equiv \int dt \int_m d^2x(\pi^{ab}\dot{g}_{ab} - NH - N^aH_a) \tag{5.21}$$

freely, we discover that the supplementary conditions (5.10) and (5.11) automatically follow from it. Indeed, due to the new structure of the super-Hamiltonian (5.19), the variation of K_{ab} gives Eq. (2.25). The variation of the N and N^a gives the constraints $H = 0 = H_a$ which, upon the substitution (2.25), turn out to be identical with the supplementary conditions (5.10) and (5.11).

This allows us to replace the conditional variational principle by the free variational principle (5.19)—(5.21) and remove thus the first of our troubles. The extrinsic curvature is to be varied freely; it enters the action (5.21) as another Lagrange multiplier, through the super-Hamiltonian (5.19). The action (5.21) is still linear in the momentum π^{ab}. However, by the well-known trick of the variational calculus,[26] any variational principle can be transformed into an equivalent one by eliminating some of the variables: We solve their Euler—Lagrange equations, express the chosen variables in terms of the remaining ones, and substitute these solutions into the original action. In particular, the Euler—Lagrange equation obtained from the action (5.19)—(5.21) by varying K_{ab}, namely, Eq. (2.25), can be solved for K_{ab} in terms of g_{ab} and π^{ab}. When this solution is substituted back into the action (5.21), we recover the action functional (2.24) with the hyperbolic super-Hamiltonian (5.1).

The kinematical action (2.10) with the linear super-Hamiltonian (2.11) is thus transformed into the geometrodynamical action (2.24) with the hyperbolic super-Hamiltonian (5.1) by the canonical transformation (5.2) and (5.18) with the generating functional (5.14), followed by the conversion of the conditional variational principle (5.10), (5.11), and (5.19)—(5.21) into the free variational principle (5.19)—(5.21), and by the elimination of the Lagrange multiplier K_{ab} from the action (5.21).

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Random linear systems and temporal homogeneity

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The class of random linear systems having stochastic Green's functions whose moments are invariant under arbitrary uniform translations of all time variables is defined and investigated. It is pointed out that this class is very broad, including, for example, virtually all treatments of wave propagation through a random medium. Proceeding by analogy with quantum field theory the quantities $\mathcal{G}$ and M, related to the first and second moments of the stochastic Green's function, are defined. Various properties of $\mathcal{G}$ and M (which in quantum field theory correspond respectively to a propagator and an elastic scattering amplitude) are discussed, and it is shown that they may be conveniently used to describe the principal physical effects induced by transmission through a randomly fluctuating system. Specific examples are given in which these quantities are explicitly calculated and used to illustrate the general results.

I. INTRODUCTION

There exists an extensive literature devoted to random dynamical systems.[1] An important class of such systems are what might be termed linear random systems, which we take to denote dynamical systems in which the outputs are linearly related to the inputs via a causal Green's function G, where G itself is taken to be a stochastic function. Thus the randomness is an intrinsic feature of the system, inducing random outputs for any nonvanishing inputs, whether random or not. Random linear systems and stochastic Green's functions have been treated from a very general viewpoint in a number of papers by Adomian.[2-5]

It is the purpose of the present work also to investigate stochastic Green's functions, but from a somewhat different point of view. Specifically, we shall confine ourselves to random linear systems which exhibit a property which we call "temporal homogeneity." There are two main reasons for doing this. First, this property, a generalization of time translational invariance for deterministic linear systems, is sufficiently general that it appears to be implicitly assumed in virtually all calculations appearing in the literature (at least those involving wave propagation through a random medium). On the other hand, it is strong enough that it enables one to derive a number of interesting implications, and provides a useful general framework within which to categorize much of the work in this area. This can be accomplished purely on the basis of an input-output or "black box" approach, completely independently of the detailed structure or even the existence of any underlying stochastic equations of motion.

An important area involving random linear systems is wave propagation through a random medium. The present work is primarily motivated by this problem, and much of the development will be made from this point of view. For scalar waves the basic relation then becomes

$$\eta(\mathbf{x}, t) = \int d^3x'\, dt'\, G(\mathbf{x}, \mathbf{x}', t, t') f(\mathbf{x}', t'), \tag{1.1}$$

where, by casuality,

$$G(\mathbf{x}, \mathbf{x}', t, t') = 0 \qquad t' > t. \tag{1.2}$$

Here η is the observed field and f is the source density, which is taken to be statistically independent of G. There is no loss of generality involved in considering this system. The only this that is affected is the nature of the nontime variables in G, and adaptation to other random systems is trivial.

In Sec. II temporal homogeneity is defined and motivated. The basic quantities $\mathcal{G}$ and M, related to the first two moments of G, are introduced and some of their properties derived in Sec. III. A description of the physical effects of fluctuations in terms of these quantities is presented in Sec. IV. In Sec. V specific examples of random linear systems are considered and some explicit calculations of $\mathcal{G}$ and M are performed. A summary and conclusions are presented in Sec. VI. Two Appendices are included.

II. TEMPORAL HOMOGENEITY

Consider as an example acoustic propagation through a fluctuating ocean. It is well known that acoustic sensors placed in the ocean often register outputs that may be interpreted as stationary random processes.[6,7] This in turn would imply that the relevant sources are themselves stationary. We are thus led to entertain the hypothesis of the preservation of stationarity: Whenever the sources consist of stationary random processes so do the observed fields.

Actually, we shall confine our attention to random linear systems having a slightly stronger property which we call *temporal homogeneity*. This is simply the requirement that the moments[8] of the Green's function be invariant under any uniform displacement of all time variables, i.e.,

$$\langle G(\mathbf{x}_1, \mathbf{x}_2, t_1 + a, t_2 + a)\rangle = \langle G(\mathbf{x}_1, \mathbf{x}_2, t_1, t_2)\rangle, \tag{2.1}$$

$$\langle G(\mathbf{x}_1, \mathbf{x}_2, t_1 + a, t_2 + a) G(\mathbf{x}_3, \mathbf{x}_4, t_3 + a, t_4 + a)\rangle = \langle G(\mathbf{x}_1, \mathbf{x}_2, t_1, t_2) G(\mathbf{x}_3, \mathbf{x}_4, t_3, t_4)\rangle, \tag{2.2}$$

valid for all values of a. Here the brackets denote an ensemble average. It is shown in Appendix B that temporal homogeneity, which will be used extensively throughout the remainder of this work, guarantees the preservation of stationarity.[9]

Put another way, we are confining our attention to random linear systems which, loosely speaking, "fluctuate" or "oscillate" as opposed to proceeding

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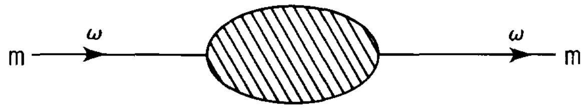

FIG. 1. Diagrammatic representation of $\mathcal{G}_m(\omega)$.

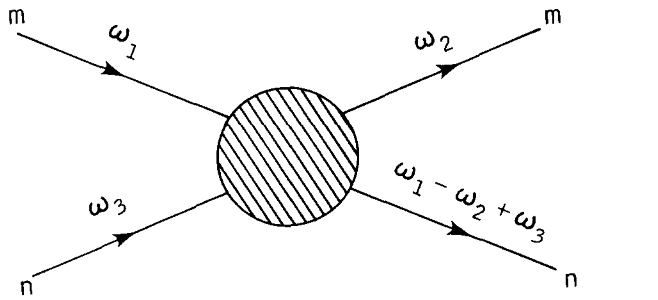

FIG. 2. Diagrammatic representation of $\mathcal{G}_{mn}(\omega_1,\omega_2,\omega_3)$.

"monotonically" as time progresses. The whole idea of temporal homogeneity is simply an attempt to capture this property in as simple and general a way as possible. It is interesting to note that in quantum field theory[10] (QFT) time translation invariance is equivalent to energy conservation. The relationship of this to temporal homogeneity will be quite apparent in succeeding sections.

It is important to understand the types of ensemble averaging which apply in Eqs. (2.1) and (2.2), as well as in succeeding sections. Since G is a function of the medium alone, the average is taken over the ensemble of all possible mediums. In practice this is often equivalent to the ensemble S of all possible realizations of the index of refraction. Many times $f(\mathbf{x}', t')$ will be treated as a random function, with the ensemble of all possible realizations denoted by F. We shall assume that f and G are completely independent. Since $\eta(\mathbf{x}, t)$ depends upon both G and f, all averages involving it must be understood to be over the ensemble $S\times F$. In calculating various averages extensive use is made of the statistical independence of G and f, enabling one to use such identities as

$$\langle Gf\rangle_{S\times F} = \langle G\rangle_S \langle f\rangle_F. \tag{2.3}$$

With these conditions in mind, we shall henceforth dispense with subscripts that specify the ensemble.

III. PROPAGATORS AND SCATTERING AMPLITUDES

A. Definitions

For notational convenience we shall write $G(\mathbf{x}, \mathbf{x}', t, t')$ as $G_m(t, t')$, where the index m is used to specify the coordinate pair $(\mathbf{x}, \mathbf{x}')$. For a general random linear system, then, m simply represents all the nontime arguments in G.

Let us consider first order moments of G. By temporal homogeneity we have

$$\langle G_m(t_1, t_2)\rangle = \langle G_m(t_1 - t_2, 0)\rangle \tag{3.1}$$

so that first-order moments are actually functions of only one time variable. Define $\mathcal{G}_m(\omega)$ by

$$\mathcal{G}_m(\omega) \equiv \int dt \exp(-i\omega t)\langle G_m(t, 0)\rangle. \tag{3.2}$$

By analogy with QFT we see that $\mathcal{G}_m(\omega)$ corresponds to a two-point function, or propagator.[11] Accordingly we represent it diagrammatically as shown in Fig. 1. We also may consider $\mathcal{G}_m(\omega)$ to be a generalization of the transfer function as it is defined for deterministic, nonfluctuating linear systems.

Again invoking temporal homogeneity, it follows immediately that second order moments of G depend only upon three time variables. Accordingly we employ a triple Fourier transform to define the quantity

$$\mathcal{G}_{mn}(\omega_1, \omega_2, \omega_3) \equiv \int dt_1\, dt_2\, dt_3 \exp(-i\omega_1 t_1 + i\omega_2 t_2 - i\omega_3 t_3) \times \langle G_m(t_1, t_2) G_n(t_3, 0)\rangle. \tag{3.3}$$

Again from QFT we see that $\mathcal{G}_{mn}(\omega_1, \omega_2, \omega_3)$ is simply a four-point function, or a two-particle elastic scattering amplitude. Specifically, $\mathcal{G}_{mn}$ represents a "collision" in which an m particle is incident with energy ω_1 and leaves with energy ω_2, while the other particle, identified by the index n, is incident with energy ω_3 and recoils with energy $\omega_1 - \omega_2 + \omega_3$, energy conservation being guaranteed by temporal homogeneity. This is shown diagrammatically in Fig. 2.

One may show directly from the definition the useful identity that

$$\mathcal{G}_{mn}(\omega_1, \omega_2, \omega_3) = \mathcal{G}_{nm}(\omega_3, \omega_1 - \omega_2 + \omega_3, \omega_1). \tag{3.4}$$

This result is immediate from a diagrammatic point of view, since both terms represent the same scattering event.

It is useful to define the quantity M by

$$\mathcal{G}_{mn}(\omega_1, \omega_2, \omega_3) = 2\pi\delta(\omega_1 - \omega_2)\mathcal{G}_m(\omega_1)\mathcal{G}_n(\omega_3) + M_{mn}(\omega_1, \omega_2, \omega_3). \tag{3.5}$$

The decomposition (3.5) is also familiar from QFT. It corresponds to separating the contributions to $\mathcal{G}_{mn}$ into a sum of "disconnected" diagrams as represented by the δ function term and a sum of "connected" diagrams given by the M function. This is shown in Fig. 3.

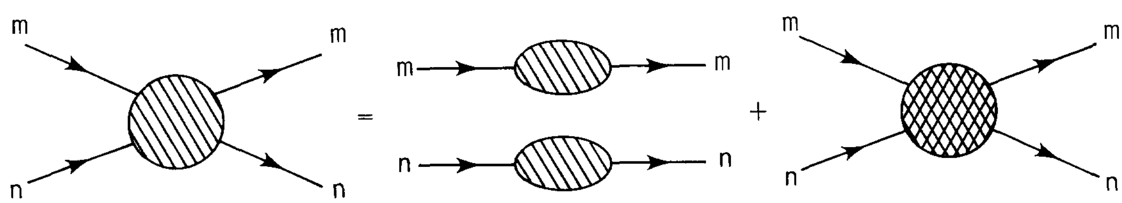

FIG. 3. Diagrammatic representation of the decomposition (3.5). The blob representing M is crosshatched to signify that only connected diagrams are included.

The quantities $\mathcal{G}$ and M have just been defined by appealing to analogies with QFT. There is nothing new, of course, about applying ideas from QFT to wave propagation through a random medium.[12] In previous work, however, these ideas have been introduced at a stage where dynamical calculations based upon some stochastic wave equation are to be performed. In the present work on the other hand, we see that the analogy actually sets in at a much earlier "black box" stage, with no reference whatever to detailed dynamics.

B. Limits

Every temporally homogeneous random system has two important limits. The first of these is the *static limit*, in which the rate at which the medium is fluctuating vanishes. Put another way, every member of the ensemble S of possible systems[13] is time invariant,[14] so that

$$G_m(t,t')=G_m(t-t'). \tag{3.6}$$

This allows us to define the random time independent Fourier transform

$$G_m(\omega)\equiv\int dt\,\exp(-i\omega t)G_m(t). \tag{3.7}$$

Therefore,

$$\langle G_m(t_1,t_2)G_n(t_3,0)\rangle = \frac{1}{4\pi^2}\int d\omega\,d\omega'\,\exp[i\omega(t_1-t_2)+i\omega' t_3]\langle G_m(\omega)G_n(\omega')\rangle. \tag{3.8}$$

Substituting into the defining relation for $\mathcal{G}_{mn}$ yields

$$\mathcal{G}_{mn}(\omega_1,\omega_2,\omega_3)=2\pi\delta(\omega_1-\omega_2)\langle G_m(\omega_1)G_n(\omega_3)\rangle, \tag{3.9}$$

from which it immediately follows that, in the static limit,

$$M_{mn}(\omega_1,\omega_2,\omega_3)=2\pi\delta(\omega_1-\omega_2)[\langle G_m(\omega_1)G_n(\omega_3)\rangle - \langle G_m(\omega_1)\rangle\langle G_n(\omega_3)\rangle]. \tag{3.10}$$

In terms of scattering, the m and n particles may no longer exchange energy with each other. The great majority of work in random wave propagation is done in the static limit.[15] The second limit of general interest is the *deterministic limit* in which $G_m(t,t')$ becomes a deterministic function. In this limit there is no distinction between $\langle G_mG_n\rangle$ and G_mG_n or between $\langle G_m\rangle$ and G_m. It follows immediately from Eq. (3.10) that in the deterministic limit

$$M_{mn}(\omega_1,\omega_2,\omega_3)=0. \tag{3.11}$$

C. Constraints

Temporal homogeneity guarantees the preservation of stationarity, and indeed the defining conditions (2.1) and (2.2) for one appear quite similar to the conditions for the other as given in Appendix A. Stationary random processes satisfy a very fundamental constraint imposed by the Wiener–Khinchin theorem.[16] We address here the question of finding corresponding constraints obeyed by the Green's functions.

Let $G_m(t_1,t_2)$ be any temporally homogeneous Green's function. Define the quantity

$$f_m(\omega;t)\equiv\int dt_1\exp(-i\omega t_1)G_m(t_1+t,t)-\mathcal{G}_m(\omega). \tag{3.12}$$

It is easy to verify that $f_m(\omega;t)$ is a zero-mean stationary random process in t, and also that it is mutually stationary in t with $f_n(\omega';t)$ for all m, n, ω, and ω'. Calculating the correlation of two such processes yields after some algebra the result

$$\langle f_m(\omega_1;\tau)f_n^*(-\omega_3;0)\rangle = \frac{\exp(i\omega_1\tau)}{2\pi}\int d\omega\,\exp(-i\omega\tau)\mathcal{G}_{mn}(\omega_1,\omega,\omega_3) - \mathcal{G}_m(\omega_1)\mathcal{G}_n(\omega_3). \tag{3.13}$$

Using Eq. (3.5) we obtain

$$Q(m,\omega_1;n,-\omega_3;\omega_1-\omega_2)=M_{mn}(\omega_1,\omega_2,\omega_3), \tag{3.14}$$

where Q is defined by

$$Q(m,\omega;n,\omega';\omega'')\equiv\int d\tau\,\exp(-i\omega''\tau)\langle f_m(\omega;\tau)f_n^*(\omega';0)\rangle. \tag{3.15}$$

From the Wiener–Khinchin theorem we know that the power spectrum of $f_m(\omega_1;t)$ must be nonnegative or

$$Q(m,\omega;m,\omega;\omega')\geq 0. \tag{3.16}$$

Likewise the coherence of $f_m(\omega_1;t)$ and $f_n(-\omega_3;t)$ must be less than or equal to unity, or

$$|Q(m,\omega;n,\omega';\omega'')|^2 \leq Q(m,\omega;m,\omega;\omega'')Q(n,\omega';n,\omega';\omega''). \tag{3.17}$$

With the help of Eq. (3.14) these translate into the basic constraints

$$M_{mm}(\omega_1,\omega_2,-\omega_1)\geq 0 \tag{3.18}$$

and

$$|M_{mn}(\omega_1,\omega_2,\omega_3)|^2 \leq M_{mm}(\omega_1,\omega_2,-\omega_1)M_{nn}(\omega_3,\omega_1-\omega_2+\omega_3,-\omega_3). \tag{3.19}$$

These constraints are valid for all real frequencies and for all m and n. Identical constraints are satisfied by $\mathcal{G}_{mn}$, as may be verified by redefining $f_m(\omega;t)$, so that the $\mathcal{G}_m(\omega)$ term is no longer subtracted off, and then going through the same steps.

IV. PHYSICAL EFFECTS OF FLUCTUATIONS

Within the context of second order statistics and the frequency domain, there appear to be three general types of effects which are produced by temporal fluctuations in a random linear system. For definiteness we again assume scalar wave propagation through a random medium. These effects are most easily illustrated when only a single point source is present. Unless specified otherwise, we shall assume that the source density in Eq. (1.1) is given by

$$f(\mathbf{x}',t')=f(t')\delta(\mathbf{x}'-\mathbf{r}), \tag{4.1}$$

where $f(t')$ is a stationary random process.

Let P_f be the source spectrum. Then the spectrum of the observed field at $\mathbf{x}$, P_η, is given by

$$P_\eta(\omega)=|\mathcal{G}_m(\omega)|^2P_f(\omega)+\frac{1}{2\pi}\int d\omega'\,M_{mm}(\omega,\omega',-\omega)P_f(\omega'), \tag{4.2}$$

where the subscript m refers to the coordinate pair $(\mathbf{x}, \mathbf{r})$. Note that the requirement $P_\eta \geq 0$ is guaranteed by Eq. (3.18). Equation (4.2) implies spectral broadening, a well-known consequence of temporal fluctuations.[2,5] This is most readily apparent when the source is monochromatic at frequency ω_0, i.e.,

$$P_f(\omega) = \delta(\omega + \omega_0) + \delta(\omega - \omega_0). \tag{4.3}$$

Then we have

$$P_\eta(\omega) = |\mathcal{G}_m(\omega_0)|^2[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)] + (2\pi)^{-1}[M_{mm}(\omega, \omega_0, -\omega) + M_{mm}(\omega, -\omega_0, -\omega)]. \tag{4.4}$$

In terms of scattering diagrams, we see that spectral broadening is contingent upon the ability of the particles to exchange energy. It follows immediately that there can be no spectral broadening in either the static or the deterministic limit. It is clear that spectral broadening can complicate the problem of analyzing the spectral content of a source on the basis of the observed spectrum, even if one has perfect knowledge of $\mathcal{G}$ and M.

The other effects we wish to discuss involve coherence. If $f(t)$ and $g(t)$ are any two mutually stationary random processes, their coherence is defined by

$$C_{fg}(\omega) = [P_f(\omega)P_g(\omega)]^{-1/2} \int d\tau \exp(-i\omega\tau)\langle f(\tau) g^*(0)\rangle, \tag{4.5}$$

where P_f and P_g denote the power spectra of f and g. It is straightforward to show on the basis of the Wiener–Khinchin theorem[16] that

$$0 \leq |C_{fg}| \leq 1. \tag{4.6}$$

If $|C_{fg}| = 1$, we say that f and g are perfectly coherent.

We first consider the coherence between source and observed field. Using the same notation and physical situation as for spectral broadening, one obtains

$$C_{f\eta}(\omega) = \left(|\mathcal{G}_m(\omega)|^2 + \frac{1}{2\pi}\int M_{mm}(\omega, \omega', -\omega)\frac{P_f(\omega')}{P_f(\omega)}\right)^{-1/2} \times \mathcal{G}_m^*(\omega). \tag{4.7}$$

Thus we see that an effect of fluctuations is to degrade the source-field coherence. As M becomes more and more dominant, the observed field becomes increasingly independent of the source. Clearly the effect persists in the static limit. This general result is indicative of fundamental restrictions upon one's ability to transmit information through a fluctuating medium.

Also of interest is the coherence between the field as observed at $\mathbf{x}_1$ and the field as observed at $\mathbf{x}_2$. The result is

$$C_{12}(\omega) = D^{-1}\left[\mathcal{G}_1(\omega)\mathcal{G}_2^*(\omega)P_f(\omega) + \frac{1}{2\pi}\int d\omega' M_{12}(\omega, \omega', -\omega)P_f(\omega')\right], \tag{4.8}$$

where

$$D = \left(|\mathcal{G}_1(\omega)|^2 P_f(\omega) + \frac{1}{2\pi}\int d\omega' M_{11}(\omega, \omega', -\omega)P_f(\omega')\right) \times \left(|\mathcal{G}_2(\omega)|^2 P_f(\omega) + \frac{1}{2\pi}\int d\omega' M_{22}(\omega, \omega', -\omega)P_f(\omega')\right). \tag{4.9}$$

Here the subscript 1 corresponds to $(\mathbf{r}, \mathbf{x}_1)$ and 2 to $(\mathbf{r}, \mathbf{x}_2)$. After some algebra, it may be shown from Eq. (3.19) that C_{12} satisfies condition (4.6). In the deterministic limit we again have perfect coherence. As before, the effect of fluctuations is to degrade the coherence, a result which persists in the static limit. One effect of this degradation is to place fundamental limits on one's ability to extract information about sources on the basis of measured cross-spectral densities.

V. EXAMPLES

In marked contrast to the situation in QFT, it is not difficult to find nontrivial random linear systems for which $\mathcal{G}$ and M may be calculated explicitly. Two such examples are given here, along with a brief discussion of the relevance of temporal homogeneity to the far more complicated situation of wave propagation through a random medium.

A. Multiplicative noise

This example is perhaps the simplest conceivable which still has enough structure to illustrate in a reasonably nontrivial fashion all the basic results derived in this work. Consider an N channel system with outputs $\eta_1(t), \eta_2(t), \ldots, \eta_N(t)$ induced by a single common input $f(t)$ such that

$$\eta_k(t) = \alpha_k(t) f(t), \tag{5.1}$$

where the $\alpha_k(t)$ are stationary random processes which specify the system. With no loss of generality one may write

$$\alpha_k(t) = 1 + \beta_k(t), \tag{5.2}$$

where $\beta_k(t)$ is a zero-mean stationary random process. Define

$$C_{mn}(t) = \langle \beta_m(t)\beta_n(0)\rangle, \tag{5.3}$$

$$P_{mn}(\omega) = \int dt \exp(-i\omega t) C_{mn}(t), \tag{5.4}$$

so that P_{mm} is the power spectrum of β_m. From the defining relation (1.1) along with Eqs. (5.1) and (5.2) it follows that

$$G_n(t, t') = [1 + \beta_n(t)]\delta(t - t') \tag{5.5}$$

or

$$\langle G_n(t, t')\rangle = \delta(t - t'). \tag{5.6}$$

Likewise

$$\langle G_m(t_1, t_2) G_n(t_3, t_4)\rangle = \langle [1 + \beta_m(t_1)]\delta(t_1 - t_2)[1 + \beta_n(t_3)]\delta(t_3 - t_4)\rangle = [1 + C_{mn}(t_1 - t_3)]\delta(t_1 - t_2)\delta(t_3 - t_4). \tag{5.7}$$

Hence this system exhibits temporal homogeneity, which is seen to result from the stationarity of the system parameters $\beta_m(t)$. Note that temporal homogeneity sets in only *after* the ensemble averages are taken. From the definitions (3.2) and (3.3) one obtains

$$\mathcal{G}_m(\omega) = 1, \tag{5.8}$$

$$\mathcal{G}_{mn}(\omega_1, \omega_2, \omega_3) = 2\pi\delta(\omega_1 - \omega_2) + P_{mn}(\omega_1 - \omega_2). \qquad (5.9)$$

This latter equation corresponds exactly to the decomposition (3.5), yielding

$$M_{mn}(\omega_1, \omega_2, \omega_3) = P_{mn}(\omega_1 - \omega_2). \qquad (5.10)$$

Since $\beta_m(t)$ has zero mean, it must vanish in the deterministic limit. Therefore, M vanishes as well. In the static limit $\beta_n(t)$ has no time dependence. From Eqs. (5.3) and (5.4) it immediately follows that

$$M_{mn}(\omega_1, \omega_2, \omega_3) = 2\pi\delta(\omega_1 - \omega_2)\langle\beta_m\beta_n\rangle \qquad (5.11)$$

in the static limit, which on the basis of Eq. (5.5) may readily be shown equivalent to Eq. (3.10).

Note that the basic inequality (3.18) is immediately satisfied. A restatement of Eq. (4.6) requires that

$$|P_{mn}(\omega_1 - \omega_2)|^2 \leq P_{mm}(\omega_1 - \omega_2)P_{nn}(\omega_2 - \omega_1) \qquad (5.12)$$

but in the context of the present model this is simply a verification of the inequality (3.19).

Similarly, the various results of Sec. IV may be explicitly illustrated. Of particular interest is spectral spreading. Assume a monochromatic input of frequency ω_0. Let S_{kk} denote the spectrum of η_k. Then from Eq. (4.4) we have

$$S_{kk}(\omega) = \delta(\omega - \omega_0) + \delta(\omega + \omega_0) + \frac{1}{2\pi}[P_{kk}(\omega - \omega_0) + P_{kk}(\omega + \omega_0)] \qquad (5.13)$$

so that the broadened portion of the output spectrum is simply a sum of displacements of the relevant system fluctuation spectrum. Unfortunately this simple result does not in general hold for other models.

B. Reflections in one dimension off a randomly moving point scatterer

In acoustics, temporal fluctuations may be thought of as the motions of various inhomogeneities in the sound speed. We treat here a drastically simplified facsimile of such a situation. In particular, consider the case where waves are allowed to interact with a randomly moving point scatterer. Let the problem be restricted to one dimension. We consider the case where the source and receiver are located at the origin, as shown in Fig. 4. Let $x(t)$ be the position of the scatterer. We assume that

$$x(t) = b/2 + \epsilon(t)/2, \qquad (5.14)$$

where $b/2$ is the average position of the scatterer and $\epsilon(t)$ is a zero-mean random process, the factor of $\frac{1}{2}$ being chosen for later convenience. Suppose that a source generating an input $f(t)$ is placed at $x = 0$ and that we observe $\eta(t)$ at $x = 0$. The direct path from source to receiver is subtracted out, so that $\eta(t)$ consists only of the echo from the scatterer. Furthermore, we take the scatterer to be perfect which is to say that, if the scatterer were immobile, $\eta(t)$ would be a time-delayed replica of $f(t)$. Accordingly, if ϵ were constant, the Green's function linking source and echo would simply be given by

$$G(t, t') = ac\delta(ct - ct' - b - \epsilon), \qquad (5.15)$$

where a is a constant specifying the scattering strength and c is the speed of propagation. Consider now the case where ϵ is changing. In order for $G(t, t')$ to be nonvanishing, the scatterer must be at exactly the right place at the instant of reflection so that a pulse emitted at t' will return at t. Clearly this reflection occurs at time $(t + t')/2$. Thus Eq. (5.15) must be generalized to

$$G(t, t') = ac\delta\left(ct - ct' - b - \epsilon\left(\frac{t + t'}{2}\right)\right). \qquad (5.16)$$

It follows from Eq. (5.16) that the stationarity of $\epsilon(t)$ in the weak sense of Appendix A is not sufficient to guarantee temporal homogeneity. Instead we impose the slightly more stringent requirements

$$g(x; t + a) = g(x; t) \qquad (5.17)$$

and

$$f(x, y; t + a, t' + a) = f(x, y; t, t'), \qquad (5.18)$$

where a is an arbitrary time translation and where $g(x, t)$ is the probability density of $\epsilon(t)$ while $f(x, y; t, t')$ is the joint probability density of $\epsilon(t)$ and $\epsilon(t')$. By virtue of Eqs. (5.17) and (5.18) these two quantities may be denoted by $g(x)$ and $f(x, y; t - t')$. As probability densities they are such that

$$\langle\alpha(\epsilon(t))\rangle = \int dx\, \alpha(x)g(x), \qquad (5.19)$$

$$\langle\beta(\epsilon(t), \epsilon(t'))\rangle = \int dx\, dy\, \beta(x, y)f(x, y; t - t'), \qquad (5.20)$$

where α and β are arbitrary functions.

With the help of Eqs. (5.19) and (5.20) it is not difficult to show that

$$\langle G(t_1, t_2)\rangle = acg(ct_1 - ct_2 - b) \qquad (5.21)$$

and

$$\langle G(t_1, t_2)G(t_3, t_4)\rangle = a^2c^2 f(ct_1 - ct_2 - b, ct_3 - ct_4 - b; \tfrac{1}{2}(t_1 + t_2 - t_3 - t_4)). \qquad (5.22)$$

Hence Eqs. (5.17) and (5.18) are sufficient to assure temporal homogeneity.

We use Fourier transform Eq. (5.21) to obtain

$$\mathcal{G}(\omega) = a\exp(-i\omega b/c)\int dx\, \exp(-i\omega x/c)g(x). \qquad (5.23)$$

From this result we see that the effect of fluctuations on $\mathcal{G}(\omega)$ is simply to introduce an attenuation factor given by the integral. The effect is negligible unless the frequency ω is high enough so that the wavelength is of the same order or less than the variations in the position of the scatterer. It is precisely at these frequencies of course that the motion of the scatterer seriously disrupts the interference pattern, and the attenuation factor simply represents the tendency of the "average pattern" to be washed out to zero. It is also interesting to note that $\mathcal{G}(\omega)$ is completely unaffected by the detailed structure of the dynamics of $\epsilon(t)$; thus it is unaffected by passage to the static limit.

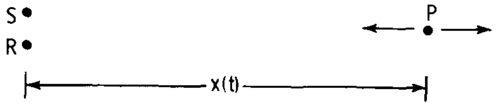

FIG. 4. Relative position of source, receiver, and point scatterer for situation considered in example of Sec. V B. The position of the scatterer varies with time.

Taking the appropriate triple Fourier transform of Eq. (5.22) yields

$$\mathcal{G}(\omega_1,\omega_2,\omega_3) = a^2 \exp[-i(\omega_1+\omega_3)b/c]\int dx\,dy\,dt \times \exp[-i(\omega_1+\omega_2)x/2c + i(\omega_2-\omega_1-2\omega_3)y/2c] \times \exp[i(\omega_2-\omega_1)t] f(x,y;t). \quad (5.24)$$

Thus in order to find $\mathcal{G}(\omega_1,\omega_2,\omega_3)$ one must specify $f(x,y;t)$, which requires a rather detailed knowledge of the dynamics. Even knowing the power spectrum of $\epsilon(t)$ will not suffice [unless $\epsilon(t)$ happens to be gaussian]. To find M note that, as $\tau\to\infty$, $\epsilon(t+\tau)$ and $\epsilon(t)$ will become independent of each other. Therefore, we have

$$f(x,y;\infty) = g(x)g(y), \quad (5.25)$$

where $f(x,y;\infty)\equiv \lim_{\tau\to\infty} f(x,y;t)$. Thus we may replace the integrand in Eq. (5.24) with $[f(x,y;t)-f(x,y;\infty)] + g(x)g(y)$. Upon doing this, the decomposition (3.5) immediately emerges, with

$$M(\omega_1,\omega_2,\omega_3) = a^2 \exp[-i(\omega_1+\omega_3)b/c]\int dx\,dy\,dt \times \exp[-i(\omega_1+\omega_2)x/2c + i(\omega_2-\omega_1+2\omega_3)y/2c] \times \exp[i(\omega_2-\omega_1)t][f(x,y;t)-f(x,y;\infty)]. \quad (5.26)$$

Thus the quantity M again emerges in a quite natural way, with the expression for it involving a better behaved integrand than the one in Eq. (5.24). Note that both $\mathcal{G}$ and M have much more structure in this example than they did for multiplicative noise.

In order to obtain a more explicit expression for M, the process $\epsilon(t)$ must be spelled out in more detail. As a very simple example, let us consider what may be termed a Poisson transition model. We shall assume a situation where, at instants determined by a Poisson distribution, the particle suddenly jumps from where it is to any point x with probability $g(x)\,dx$ of landing in dx. If we let $\mu\,dt$ be the probability of a transition during dt one immediately obtains

$$f(x,y;t) = \delta(x-y)g(x)\exp(-\mu|t|) + [1-\exp(-\mu|t|)]g(x)g(y) \quad (5.27)$$

from which M may be calculated explicitly. The result is

$$M(\omega_1,\omega_2,\omega_3) = \frac{2\mu}{\mu^2+(\omega_1-\omega_2)^2}\left[a\mathcal{G}(\omega_1+\omega_3) - \mathcal{G}\left(\frac{\omega_1+\omega_2}{2}\right)\mathcal{G}\left(\frac{\omega_1-\omega_2+2\omega_3}{2}\right)\right]. \quad (5.28)$$

Letting $\omega_3=-\omega_1$, M becomes

$$M(\omega_1,\omega_2,-\omega_1) = \frac{2\mu}{\mu^2+(\omega_1-\omega_2)^2}\times\left[a^2 - \left|\mathcal{G}\left(\frac{\omega_1+\omega_2}{2}\right)\right|^2\right]. \quad (5.29)$$

On the basis of Eq. (4.4) we see that for a monochromatic input of frequency ω_0, the spectrum of the output will be a Lorentzian centered about ω_0, but modified by a form factor. The spectrum of $\epsilon(t)$ itself is easily calculated to be

$$P_\epsilon(\omega) = \frac{2\langle\epsilon^2\rangle\mu}{\mu^2+\omega^2} \quad (5.30)$$

so that in this case the shape of the line broadening is similar to the shape of the underlying fluctuation spectrum as in the multiplicative noise example only if the form factor does not have much effect. Note that in the static limit $\mu\to 0$ the Lorentzian in Eq. (5.29) becomes proportional to $\delta(\omega_1-\omega_2)$ and one may readily verify that that M has the required limit as discussed in Sec. III. In the deterministic limit we have $g(x)\to\delta(x)$ and M vanishes, as is also required.

C. Wave propagation through a random medium

Wave propagation through a random medium is, of course, orders of magnitude more difficult than the two examples just considered, and no attempt is made here to perform detailed calculations. Instead a general discussion is given concerned primarily with temporal homogeneity.

The wave equation that suggests itself in the presence of a time varying index of refraction is of the form[17]

$$\nabla^2\eta(\mathbf{x},t) - \frac{1}{c_0^2}n^2(\mathbf{x},t)\frac{\partial^2}{\partial t^2}\eta(\mathbf{x},t) = f(\mathbf{x},t), \quad (5.31)$$

where $n(\mathbf{x},t)$ is the index of refraction and c_0 is some representative speed of propagation. We write

$$n(\mathbf{x},t) = m(\mathbf{x}) + l(\mathbf{x},t), \quad (5.32)$$

where $m(\mathbf{x})$ is a deterministic function defining the unperturbed system while $l(\mathbf{x},t)$ is a random function describing spatial and temporal fluctuations. Thus Eq. (5.31) becomes a stochastic partial differential equation. Keeping Eq. (1.1) in mind we see that G satisfies the relation

$$\left(\nabla_x^2 - \frac{n^2(\mathbf{x},t)}{c_0^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{x},\mathbf{x}',t,t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \quad (5.33)$$

whereas G_0, the deterministic unperturbed Green's function, satisfies

$$\left[\nabla_x^2 - \frac{m^2(\mathbf{x})}{c_0^2}\frac{\partial^2}{\partial t^2}\right]G_0(\mathbf{x},\mathbf{x}',t-t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \quad (5.34)$$

Following standard procedure,[18] one may show that G and G_0 satisfy an integral equation of the form

$$G(\mathbf{x},\mathbf{x}',t,t') = G_0(\mathbf{x},\mathbf{x}',t-t') + \int d^3x_1\,dt_1\,G(\mathbf{x},\mathbf{x}_1,t,t_1) \times V(\mathbf{x}_1,t_1)\ddot{G}_0(\mathbf{x}_1,\mathbf{x}',t_1-t') \quad (5.35)$$

where

$$\ddot{G}_0(\mathbf{x}_1,\mathbf{x}',\tau) = \frac{\partial^2}{\partial\tau^2}G_0(\mathbf{x}_1,\mathbf{x}',\tau) \quad (5.36)$$

and where V is a stochastic "interaction potential" given by

$$V(\mathbf{x},t) = \frac{1}{c_0^2}[l^2(\mathbf{x},t) + 2m(\mathbf{x})l(\mathbf{x},t)]. \quad (5.37)$$

Equation (5.35) can be iterated. Symbolically one has

$$G = G_0 + \int G_0V\ddot{G}_0 + \int\int G_0V\ddot{G}_0V\ddot{G}_0 + \cdots. \quad (5.38)$$

Suppose now that $l(\mathbf{x},t)$ is strong sense stationary[19] in time. It immediately follows from taking the ensemble average of Eq. (5.38), that Eq. (2.1) is satisfied term by term. Similarly one may show that all higher moments of G satisfy the homogeneity criteria. Thus we see that temporal stationarity in the index of refraction guarantees temporal homogeneity in the Green's function. As pointed out in Sec. III, the great majority of calculations are performed in the static limit, which in present notation simply amounts to removing any time dependence from l. Temporal homogeneity then follows trivially.

In the evaluation of the various moments of G it is often useful to employ diagrammatic technique based on Eq. (5.38). The topology of these diagrams is then highly dependent upon what assumptions one makes about the higher moments of $l(\mathbf{x}, t)$. This is analogous to the fact that the precise nature of the Feynman diagrams in a quantum field theory depend upon the details of the interaction terms. Feynman diagrams along with other field theoretic analogies have been employed recently in a number of papers.[20-23]

VI. DISCUSSION

In this paper we have proposed the idea of temporal homogeneity. This property is sufficiently general that it is probably safe to say that it has been tacitly assumed in virtually all investigations of acoustic propagation through a random medium. This is obvious for work done in the static limit. Also, in those instances where explicit time fluctuations have been considered the object is usually to calculate spectral broadening,[24] but it is difficult to see how a meaningful observed spectrum could even exist without temporal homogeneity.

Proceeding by analogy with quantum field theory, the main result has been the emergence of the quantities $\mathcal{G}$ and M. They have been studied in detail with emphasis on general limits and constraints. Their usefulness in describing the effects induced by random fluctuations has been demonstrated. Since these quantities are intrinsic properties of the system itself and are closely related to phenomena of interest, they may be used as a basis for comparing different calculations and systems. This has been demonstrated in the examples.

Accordingly, our principal conclusion is that stochastic Green's functions in conjunction with temporal homogeneity provide a general, unified framework within which to treat a large class of random linear systems, and within which to compare various approaches and results.

We have seen that coherence degradation effects are present whether or not the static limit is taken. An interesting question for further investigation is the extent to which these effects are altered by the static approximation, and more generally the extent to which this approximation is valid in specific situations.

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APPENDIX A: STATIONARY RANDOM PROCESSES; WIENER-KHINCHIN THEOREM

We collect here a few of the properties of stationary random processes utilized in the present work. Unless specifically stated otherwise, we shall say that a random function $f(t)$ is stationary if for all time displacements a

$$\langle f(t+a)\rangle = \langle f(t)\rangle \tag{A1}$$

and

$$\langle f(t_1+a)f^*(t_2+a)\rangle = \langle f(t_1)f^*(t_2)\rangle. \tag{A2}$$

Conditions (A1) and (A2) are also said to constitute wide-sense or weak stationarity. Twice in Sec. V stronger forms of stationarity to be obeyed by system parameters are imposed which are spelled out in detail in the text. If f and g are two stationary random processes we say that they are stationary with respect to each other, or mutually stationary, if

$$\langle f(t_1+a)g^*(t_2+a)\rangle = \langle f(t_1)g^*(t_2)\rangle. \tag{A3}$$

Much of the present work rests upon the Wiener—Khinchin theorem which states that, for a stationary random process $f(t)$, the power spectrum $P(\omega)$ is such that

$$P(\omega) \geqslant 0 \tag{A4}$$

for all real ω, where

$$P(\omega) \equiv \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau)\langle f(\tau)f^*(0)\rangle. \tag{A5}$$

Let $f(t)$ and $g(t)$ be two mutually stationary random processes. Then

$$h(t) = \alpha f(t) + \beta g(t) \tag{A6}$$

is also a stationary random process where α and β are arbitrary. It is a standard exercise to show that the imposition of the Wiener—Khinchin theorem on $h(t)$ for all α and β produces inequality (4.6).

APPENDIX B; TEMPORAL HOMOGENEITY AND STATIONARITY PRESERVATION

We show here that temporal homogeneity is sufficient to guarantee preservation of stationarity. Without loss of generality we may consider wave propagation, in which case the input function may be denoted by $f(\mathbf{x}, t)$. Input functions at all pairs of points are taken to be mutually stationary in time, so that we may define the quantities

$$F(\mathbf{x}) \equiv \langle f(\mathbf{x}, t)\rangle \tag{B1}$$

and

$$S(\mathbf{x}, \mathbf{y}; t-t') \equiv \langle f(\mathbf{x}, t)f(\mathbf{y}, t')\rangle. \tag{B2}$$

The output field η, whose stationarity is to be established, is given by

$$\eta(\mathbf{x}, t) = \int d^3x' \int_{-\infty}^{\infty} dt'\, G(\mathbf{x}, \mathbf{x}', t, t')f(\mathbf{x}', t'). \tag{B3}$$

Note that the time integration extends from $-\infty$ to ∞. Because of causality [Eq. (1.2)], however, the integrand actually vanishes over the range $t < t' < \infty$. On the basis of Eqs. (B2) and (B3) one obtains

$$\langle\eta(\mathbf{x},t+\tau)\eta(\mathbf{y},t)\rangle$$
$$=\int d^3x'\,d^3y'\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt''\langle G(\mathbf{x},\mathbf{x}',t+\tau,t'')\times G(\mathbf{y},\mathbf{y}',t,t')\rangle S(\mathbf{x}',\mathbf{y}';t''-t'). \tag{B4}$$

Now invoke Eq. (2.2) with $a=-t$, yielding

$$\langle\eta(\mathbf{x},t+\tau)\eta(\mathbf{y},t)\rangle$$
$$=\int d^3x'\,d^3y'\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt''\langle G(\mathbf{x},\mathbf{x}',\tau,t''-t)\times G(\mathbf{y},\mathbf{y}',0,t'-t)\rangle S(\mathbf{x}',\mathbf{y}';t''-t'). \tag{B5}$$

Change variables of integration to v and w where

$$v=t''-t,\quad w=t'-t. \tag{B6}$$

Then

$$\langle\eta(\mathbf{x},t+\tau)\eta(\mathbf{y},t)\rangle$$
$$=\int d^3x'\,d^3y'\int_{-\infty}^{\infty}dw\int_{-\infty}^{\infty}dv\,\langle G(\mathbf{x},\mathbf{x}',\tau,v)G(\mathbf{y},\mathbf{y}';0,w)\rangle\times S(\mathbf{x}',\mathbf{y}';v-w). \tag{B7}$$

Therefore, $\langle\eta(\mathbf{x},t+\tau)\eta(\mathbf{y},t)\rangle$ is independent of t. Going through similar steps yields

$$\langle\eta(\mathbf{x},t)\rangle=\int d^3x'\int_{-\infty}^{\infty}dt'\,\langle G(\mathbf{x},\mathbf{x}',0,t')\rangle F(\mathbf{x}'). \tag{B8}$$

Therefore, stationarity is preserved.

[1]See, for example, L. A. Chernov, *Wave Propagation in a Random Medium* (McGraw-Hill, New York, 1960); V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961); J. W. Stobehn, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1971), Vol. 4, p. 75; N. G. VanKampen, Phys. Rep. C **24**, 171 (1976); and references therein.

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[6]For an account of stationary random processes see A. M. Yaglom, *Theory of Stationary Random Processes* (Prentice-Hall, Englewood Cliffs, New Jersey, 1962).

[7]See Appendix A for a brief summary of the properties of stationary random processes required for the present work.

[8]This definition can obviously be extended to higher moments of G. The present work, however, is concerned only with the first two moments.

[9]For systems obeying a stochastic differential equation the question of whether or not stationarity is preserved depends, of course, upon how the initial conditions are handled. When they are treated in such a way that Eqs. (1.1) and (1.2) are valid, stationarity is preserved, as is shown in Appendix B.

[10]See, for example, J. D. Bjorken, and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[11]In this work we restrict the analogy with quantum field theory to time and energy (frequency) variables. Momentum variables can be introduced only with corresponding assumptions of spatial translational invariance, which we wish to avoid. When momenta are introduced, they can be handled in a manner completely analogous to the energy variable.

[12]See Sec. V C.

[13]Defined in the last paragraph of Sec. II.

[14]Temporal homogeneity is required to remain valid during passage to either limit.

[15]For a discussion of this point see M. J. Beran and G. B. Parrent, *Theory of Partial Coherence* (Prentice-Hall, Englewood Cliffs, New Jersey, 1964), Chapter 6.

[16]See Appendix A.

[17]It might be argued that actually the positions of $\partial^2/\partial t^2$ and $\eta^2(\mathbf{x},t)$ should be interchanged in Eq. (5.31). The main results of this section are unaffected by such a complication, however.

[18]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chapter 6.

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Lower bounds on the total cross section and the slope parameter for some measurable sequences of $s \to \infty$

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The lower bounds on the total cross section and the slope parameter are obtained on the basis of the analyticity and polynomial upper boundedness of the scattering amplitude and the unitarity of the S matrix: $\sigma_{tot} \geq Cs^{-6}(\log s)^{-2}$, $B \geq Cs^{-5}(\log s)^{-4}$ for some measurable sequences of $s \to \infty$. These bounds hold for any t in $0 \leq t < 4m_\pi^2$. It is unnecessary in order to obtain our bounds that the scattering amplitude has the crossing even property. If we assume this property, we can suppress the logarithmic factors of our bounds. Also we obtain our lower bounds for any sequence of $s \to \infty$, if we take the average scattering amplitude.

I. INTRODUCTION

There are many studies of the lower and upper bounds on the observable quantities. We will take the following principles as the basis: the analyticity and polynomial upper boundedness of the scattering amplitude and the unitarity of the S matrix. The first result is the Froissart bound[1]

$$\sigma_{tot}(s) \leq C(\log s)^2 \quad \text{as } s \to \infty. \tag{1.1}$$

Here C is a certain positive constant. Later C is constrained[2] as

$$\sigma_{tot}(s) \leq \frac{\pi}{m_\pi^2 - \epsilon}\left[\log\left(\frac{s}{f(\epsilon)}\right)\right]^2 \quad \text{as } s \to \infty, \tag{1.2}$$

where ϵ is any small positive constant and $f(\epsilon) \to 0$ as $\epsilon \to 0$. So this upper bound diverges, if we take the limit $\epsilon \to 0$. The removal of this superfluous ϵ is attained[3] and all the terms tending to infinity for $s \to \infty$ are obtained[3] in the asymptotic expansion of the Froissart bound.

First we consider the lower bound on the total cross section. Jin and Martin[4] obtained the bound based on the Herglotz function technique,

$$\sigma_{tot}(s) \geq Cs^{-6}(\log s)^{-3} \quad \text{as } s \to \infty. \tag{1.3}$$

Then on the basis of the unsubtracted dispersion relation and the Phragmèn–Lindelöf theorem in the complex function theory, Simon[5] showed the simplified derivation of this result in a slightly weakened form to hold either in an s channel or in a u channel. He also improved the logarithmic factor in (1.3) to $(\log s)^{-2}$. And Cornille[6] pointed out the fact, based on the integral expression of the elastic cross section

$$\sigma_{el} = \int_{-4k^2}^{0} dt\,\frac{d\sigma}{dt} = \int_{-4k^2}^{0} dt\,|F(s,t)|^2/(64\pi k^2 s), \tag{1.4}$$

that the logarithmic factor in (1.3) could be omitted in the case of crossing even scattering amplitude. Simon and Cornille made use of the Phragmèn–Lindelöf theorem and gave the lower bound like (1.3) for at least one sequence of $s \to \infty$. In this paper, we will prove the lower bound on the total cross section for some measurable sequences of $s \to \infty$ without use of the Phragmèn–Lindelöf theorem and the crossing even property of the scattering amplitude.

Secondly, let us research into the bound on the slope parameter of the absorptive part of the scattering amplitude,

$$B \equiv B(s,t) \equiv \frac{d}{dt}\mathrm{Im}F(s,t)/\mathrm{Im}F(s,t). \tag{1.5}$$

The upper bound on the slope parameter is easily obtained using the cutting of the partial waves up to $C\sqrt{s}\log s$,

$$B \leq C(\log s)^2 \quad \text{as } s \to \infty \quad \text{in } 0 \leq t < 4m_\pi^2. \tag{1.6}$$

It is noted that this cutting is a usual technique to obtain the Froissart bound, and the analyticity domain in t of the scattering amplitude includes[7] the region $0 \leq t < 4m_\pi^2$. Here C refers only to a certain positive constant and C should not be taken as the same value. Since the Regge theory gives the slope parameter $C\log s$, the improvement of this upper bound (1.6) is desirable. This is attained[8] in use of the above lower bound (1.3) on the total cross section,

$$B \leq C\log s \quad \text{as } s \to \infty \quad \text{in } 0 < t < 4m_\pi^2. \tag{1.7}$$

Next on the basis of Simon's technique, the lower bound on the slope parameter is obtained[9]

$$B(s, t=0) \geq Cs^{-6}(\log s)^{-6} \quad \text{for at least one sequence of } s \to \infty. \tag{1.8}$$

Although the Froissart bound was used in this case, it was unnecessary[10] and so the bound (1.8) was improved by the factor $s(\log s)^2$. The bound (1.8) was proved only for at least one sequence of $s \to \infty$. In this paper it will be also proved that the lower bound on the slope parameter is obtained for some measurable sequences of $s \to \infty$ without use of the Phragmen–Lindelöf theorem and the crossing even property of the scattering amplitude.

The origin of the replacement of the condition "for at least one sequence of $s \to \infty$" by "for some measurable sequences of $s \to \infty$" is that the following formula is available:

$$\lim_{z\to\infty}\int_a^\infty \frac{g(x)}{x-z}\,dx = 0 \quad \text{for one sequence of } z \to \infty, \tag{1.9}$$

if

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$$\int_a^\infty \frac{g(x)}{x}\, dx < \infty, \qquad (1.10)$$

when a is an arbitrary finite number, $g(x) \geq 0$, and $g(z)$ approaches to zero for one sequence of $z \to \infty$. So we can avoid the use of the Phragmen—Lindelöf theorem, and it is the origin of the unnecessity of the crossing even property of the scattering amplitude that we can make use of the formula[9]

$$\lim_{z\to\infty} z\int_a^\infty dx \frac{h(x)}{x+z} = \infty \quad \text{for any sequence of } z \to \infty, \qquad (1.11)$$

if

$$\int_a^\infty dx\, h(x) = \infty, \qquad (1.12)$$

where $h(x)$ is any positive definite function of x. These formulas are proved in the appendices.

It is also proved that we can suppress the logarithmic factors in the bounds (1.3) and (1.8), if we assume the crossing even property of the scattering amplitude. When we take the average scattering amplitude, we obtain lower bounds for any sequence of $s \to \infty$.

This paper is planned as follows: In Sec. II, the general framework is given. We derive lower bounds on the total cross section in Sec. III and on the slope parameter in Sec. IV. We also consider the crossing even case in Sec. V. In Sec. VI, we give discussions on our results and make comparisons between ours and the other bounds.

II. FORMULATION OF GENERAL PRINCIPLES

In this section we present the formulation of the analyticity and polynomial upper boundedness of the scattering amplitude and the unitarity of the S matrix. The dispersion relation is also given, which is essential in order to obtain our bounds. For simplicity we consider the elastic scattering with unit mass and neglect the spin complications. If we include the spin, the result is the same with this paper for the imaginary part of the helicity nonflip scattering amplitude.

The analyticity in t of the scattering amplitude $F(s,t)$ leads to the following expansion:

$$F(s,t) = 8\pi \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(\cos\theta). \qquad (2.1)$$

Here s is the c.m. energy squared, t the c.m. momentum transfer squared, k the c.m. momentum, θ the c.m. scattering angle,

$$\cos\theta = 1 + \frac{t}{2k^2}, \qquad (2.2)$$

and

$$s = 4(k^2+1). \qquad (2.3)$$

The polynomial upper boundedness of the scattering amplitude is

$$|F(s,t)| < C|s|^N \quad \text{as } s \to \infty. \qquad (2.4)$$

From the unitarity of the S matrix, we have the constraints

$$0 \leq |f_l(s)|^2 \leq \mathrm{Im} f_l(s) \leq 1, \qquad (2.5)$$

in the s channel physical region. Then the imaginary part of the scattering amplitude and its derivative in t are positive definite at the forward angle in the s channel physical region,

$$\mathrm{Im}F(s,0) \geq 0, \quad \partial_t \mathrm{Im}F(s,0) \geq 0 \quad \text{for } s \geq 4. \qquad (2.6)$$

Similarly we have the constraints in the u channel physical region,

$$\mathrm{Im}F_3(u,0) \geq 0, \quad \partial_t \mathrm{Im}F_3(u,0) \geq 0 \quad \text{for } u \geq 4. \qquad (2.7)$$

Here

$$\partial_t F(s,0) \equiv \frac{\partial}{\partial t} F(s,t)\Big|_{t=0}. \qquad (2.8)$$

For simplicity we consider the case $t=0$. The same analysis holds for $F(s,t)$ and $\partial_t F(s,t)$ in any t, $0 \leq t < 4$.

We will essentially make use of the dispersion relation in order to obtain the lower bounds on the total cross section and the slope parameter. It is known[11] that the dispersion relation holds in general with twice subtractions on the above assumptions and the analyticity in s of the scattering amplitude,

$$F(s,t) = A(t) + (s-s_0)B(t) + \frac{1}{\pi}\int_4^\infty ds' \frac{(s-s_0)^2}{(s'-s_0)^2(s'-s)} \times \mathrm{Im}F(s',t) + \frac{1}{\pi}\int_4^\infty du' \frac{(u-u_0)}{(u'-u_0)^2(u'-u)} \times \mathrm{Im}F_3(u',t). \qquad (2.9)$$

Here

$$\mathrm{Im}F_3(u,t) = -\mathrm{Im}F(4-s-t,t). \qquad (2.10)$$

For simplicity we set $s_0 = u_0 = 0$. When we take s and t as the independent variables and make use of the formula

$$u = 4 - s - t, \qquad (2.11)$$

we get

$$\partial_t F(s,t) = A'(t) + sB'(t) + \frac{1}{\pi}\int_4^\infty ds' \frac{s^2}{s'^2(s'-s)} \partial_t \mathrm{Im}F(s',t) + \frac{1}{\pi}\int_4^\infty du' \left[\frac{u^2}{u'^2(u'-u)} \partial_t \mathrm{Im}F_3(u',t) - \left(\frac{1}{(u'-u)^2} - \frac{1}{u'^2}\right)\mathrm{Im}F_3(u',t)\right]. \qquad (2.12)$$

III. DERIVATION OF THE LOWER BOUND ON THE TOTAL CROSS SECTION

Now let us derive the lower bound on the total cross section. For this purpose, it is sufficient to show the following:

$$\lim_{s\to\infty} s^2 F(s,0) = C \neq 0 \quad \text{for any sequence of } s \to \infty, \qquad (3.1)$$

or

$$\int_4^\infty ds' s' \,\mathrm{Im}F(s',0) = \infty. \qquad (3.2)$$

The reason is as follows. The case (3.1) gives the lower bound

$$|F(s,0)| \geq Cs^{-2} \text{ for any sequence of } s\to\infty. \quad (3.3)$$

Also we have the estimates using the Schwarz inequality,

$$|F(s,0)|^2 \leq \left[\sum_{l=0}^{L}(2l+1)|f_l(s)|\right]^2$$
$$\leq \sum_{l=0}^{L}(2l+1)\sum_{l=0}^{L}(2l+1)|f_l(s)|^2$$
$$\leq L^2 \,\mathrm{Im}F(s,0) \text{ as } s\to\infty. \quad (3.4)$$

Here we omit the constant factor and use the unitarity constraint (2.5) and the fact that the expansion (2.1) can be cut at

$$L = K\sqrt{s}\log s, \quad (3.5)$$

if we consider sufficiently high energies and K is a sufficiently large constant. Hence we attain the lower bound on the total cross section due to the optical theorem for the case (3.1),

$$\sigma_{tot}(s) \geq Cs^{-6}(\log s)^{-2} \text{ for any sequence of } s\to\infty \quad (3.6)$$

and for the case (3.2),

$$\sigma_{tot}(s) \geq Cs^{-3}(\log s)^{-2} \quad (3.7)$$

for some measurable sequences of $s\to\infty$. These are summarized as follows:

$$\sigma_{tot}(s) \geq Cs^{-6}(\log s)^{-2} \quad (3.8)$$

for some measurable sequences of $s\to\infty$. Therefore, we obtain the lower bound (3.8) on the total cross section from both of the estimates (3.1) and (3.2).

In order to get the estimate (3.1) or (3.2), we negate this

$$\lim_{s\to\infty} s^2F(s,0)=0 \quad (3.9)$$

for at least one sequence of $s\to\infty$ and

$$\int_4^\infty ds'\, s'\mathrm{Im}F(s',0)<\infty. \quad (3.10)$$

Hereafter we will see that two conditions (3.9) and (3.10) lead to a contradiction.

First we derive the lower bound for the simple case in which the unsubtracted dispersion relation holds

$$F(s,0) = \frac{1}{\pi}\int_4^\infty ds'\,\frac{\mathrm{Im}F(s',0)}{s'-s} + \frac{1}{\pi}\int_4^\infty du'\,\frac{\mathrm{Im}F_3(u',0)}{u'-u}. \quad (3.11)$$

From (3.9), (3.10), and (A1) we have the estimates

$$\lim_{s\to\infty} sF(s,0)=0 \quad (3.12)$$

and

$$\lim_{s\to\infty}\int_4^\infty ds'\,\frac{s'\,\mathrm{Im}F(s',0)}{s'-s}=0 \quad (3.13)$$

for at least one sequence of $s\to\infty$.

So the dispersion relation gives

$$\lim_{s\to\infty} s\int_4^\infty du'\,\frac{\mathrm{Im}F_3(u',0)}{u'-4+s} = \int_4^\infty du'\,\mathrm{Im}F(s',0)<\infty. \quad (3.14)$$

This leads to

$$\int_4^\infty du'\,\mathrm{Im}F_3(u',0)<\infty, \quad (3.15)$$

because the negation of (3.15) leads to a divergence of the left-hand side of (3.14) by formula (B2) and thus to a contradiction with the finiteness of the right-hand side of (3.14). From (3.14), (3.15), and (B2), we have the equality

$$\int_4^\infty du'\,\mathrm{Im}F_3(u',0)=\int_4^\infty ds'\,\mathrm{Im}F(s',0)<\infty. \quad (3.16)$$

Then

$$s^2F(s,0) = -\frac{1}{\pi}\left(\int_4^\infty ds'\,s'\,\mathrm{Im}F(s',0) + s\int_4^\infty du'\,\frac{u'-4}{u'-4+s}\times\mathrm{Im}F_3(u',0)\right) + \frac{1}{\pi}\int_4^\infty ds'\,\frac{s'^2}{s'-s}\mathrm{Im}F(s',0)$$
$$\equiv I_1+I_2. \quad (3.17)$$

Hence we find a contradiction, because the integral I_2 and the left-hand side of (3.17) approach zero due to (A1) and (3.9), and I_1 is negative definite from the unitarity constraints (2.6) and (2.7).

Next we consider the general case, that is, the twice subtracted dispersion relation (2.9). This relation can be rewritten as

$$F(s,0) = A(0)+sB(0)+\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}-\frac{1}{s'}-\frac{s}{s'^2}\right)\times\mathrm{Im}F(s',0)+\frac{1}{\pi}(s-4)^2\int_4^\infty du'\,\frac{1}{(u'-4+s)u'^2}\times\mathrm{Im}F_3(u',0). \quad (3.18)$$

Since we have the relations (A1), (B1), (B2), and $s^{-1}F(s,0)\to 0$ for at least one sequence of $s\to\infty$ from (3.9), we get

$$B(0)-\frac{1}{\pi}\int_4^\infty ds'\,\frac{1}{s'^2}\mathrm{Im}F(s',0)+\frac{1}{\pi}\int_4^\infty du'\,\frac{1}{u'^2}\mathrm{Im}F_3(u',0)=0. \quad (3.19)$$

Then relation (3.18) is rewritten as

$$F(s,0)=A(0)+\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}-\frac{1}{s'}\right)\mathrm{Im}F(s',0) - \frac{1}{\pi}(s-4)\int_4^\infty du'\,\frac{\mathrm{Im}F_3(u',0)}{(u'-4+s)u'}. \quad (3.20)$$

From (3.9), (3.10), and (A1), we have the estimates

$$\lim_{s\to\infty} F(s,0)=0 \tag{3.21}$$

and

$$\lim_{s\to\infty} \int_4^\infty ds' \frac{\operatorname{Im}F(s',0)}{s'-s}=0 \tag{3.22}$$

for at least one sequence of $s\to\infty$.

So the relation gives

$$\lim_{s\to\infty}(s-4)\int_4^\infty du' \frac{\operatorname{Im}F_3(u',0)}{(u'-4+s)u'}$$

$$=\pi A(0)-\int_4^\infty ds' \frac{1}{s'}\operatorname{Im}F(s',0). \tag{3.23}$$

Since the right-hand side of (3.23) is finite from (3.10), we have the estimate from formula (B2)

$$\lim_{s\to\infty}(s-4)\int_4^\infty du' \frac{\operatorname{Im}F_3(u',0)}{(u'-4+s)u'}$$

$$=\int_4^\infty du' \frac{1}{u'}\operatorname{Im}F_3(u',0)<\infty. \tag{3.24}$$

Then

$$A(0)-\frac{1}{\pi}\int_4^\infty ds' \frac{1}{s'}\operatorname{Im}F(s',0)-\frac{1}{\pi}\int_4^\infty du'$$

$$\times\frac{1}{u'}\operatorname{Im}F_3(u',0)=0. \tag{3.25}$$

Thus we return to the unsubtracted dispersion relation case (3.11). Therefore, we proved the lower bound (3.8) on the total cross section.

IV. DERIVATION OF THE LOWER BOUND ON THE SLOPE PARAMETER

In this section we derive the lower bound on the slope parameter of the imaginary part of the scattering amplitude for some measurable sequences of $s\to\infty$. For the derivation it is sufficient to show the following:

$$\lim_{s\to\infty} s^2\partial_t F(s,0)=C\neq 0 \quad \text{for any sequence of } s\to\infty, \tag{4.1}$$

or

$$\int_4^\infty ds'\, s'\partial_t \operatorname{Im}F(s',0)=\infty. \tag{4.2}$$

The reason is as follows. Since we have the estimate

$$|\partial_t F(s,0)|^2$$

$$\leq\left(\frac{1}{s}\sum_{l=1}^\infty (2l+1)(l^2+l)|f_l(s)|\right)^2$$

$$\leq\frac{1}{s^2}\sum_{l=1}^L(2l+1)(l^2+l)\sum_{l=1}^L(2l+1)(l^2+l)|f_l(s)|^2$$

$$\leq\frac{1}{s}L^4\partial_t \operatorname{Im}F(s,0) \quad \text{as } s\to\infty, \tag{4.3}$$

we attain the bound for the case (4.1),

$$\partial_t \operatorname{Im}F(s,0)\geq Cs^{-5}(\log s)^{-4} \quad \text{for any sequence of } s\to\infty, \tag{4.4}$$

and for the case (4.2),

$$\partial_t \operatorname{Im}F(s,0)\geq Cs^{-2}(\log s)^{-2} \tag{4.5}$$

for some measurable sequences of $s\to\infty$. These are summarized as follows:

$$\partial_t \operatorname{Im}F(s,0)\geq Cs^{-5}(\log s)^{-4} \tag{4.6}$$

for some measurable sequences of $s\to\infty$. Then we obtain the lower bound on the slope parameter (1.4),

$$B=\frac{[1/(s-4)]\sum_{l=1}^\infty(2l+1)(l^2+l)\operatorname{Im}f_l(s)}{\sum_{l=0}^\infty(2l+1)\operatorname{Im}f_l(s)}$$

$$\geq\frac{\sum_{l=1}^\infty(2l+1)(l^2+l)\operatorname{Im}f_l(s)}{(s-4)2\operatorname{Im}f_0(s)}$$

$$\geq\tfrac{1}{2}\partial_t \operatorname{Im}F(s,0) \tag{4.7}$$

for the case

$$\operatorname{Im}f_0(s)\geq\sum_{l=1}^\infty(2l+1)\operatorname{Im}f_l(s). \tag{4.8}$$

Here we also used the unitarity constraint $0\leq\operatorname{Im}f_0(s)\leq 1$ to attain the last part of the estimate (4.7). We get the lower bound

$$B\geq Cs^{-1} \tag{4.9}$$

for the other case,

$$\operatorname{Im}f_0(s)\leq\sum_{l=1}^\infty(2l+1)\operatorname{Im}f_l(s). \tag{4.10}$$

Hence from (4.6), we attain the lower bound on the slope parameter,

$$B\geq Cs^{-5}(\log s)^{-4} \tag{4.11}$$

for some measurable sequences of $s\to\infty$.

Now let us prove the estimate (4.1) or (4.2). For this purpose, we negate this

$$\lim_{s\to\infty} s^2\partial_t F(s,0)=0 \tag{4.12}$$

for at least one sequence of $s\to\infty$ and

$$\int_4^\infty ds'\, s'\partial_t \operatorname{Im}F(s',0)<\infty. \tag{4.13}$$

Hereafter we will see that the estimates (4.12) and (4.13) lead to a contradiction.

First we consider the unsubtracted dispersion relation (3.11). Then the dispersion relation for the derivative in t of the scattering amplitude is

$$\partial_t F(s,t)=\frac{1}{\pi}\int_4^\infty ds' \frac{\partial_t \operatorname{Im}F(s',t)}{s'-s}$$

$$+\frac{1}{\pi}\int_4^\infty du'\left(\frac{\partial_t \operatorname{Im}F_3(u',t)}{u'-u}-\frac{\operatorname{Im}F_3(u',t)}{(u'-u)^2}\right). \tag{4.14}$$

Here we used the relation $u=4-s-t$, and s and t were taken as the independent variables. Then

$$s\partial_t F(s,0)$$

$$= -\frac{1}{\pi}\int_4^\infty ds'\,\partial_t \mathrm{Im}F(s',0)$$

$$+\frac{1}{\pi}\int_4^\infty du'\,\frac{s}{u'-4+s}\,\partial_t\mathrm{Im}F_3(u',0)$$

$$+\frac{1}{\pi}\int_4^\infty ds'\,\frac{s'\partial_t\mathrm{Im}F(s',0)}{s'-s}$$

$$-\frac{1}{\pi}\int_4^\infty du'\,\frac{s\,\mathrm{Im}F_3(u',0)}{(u'-4+s)}$$

$$\equiv J_1+J_2+J_3+J_4. \qquad (4.15)$$

Since we assumed the unsubtracted dispersion relation, the integral,

$$\int_4^\infty du'\,\frac{\mathrm{Im}F_3(u',0)}{u'-4} \qquad (4.16)$$

is finite, so the formulas (B1) and (B3) lead the integral J_4 to zero as $s\to\infty$. On the other hand, assumptions (4.12) and (4.13) give the value zero to the left-hand side of (4.15) and to the integral J_3 for at least one sequence of $s\to\infty$, from formula (A1), and make J_1 finite. Hence we have the estimate

$$\lim_{s\to\infty}\int_4^\infty du'\,\frac{s}{u'-4+s}\,\partial_t\mathrm{Im}F_3(u',0)$$

$$=\int_4^\infty ds'\,\partial_t\mathrm{Im}F(s',0)<\infty. \qquad (4.17)$$

Then formulas (B1) and (B2) lead to

$$\int_4^\infty ds'\,\partial_t\mathrm{Im}F(s',0)=\int_4^\infty du'\,\partial_t\mathrm{Im}F_3(u',0)<\infty. \qquad (4.18)$$

Now we can write the dispersion relation as follows:

$$s^2\partial_t F(s,0)$$

$$=\frac{1}{\pi}\int_4^\infty ds'\,\frac{s'^2}{s'-s}\,\partial_t\mathrm{Im}F(s',0)$$

$$-\frac{1}{\pi}\Bigg[\int_4^\infty ds'\,s'\partial_t\mathrm{Im}F(s',0)$$

$$+s\int_4^\infty du'\Bigg(\frac{u'-4}{u'-4+s}\partial_t\mathrm{Im}F_3(u',0)$$

$$+\frac{s}{(u'-4+s)^2}\mathrm{Im}F_3(u',0)\Bigg)\Bigg]. \qquad (4.19)$$

This equation tells us that the left-hand side and the first integral of the right-hand side approach zero for at least one sequence of $s\to\infty$, from (4.12) and (4.13), and all the other parts negative definite for any $s\geq 4$ from the unitarity constraints (2.6) and (2.7). Hence we have the contradiction.

Next we consider the twice subtracted dispersion relation (2.12) for the derivative in t of the scattering amplitude. From the formula

$$\frac{s^2}{s'^2(s'-s)}=\frac{1}{s'-s}-\frac{1}{s'}-\frac{s}{s'^2}, \qquad (4.20)$$

relation (2.12) is rewritten as

$$\partial_t F(s,0)=A'(0)+sB'(0)$$

$$+\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}-\frac{1}{s'}-\frac{s}{s'^2}\right)\partial_t\mathrm{Im}F(s',0)$$

$$+\frac{1}{\pi}\int_4^\infty du'\Bigg[\frac{(s-4)^2}{u'^2(u'-4+s)}\,\partial_t\mathrm{Im}F_3(u',0)$$

$$-\left(\frac{1}{(u'-4+s)^2}-\frac{1}{u'^2}\right)\mathrm{Im}F_3(u',0)\Bigg]. \qquad (4.21)$$

Since we have the relations (A1), (B1), (B2), and $s^{-1}F(s,0)\to 0$ for at least one sequence of s from (4.12), we get

$$B'(0)-\frac{1}{\pi}\int_4^\infty ds'\,\frac{1}{s'^2}\partial_t\mathrm{Im}F(s',0)$$

$$+\frac{1}{\pi}\int_4^\infty du'\,\frac{1}{u'^2}\partial_t\mathrm{Im}F_3(u',0)=0. \qquad (4.22)$$

Hence we attain

$$\partial_t F(s,0)$$

$$=A'(0)+\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}-\frac{1}{s'}\right)\partial_t\mathrm{Im}F(s',0)$$

$$+\frac{1}{\pi}\int_4^\infty du'\Bigg[\left(\frac{1}{u'-4+s}-\frac{1}{u'}\right)\partial_t\mathrm{Im}F_3(u',0)$$

$$-\left(\frac{1}{(u'-4+s)^2}-\frac{1}{u'^2}\right)\mathrm{Im}F_3(u',0)\Bigg].$$

We have the unitarity constraint

$$\partial_t\mathrm{Im}F_3(u,0)=8\pi\,\frac{\sqrt{u}}{k_u}\,\frac{1}{u-4}\sum_{l=1}^{\infty}(2l+1)(l^2+l)\mathrm{Im}f_l(u)$$

$$\geq\frac{2}{u-4}\left(\mathrm{Im}F_3(u,0)-8\pi\,\frac{\sqrt{u}}{k_u}\,\mathrm{Im}f_0(u)\right)$$

and so

$$\int_4^\infty du'\,\frac{1}{u'}\partial_t\mathrm{Im}F_3(u',0)$$

$$\geq\int_4^\infty du'\left(\frac{1}{u'}\partial_t\mathrm{Im}F_3(u',0)-\frac{1}{u'^2}\mathrm{Im}F_3(u',0)\right)$$

$$\geq\frac{1}{2}\int_4^\infty du'\,\frac{1}{u'}\partial_t\mathrm{Im}F_3(u',0)-16\pi\int_4^\infty du'\,\frac{1}{u'}\mathrm{Im}f_0(u').$$

Then Theorem B and the relation "$\partial_t F(s,0)\to 0$ for at least one sequence of $s\to\infty$" lead to

$$A'(0)-\frac{1}{\pi}\int_4^\infty ds'\,\frac{1}{s'}\partial_t\mathrm{Im}F(s',0)$$

$$-\frac{1}{\pi}\int_4^\infty du'\left[\frac{1}{u'}\partial_t \mathrm{Im}F_3(u',0)-\frac{1}{u'^2}\mathrm{Im}F_3(u',0)\right]=0. \tag{4.23}$$

Thus we return to the unsubtracted case (4.14). Therefore, we proved the lower bound (4.11).

Next we derive the lower bound on the slope parameter for the case $0<t<4$. The slope parameter is

$$B(s,t)=\frac{\sum_{l=1}^\infty (2l+1)\mathrm{Im}f_l(s)P_l'(z)}{2k^2\sum_{l=0}^\infty (2l+1)\mathrm{Im}f_l(s)P_l(z)}, \tag{4.24}$$

where

$$z=1+t/2k^2>1. \tag{4.25}$$

Then we have the bound

$$B(s,t)\geq \tfrac{1}{2}\partial_t \mathrm{Im}F(s,t)\geq Cs^{-5}(\log s)^{-4} \tag{4.26}$$

for some measurable sequences of $s\to\infty$ in the case

$$1\geq \mathrm{Im}f_0(s)\geq \sum_{l=1}^\infty (2l+1)\mathrm{Im}F_l(s)P_l(z). \tag{4.27}$$

On the other hand, more discussions are necessary in the case

$$\mathrm{Im}f_0(s)<\sum_{l=1}^\infty (2l+1)\mathrm{Im}f_l(s)P_l(z). \tag{4.28}$$

We define the value v as

$$P_v(z)=2. \tag{4.29}$$

Here v is not necessary to be the integer. We make use of the following properties of the Legendre function $P_l(z)$:

$$P_l(z) \text{ is an increasing function of } l, \tag{4.30}$$

$$P_l'(z)\geq \tfrac{1}{2}P_l(z) \quad \text{for } P_l(z)\leq 2, \tag{4.31}$$

$$P_l'(z)\geq \frac{1}{2(z-1)}P_l(z) \quad \text{for } P_l(z)\geq 2 \tag{4.32}$$

for $z>1$. These properties are shown in Theorem C of the Appendix. The case (4.28) is moreover divided into two parts:

$$\mathrm{Im}f_0(s)\leq \sum_{l=1}^\infty (2l+1)\mathrm{Im}f_l(s)P_l(z) \leq 2\sum_{1\leq l\leq v}(2l+1)\mathrm{Im}f_l(s)P_l(z), \tag{4.33}$$

$$\mathrm{Im}f_0(s)\leq \sum_{l=1}^\infty (2l+1)\mathrm{Im}f_l(s)P_l(z) \leq 2\sum_{v<l}(2l+1)\mathrm{Im}f_l(s)P_l(z). \tag{4.34}$$

For the case (4.33), we get the bound

$$B(s,t)\geq \frac{\sum_{1\leq l\leq v}(2l+1)\mathrm{Im}f_l(s)P_l'(z)}{8k^2\sum_{1\leq l\leq v}(2l+1)\mathrm{Im}f_l(s)P_l(z)} \geq \frac{1}{16k^2} \tag{4.35}$$

from (4.24), (4.31), and the unitarity constraint $\mathrm{Im}f_l(s)\geq 0$. For the case (4.34), we have the bound

$$B(s,t)\geq \frac{\sum_{l>v}(2l+1)\mathrm{Im}f_l(s)P_l'(z)}{8k^2\sum_{l>v}(2l+1)\mathrm{Im}f_l(s)P_l(z)}\geq \frac{1}{8t} \tag{4.36}$$

from (4.24) and (4.32). Hence we obtain the lower bound (4.11) on the slope parameter for any case of t in $0\leq t<4$.

V. CROSSING EVEN CASE

In this section it is shown that the logarithmic factors can be suppressed in our lower bounds (3.8) and (4.11) on the total cross section and the slope parameter respectively, if we assume the crossing even property of the scattering amplitude. In this case we have

$$\mathrm{Im}F_3(\omega+2-t/2,t)=-\mathrm{Im}F(-\omega+2-t/2,t) =\mathrm{Im}F(\omega+2-t/2,t), \tag{5.1}$$

where

$$\omega=\frac{s-u}{2}=s-2+\frac{t}{2}=-u+2-\frac{t}{2}. \tag{5.2}$$

Then the unsubtracted dispersion relation is

$$F(s,0)=\frac{1}{\pi}\int_2^\infty d\omega'\,\frac{2\omega'\mathrm{Im}F(\omega'+2,0)}{\omega'^2-\omega^2}. \tag{5.3}$$

This gives

$$\omega^2F(s,0)=-\frac{1}{\pi}\int_2^\infty d\omega'\,2\omega'\mathrm{Im}F(\omega'+2,0) +\frac{1}{\pi}\int_2^\infty d\omega'\,\frac{2\omega'^3}{\omega'^2-\omega^2}\mathrm{Im}F(\omega'+2,0) \equiv I_1+I_2. \tag{5.4}$$

First we consider the case that the first integral is finite. Since the second integral approaches zero for any sequence of $\omega\to\infty$ from (A1), we have the bound

$$|s^2F(s,0)|\geq C\neq 0 \quad \text{for any sequence of } s\to\infty. \tag{5.5}$$

If we take the twice subtracted dispersion relation, the similar argument in Sec. III leads to the bound (5.5) in the case of finite I_1.

Now we can see that the above arguments apply to a finite region $t_1<t<0$ such that we have the estimates

$$\int_4^\infty d\omega'\,2\omega'\mathrm{Im}F(\omega'+2-t/2,t)>0, \tag{5.6}$$

from the continuity in t of $\mathrm{Im}F(\omega'+2-t/2,t)$ and the unitarity constraint

$$|\mathrm{Im}F(\omega'+2-t/2,t)|\leq \mathrm{Im}F(\omega'+2,0). \tag{5.7}$$

Hence in the finite region $t_1<t<0$, we obtain the lower bound

$$|F(s,t)|\geq Cs^{-2} \quad \text{for any sequence of } s\to\infty \tag{5.8}$$

in the case of finite I_1.

From the unitarity constraint (2.5), we have the estimates

$$\mathrm{Im}F(s,0)\geq \int_{-1}^1 dz\left|\sum_{l=0}^\infty (2l+1)f_l(s)P_l(z)\right|^2$$

$$= \frac{1}{s}\int_{-s}^{0} dt\,|F(s,t)|^2 \geq \frac{1}{s}\int_{t_1}^{0} dt\,|F(s,t)|^2. \qquad (5.9)$$

Here we used Eq. (2.2), the formula

$$\int_{-1}^{1} dz\, P_l(z)P_m(z) = \frac{2}{2l+1}\delta_{lm}, \qquad (5.10)$$

and for simplicity we are omitting the constant factor. Hence we have the bound

$$\mathrm{Im}F(s,0) \geq Cs^{-5} \text{ for any sequence of } s\to\infty \qquad (5.11)$$

in the case of finite I_1. On the other hand, we of course have the bound (5.11) for some measurable sequences of $s\to\infty$ in the case of infinite I_1. Therefore, we obtain the lower bound on the total cross section in the crossing even case

$$\sigma_{\mathrm{tot}}(s) \geq Cs^{-6} \qquad (5.12)$$

for some measurable sequences of $s\to\infty$.

Next we consider the derivative in t of the scattering amplitude in the crossing even case. The unsubtracted dispersion relation is

$$\partial_t F(s,0) = \frac{1}{\pi}\int_2^\infty d\omega'\left(\frac{2\omega'}{\omega'^2-\omega^2}\partial_t \mathrm{Im}F(\omega'+2,0) - \frac{1}{(\omega'+\omega)^2}\mathrm{Im}F(\omega'+2,0)\right).$$

This gives

$$\omega^2\partial_t F(s,0) = -\frac{1}{\pi}\int_2^\infty d\omega'\left(2\omega'\partial_t\mathrm{Im}F(\omega'+2,0) + \frac{\omega^2}{(\omega'+\omega)^2}\mathrm{Im}F(\omega'+2,0)\right) + \frac{1}{\pi}\int_2^\infty d\omega'\,\frac{2\omega'^3}{\omega'^2-\omega^2}\partial_t\mathrm{Im}F(\omega'+2,0)$$

$$\equiv J_1+J_2+J_3. \qquad (5.13)$$

As shown in the case of (5.5), similar arguments lead to the lower bound at $t=0$ in the case of finite J_1,

$$|\partial_t F(s,t)| \geq Cs^{-2} \text{ for any sequence of } s\to\infty, \qquad (5.14)$$

and there exists a finite region $t_2 < t < 0$ such that the lower bound (5.14) holds for the finite J_1. From the unitarity constraint (2.5) and the formula

$$\int_{-1}^{1} dz\,(1-z^2)P_l'(z)P_m'(z) = \frac{2}{2l+1}(l^2+l)\delta_{lm}, \qquad (5.15)$$

we have the estimate

$$\partial_t \mathrm{Im}F(s,0) = \frac{1}{s}\sum_{l=1}^\infty (2l+1)(l^2+l)\mathrm{Im}f_l(s) \geq \frac{1}{s}\int_{-1}^{1} dz\,(1-z^2)\left|\sum_{l=1}^\infty (2l+1)f_l(s)P_l'(z)\right|^2 \geq \frac{1}{s}\int_{t_1}^{0} dt\,|\partial_t F(s,t)|^2. \qquad (5.16)$$

Hence we have the lower bound in the case of finite J_1,

$$\partial_t \mathrm{Im}F(s,0) \geq Cs^{-5} \text{ for any sequence of } s\to\infty. \qquad (5.17)$$

Also we have a bound (5.17) for some measurable sequences of $s\to\infty$ in the case of infinite J_1. Hence we can suppress the logarithmic factors in the lower bounds (3.8), (4.4), and (4.11), when we assume the crossing even property of the scattering amplitude.

VI. DISCUSSION

In this paper we derived the lower bounds (3.8) and (4.11) on the total cross section and the slope parameter respectively on the basis of the analyticity and polynomial upper boundedness of the scattering amplitude and the unitarity of the S matrix. It is unnecessary to derive our bounds that the scattering amplitude has the crossing even property. The origin is on formula (B2). This was used[9] already in order to derive the lower bound on the slope parameter for at least one sequence of $s\to\infty$. Our bounds hold for some measurable sequences of $s\to\infty$. It is due to formula (A1) that we can replace the previous condition[9] "for at least one sequence of $s\to\infty$" by "for some measurable sequences of $s\to\infty$." Also it is noted that we did not use the Phragmén–Lindelöf theorem.

It is noticeable that our bounds (3.8) and (4.11) on the total cross section and the slope parameter hold for any sequence of $s\to\infty$ in the case of (3.10) and (4.13) respectively. If we define the average scattering amplitude as

$$\bar{F}(s,t) = \frac{1}{s}\int_4^s ds'\,F(s',t), \qquad (6.1)$$

we can obtain the lower bound (3.8) on the average total cross section for any sequence of $s\to\infty$ irrespective of case (3.10). The reason is as follows: We have the estimate in case (3.2)

$$s^2\mathrm{Im}\bar{F}(s,0) \geq \int_4^s ds'\,s'\mathrm{Im}F(s',0) \to \infty \qquad (6.2)$$

for any sequence of $s\to\infty$. Since we observe the integrated quantity in experiment, it may be accepted experimentally that we have the lower bound (3.8) on the total cross section for any sequence of $s\to\infty$. Similar arguments of course apply to the lower bound (4.11) on the slope parameter. It is noted that the more complicated average scattering amplitude was constructed[12] in order to derive the lower bound on the elastic cross section.

It was pointed out by Cornille[6] that the logarithmic factor of the lower bound (1.3) can be suppressed for the crossing even scattering amplitude. He made use of the estimates

$$\sigma_{\mathrm{tot}}(s) \geq \sigma_{\mathrm{el}}(s) = \int_{-s}^{0} dt\left|\frac{F(s,t)}{s}\right|^2 \geq \int_{t_1}^{0} dt\left|\frac{F(s,t)}{s}\right|^2 \qquad (6.3)$$

and the lower bound (5.8) in a finite region $t_1 < t < 0$ for at least one sequence of $s\to\infty$ in the crossing even case. If we try to apply his argument to our case without the

$s-u$ crossing even property, we must treat the case

$$\int_4^\infty du'\,(u'-4)\,|\mathrm{Im}F_3(u',t)|=\infty \quad \text{for } t_1<t<0, \tag{6.4}$$

where $\mathrm{Im}F_3(u',t)$ is not assured to be positive. Then we cannot use formula (B2), so we don't suppress the log factor in our lower bound (3.8). The lower bound (3.8) without the log factor holds for some measurable sequences of $s\to\infty$, if we assume the crossing even property of the scattering amplitude. For the slope parameter there is the same situation with the total cross section case.

It appears that our bounds (3.8) and (4.11) are far away from the experimental situation. Now let us take the dynamical assumption that the imaginary part of the scattering amplitude dominates the magnitude of the amplitude at high energies. Then we have the estimate

$$F(s,0)\sim \mathrm{Im}F(s,0). \tag{6.5}$$

From (6.5), (3.1) and (3.2), we have the lower bound on the total cross section

$$\sigma_{\mathrm{tot}}(s)\geq Cs^{-3}(\log s)^{-2} \tag{6.6}$$

for some measurable sequences of $s\to\infty$.

This bound of course holds for the weaker assumption

$$|\mathrm{Re}F(s,0)/\mathrm{Im}F(s,0)|\leq C \text{ as } s\to\infty. \tag{6.7}$$

For the slope parameter, if we take the dynamical assumption

$$\sigma_{\mathrm{tot}}(s)\geq 32\pi/s \quad \text{as } s\to\infty, \tag{6.8}$$

then this corresponds to the case (4.10) and we have the bound (4.9). If we take the assumption

$$\sigma_{\mathrm{tot}}(s)\geq C, \tag{6.9}$$

then we get the estimate

$$B\geq \frac{\sum_{l\geq M}(2l+1)(l^2+l)\mathrm{Im}f_l(s)}{2(s-4)\sum_{l\geq M}(2l+1)\mathrm{Im}f_l(s)}\geq \frac{C}{64\pi}, \tag{6.10}$$

where

$$D=[(s-4)C/32\pi]^{1/2} \tag{6.11}$$

and so

$$\frac{16\pi}{s-4}\sum_{l=0}^{D}(2l+1)\mathrm{Im}f_l(s)\leq \frac{C}{2}. \tag{6.12}$$

These bounds (6.6), (4.9), and (6.10) are the improved results owing to the dynamical assumptions.

Lastly we discuss the other bounds. The lower bound on the slope parameter was shown by MacDowell and Martin[13] as follows:

$$B\geq \frac{\sigma_{\mathrm{tot}}^2}{36\pi\sigma_{\mathrm{el}}}-O\left(\frac{1}{k^2}\right). \tag{6.13}$$

The equality of this bound is attained,[14] if the scattering amplitude have the exponential behavior with respect to the momentum transfer squared t. Therefore, this lower bound (6.13) is very nearly realized for the experimental values of the total and elastic cross sections and the slope parameter. This bound does not give the lower bound with respect to s, since the value $B=0$ may be allowed.

Also the lower bound on the imaginary part of the scattering amplitude has been known[15] for any physical energy region and a certain physical and unphysical continuum region of t,

$$\mathrm{Im}F(s,t)\geq \frac{k\sqrt{s}}{8\pi}\sigma_{\mathrm{tot}}(s)P_L\left(1+\frac{t}{2k^2}\right), \tag{6.14}$$

where

$$L=[\tfrac{1}{2}(kR)^2+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2} \tag{6.15}$$

and

$$R=[8B(s,0)]^{1/2}. \tag{6.16}$$

This bound (6.14) holds for

$$-\frac{(3.83)^2}{2R^2}\leq t\leq 4 \tag{6.17}$$

at high energies. Since this bound includes the total cross section and the slope parameter as the input information, we can obtain the lower bound on the second derivative in t at $t=0$ of the imaginary part of the scattering amplitude,

$$\frac{\partial^2\mathrm{Im}F(s,0)}{\mathrm{Im}F(s,0)}\geq \frac{B^2}{4}\left(1-\frac{1}{2k^2B}\right). \tag{6.18}$$

APPENDIX A

In this appendix we prove several mathematical formulas.

Theorem A:

$$\lim_{z\to\infty}\int_a^\infty dx\,\frac{g(x)}{x-z-i\epsilon}=0 \quad \text{for one sequence of } z\to\infty, \tag{A1}$$

when a is a certain positive constant and the following conditions hold:

$$\lim_{z\to\infty} g(z)=0 \quad \text{for one sequence of } z\to\infty, \tag{A2}$$

$$\int_a^\infty dx\,\frac{|g(x)|}{x}<\infty. \tag{A3}$$

Proof: In order to derive (A1), we divide the integral into four parts:

$$\int_a^\infty dx\,\frac{g(x)}{x-z-i\epsilon}=\left(\int_a^{\sqrt{z}}+\int_{\sqrt{z}}^{z(1-b)}+\int_{z(1-b)}^{z(1+b)}+\int_{z(1+b)}^{\infty}\right)dx\times\frac{g(x)}{x-z-i\epsilon}$$

$$\equiv I_1+I_2+I_3+I_4. \tag{A4}$$

Here b is a certain positive constant such that $0<b<1$. Then we get the following estimates based on the conditions (A2) and (A3):

$$|I_1|\leq \int_a^{\sqrt{z}} dx\,\frac{x|g(x)|}{(z-x)x}\leq \frac{\sqrt{z}}{z-\sqrt{z}}\int_a^{\sqrt{z}}dx\,\frac{|g(x)|}{x}\to 0 \text{ as } z\to\infty, \tag{A5}$$

$$|I_2| \le \frac{1-b}{b} \int_{\sqrt{z}}^{z(1-b)} dx \frac{|g(x)|}{x} \to 0 \quad \text{as } z\to\infty, \tag{A6}$$

$$|I_4| \le \frac{1+b}{b} \int_{z(1+b)}^{\infty} dx \frac{|g(x)|}{x} \to 0 \quad \text{as } z\to\infty, \tag{A7}$$

$$I_3 = P\int_{-1}^{1} dy \frac{1}{y} g(z+bzy) + i\pi g(z)$$

$$= P\int_0^1 dy \frac{1}{y}[g(z+bzy) - g(Z-bzy)] + i\pi g(z)$$

$$\to bzg'(z) + i\pi g(z) \to 0 \quad \text{for one sequence of } z\to\infty. \tag{A8}$$

Here P denotes the principal value of the integral. It is noted that the justification of the estimate

$$zg'(z)\to 0 \quad \text{as } z\to\infty \tag{A9}$$

is seen in Ref. 16, when $g(z)$ approaches to zero.

Q.E.D.

APPENDIX B

Theorem B: For any positive number a and any function $h(x)$, we obtain the formula

$$\lim_{z\to\infty} z\int_a^{\infty} dx \frac{h(x)}{x+z} = \int_a^{\infty} dx\, h(x) \quad \text{if} \int_a^{\infty} dx\, |h(x)| < \infty, \tag{B1}$$

$$= \infty \quad \text{if} \int_a^{\infty} dx\, h(x) = \infty \text{ and } h(x)>0, \tag{B2}$$

$$\lim_{z\to\infty} \int_a^{\infty} dx \frac{xh(x)}{x+z} = 0 \quad \text{if} \int_a^{\infty} dx\, |h(x)| < \infty. \tag{B3}$$

These hold for any sequence of $z\to\infty$.

Proof: For the cases (B1) and (B3), we can get the following:

$$0 \le \left| \int_a^{\infty} dx\, h(x) - z\int_a^{\infty} dx \frac{h(x)}{x+z} \right|$$

$$= \left| \int_a^{\infty} dx \frac{xh(x)}{x+z} \right| \le \int_a^{\infty} dx \frac{x}{x+z} |h(x)|$$

$$\le \int_a^{\sqrt{z}} dx \frac{\sqrt{z}}{\sqrt{z}+z} |h(x)| + \int_{\sqrt{z}}^{\infty} dx\, |h(x)| \to 0 \tag{B4}$$

for any sequence of $z\to\infty$.

For the case (B2), the procedure is as follows:

$$z\int_a^{\infty} dx \frac{h(x)}{x+z} \ge z\int_a^{\sqrt{z}} dx \frac{h(x)}{x+z} \ge \frac{z}{\sqrt{z}+z} \int_a^{\sqrt{z}} dx\, h(x)$$

$$\to \int_a^{\infty} dx\, h(x) = \infty \tag{B5}$$

for any sequence of $z\to\infty$. Q.E.D.

It is noted that Theorem B was given in Ref. 9 and was recapitulated for self-consistency.

APPENDIX C

Theorem C: The Legendre function $P_l(z)$ has the following properties for $z>1$:

$$P_l(z) \text{ is an increasing function of } l, \tag{C1}$$

$$\frac{\partial}{\partial z} P_l(z) \ge \tfrac{1}{2} P_l(z) \quad \text{for } P_l(z) \le 2 \text{ and } l \ge 1, \tag{C2}$$

$$\frac{\partial}{\partial z} P_l(z) \ge \frac{1}{2(z-1)} P_l(z) \text{ for } P_l(z) \ge 2. \tag{C3}$$

Proof: We know[15] the formula

$$P_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \left(\frac{z-1}{2}\right)^n \prod_{m=1}^{n} [\nu^2+\nu-(m-1)^2 - (m-1)]. \tag{C4}$$

This formula holds for any number ν. Then we can clearly see that

$$\frac{\partial}{\partial \nu} p_\nu(z) > 0 \quad \text{for } z>1, \tag{C5}$$

so we get (C1). From (C4) we have

$$\frac{\partial}{\partial z} P_l(z) = \sum_{n=1}^{l} \frac{1}{(n!)^2} \frac{n}{2} \left(\frac{z-1}{2}\right)^{n-1} \times \prod_{m=1}^{n} [l^2+l-(m-1)^2-(m-1)]. \tag{C6}$$

so we get the following for $P_l(z)\le 2$ and $l\ge 1$:

$$\frac{\partial}{\partial z} P_l(z) \ge \tfrac{1}{2}(l^2+l) \ge \tfrac{1}{4}(l^2+l)P_l(z) \ge \tfrac{1}{2}P_l(z). \tag{C7}$$

Here we used the positivity of each term in (C6) for $z>1$. Since we have the following for $P_l(z)\ge 2$:

$$P_l(z) < 2\sum_{n=1}^{l} \frac{1}{(n!)^2}\left(\frac{z-1}{2}\right)^n \prod_{m=1}^{n} [l^2+l-(m-1)^2-(m-1)] \tag{C8}$$

and

$$\frac{\partial}{\partial z} P_l(z) \ge \frac{1}{(z-1)} \sum_{n=1}^{l} \frac{1}{(n!)^2}\left(\frac{z-1}{2}\right)^n \times \prod_{m=1}^{n} [l^2+l-(m-1)^2-(m-1)], \tag{C9}$$

we obtain the inequality (C3). Q.E.D.

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Isotropy subgroups of SO(3) and Higgs potentials[a)]

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A method is given for determining the isotropy subgroups of an arbitrary, irreducible representation of SO(3). These subgroups are explicitly worked out for low-dimensional representations. As an application of these results we contruct the most general, renormalizable, SO(3) invariant Higgs potentials for these representations, determine the local minima of the potentials and discuss patterns of spontaneous symmetry breaking.

I. INTRODUCTION

A renormalizable quantum field theory involving massive, intermediate vector bosons can be obtained by coupling self-interacting scalar bosons (Higgs bosons) to an appropriate gauge theory. The self-coupling of the Higgs bosons is so chosen that, at any point of specetime, the vacuum expectation value (VEV) of this field is nonzero. This has the effect of reducing the symmetry group of the theory from the original group G of the gauge theory down to the "symmetry group" of the VEV of the Higgs field. Since this VEV is an element of a vector space carrying a representation (usually irreducible) of G, it is clear that the "symmetry group" of the VEV is precisely its isotropy subgroup (see Sec. 2). For convenience, the Higgs bosons are usually supposed to be in the lowest-dimensional (nontrivial), irreducible representation of G. Such a representation has a very simple isotropy subgroup structure, namely all nonzero elements have isomorphic isotropy subgroups. Therefore, in these theories, the direction of symmetry breaking (i.e., the direction of the VEV in the vector space carrying the representation) is unimportant. If, however, the Higgs bosons are in higher-dimensional irreducible representations of G the isotropy subgroup structure becomes very complicated. There are, as a rule, many non-isomorphic isotropy subgroups for such representations. Therefore, the direction of symmetry breaking becomes important (since different directions may mean different isotropy subgroups and, therefore, completely different physics). The Lagrangian for the Higgs bosons always has the form $L=T-V$, where T is the kinetic term and V is a polynomial in the fields called the Higgs potential. The direction of symmetry breaking can be found (to lowest order) by determining the local minima of the Higgs potential. Symmetry breaking occurs in the directions of these local minima.

In this paper we consider theories where $G=\mathrm{SO}(3)$. A method is given for determining the isotropy subgroups for any irreducible representation of SO(3), and the isotropy subgroups for low-dimensional representations are tabulated. Finally, we construct the most general, renormalizable, SO(3) invariant Higgs potential for both the five- and seven-dimensional irreducible representations. We then determine the local minima of these potentials and discuss the pattern of symmetry breaking. To carry through this program, we find it necessary to determine the isotropy subgroups for certain representations of SO(2).

2. ISOTROPY SUBGROUPS

Let G be a Lie group. Let V be a real, finite-dimensional vector space with positive definite metric g_{ab} such that V carries an orthogonal representation of G. Denote, for any element x of G, the corresponding linear operator on V by $\Lambda^i{}_j(x)$. Fix a vector ξ^i in V. The set H_ξ of all elements of G whose corresponding linear operators leave ξ^i invariant (easily shown to be a subgroup of G) is called the isotropy subgroup of the vector ξ^i. We note, for example that $H_0=G$ and that, for any ξ^i in V and for any nonzero real number k, $H_{k\xi}=H_\xi$. We emphasize, however, that the isotropy subgroups for different ξ^i's need not be identical, or even isomorphic.

Let ξ^i be in V and H_ξ be its isotropy subgroup. The set of vectors of the form $\eta^i=\Lambda^i{}_j(x)\xi^j$, for all x in G, is called the orbit of ξ^i, and is denoted O_ξ. If $\eta^i=\Lambda^i{}_j(x)\xi^j$, then it is easy to show that $H_\eta=xH_\xi x^{-1}$. H_η is isomorphic to H_ξ but is not necessarily identical to it. Consider vector ξ^i with norm r. Since the representation is orthogonal, any vector η^i contained in O_ξ has norm r, and therefore O_ξ lies on the sphere in V of radius r. O_ξ will be called an isolated orbit if there is a neighborhood of O_ξ on the sphere in which no vector, not in O_ξ, has isotropy subgroup H_ξ. It is clear from this definition that O_0 is trivially an isolated orbit. For a nonzero vector ξ^i it is not hard to see that O_ξ is an isolated orbit if and only if $O_{k\xi}$ is an isolated orbit, where k is a nonzero real number.

H_ξ is called a principal isotropy subgroup if, given any isotropy subgroup H_λ, H_ξ is conjugate to a subgroup of H_λ. If H_ξ is a principal isotropy subgroup then O_ξ is called a principal orbit. It has been shown[1] that if G is a compact Lie group, then a principal isotropy subgroup must exist and the principal orbits are not isolated orbits (except in the trivial case when G acts transitively on spheres).

Isotropy subgroups have the property that they are, topologically, closed subsets of G. To see this, let H_ξ be an isotropy subgroup, and let $\{g_k\}$ be any sequence of elements of H_ξ converging to g in G. The sequence $\{\Lambda^i{}_j(g_k)\}$ must converge to $\Lambda^i{}_j(g)$, and $\Lambda^i{}_j(g)$ must also

a)Work supported in part by the NSF under Grant MPS 74-17456.

leave ξ^i invariant. Hence, g is in H_ξ, and H_ξ is a closed subset of G. In practice this result greatly reduces the number of possible isotropy subgroups of G.

3. ISOTROPY SUBGROUPS OF SO(2)

A. SO(2) and its closed subgroups

Let V be a real, two-dimensional vector space with positive definite metric g_{ab}. Denote by SO(2) the (compact, Abelian) group of all linear mappings $\Lambda^i{}_j$ from V to V which leave g_{ab} invariant and which, with respect to an orthonormal basis, have determinant 1. The closed subgroups of SO(2) are SO(2) itself and, for each $n \geq 1$, the cyclic group of order n, denoted Z_n. For example, Z_1 is the trivial group of one element.

B. Representations of SO(2)

Denote by V_n ($n \geq 1$) the vector space of all symmetric, nth rank tensors over V. It is easy to show that $\dim V^n = n+1$. To $\Lambda^i{}_j$ in SO(2) assign the linear mapping from V^n to V^n which sends $T^{a\cdots c}$ to $\Lambda^i{}_a \cdots \Lambda^j{}_c T^{a\cdots c}$. By these assignments V^n becomes the carrier space for an orthogonal representation of SO(2). Such representations are reducible for $n \geq 2$. Let V_0^n denote the two-dimensional subspace of V^n consisting of all symmetric, traceless, nth rank tensors over V. V_0^n is a stable subspace under the group action on V^n and therefore V_0^n is a carrier space for an orthogonal representation of SO(2). It is well-known that these representations are irreducible and that, with the exception of the one-dimensional, trivial representation, all irreducible representations of SO(2) are in this class. It is important to note that, for all n, these representations act transitively on spheres in V_0^n, i.e., for any two elements $A^{a\cdots c}$, $B^{a\cdots c}$ with the same norm there exists some $\Lambda^i{}_j$ in SO(2) such that $A^{a\cdots c} = \Lambda^a{}_i \cdots \Lambda^c{}_j B^{i\cdots j}$. Since SO(2) is Abelian, this implies that all elements of V_0^n (with the exception of the zero element) will have identical isotropy subgroups.

The above representations of SO(2) on V^n are reducible. Since SO(2) is compact, these representations are completely reducible into a direct sum of irreducible ones. The decomposition of an element of V^n into a linear combination of irreducible tensors is well known and given by

$$T^{ab\cdots de} = A^{ab\cdots de} + B^{(ab\cdots}g^{de)} + \cdots + \tilde{T}^{ab\cdots de} \tag{1}$$

where $\tilde{T}^{ab\cdots de}$ is a multiple of $g^{(ab}\cdots g^{de)}$ when n is even or $k^{(a}g^{bc}\cdots g^{de)}$ when n is odd. $A^{ab\cdots e}$, $B^{ab\cdots}$, etc., are all traceless, symmetric tensors. This decomposition is unique.

C. Isotropy subgroups

The elements of V_0^n have an interesting geometrical property that will allow us to determine the isotropy subgroups of SO(2) and will be of use to us later on. Let $A^a_{(1)}, \ldots, A^a_{(n)}$ be n unit vectors in V such that the angle between neighbors is $2\pi/n$. The symmetrized product $A^{(a}_{(1)} \cdots A^{c)}_{(n)}$ will be called an n-star and denoted by $\chi^{a\cdots c}_{(n)}$ (see Fig. 1). An n-star obviously has isotropy subgroup Z_n. We note that any element in the orbit of an n-star is itself an n-star. Stars have the following properties.

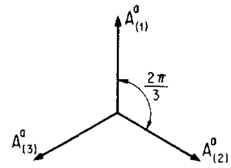

FIG. 1. $A^{(a}_{(1)}A^{b}_{(2)}A^{c)}_{(3)}$ is a 3-star. Its isotropy subgroup [with respect to SO(2)] is Z_3.

Theorem 1: $\chi^{a\cdots c}_{(n)}$ is traceless if n is odd. Furthermore, when n is even, the traceless part of the n-star is precisely that n-star minus some multiple of a symmetrized product of metric tensors.

We will use Theorem 1 to determine the isotropy subgroups of SO(2) and return to the proof of the theorem later. Consider V_0^n. For n odd $\chi^{a\cdots c}_{(n)}$ is traceless and therefore an element of V_0^n. By the transitivity of the SO(2) action on V_0^n we see that any element of V_0^n is (to within sign) a unique real multiple of an n-star. Therefore, every nonzero element has isotropy subgroup Z_n. For n even the traceless part of $\chi^{a\cdots c}_{(n)}$ is simply $\chi^{a\cdots c}_{(n)}$ minus some multiple of a symmetrized product of metric tensors. For example, if $\chi^{abcd}_{(4)}$ is a 4-star, then its traceless part is given by $\chi^{abcd}_{(4)} - \frac{1}{8} g^{(ab}g^{cd)}$. Since all elements of SO(2) leave g_{ab} invariant we see that the traceless part of $\chi^{a\cdots c}_{(n)}$ also has isotropy subgroup Z_n. By transitivity we see that any element of V_0^n is (to within sign) a unique real multiple of the traceless part of an n-star. Therefore, every nonzero element has isotropy subgroup Z_n.

Now consider V^n. Any element of V^n can be decomposed into the linear combination of irreducible tensors given in Eq. (1). Remembering that elements of V_0^m are multiples of the traceless parts of m-stars, we have the unique decomposition

$$T^{ab\cdots de} = a_n \chi^{ab\cdots de}_{(n)} + a_{n-2}\chi^{(ab\cdots de)}_{(n-2)} + \cdots + \tilde{T}'^{ab\cdots de}, \tag{2}$$

where $\tilde{T}'^{ab\cdots de}$ is $a_0 g^{(ab}\cdots g^{de)}$ when n is even and $a_1 \chi^{(a}_{(1)} g^{bc}\cdots g^{de)}$ when n is odd. The a_i's are constants. We can now, by inspection, determine all isotropy subgroups for the reducible representations of SO(2) carried by V^n. For example, consider V^3. Then for any T^{abc} in V^3 we have

$$T^{abc} = a_3 \chi^{abc}_{(3)} + a_1 \chi^{(a}_{(1)} g^{bc)}.$$

The possible isotropy subgroups are as follows. For $a_1 = a_3 = 0$, i.e., for $T^{abc} = 0$ the isotropy subgroup is SO(2). For $a_3 \neq 0$, $a_1 = 0$ the isotropy subgroup is Z_3. For $a_1 \neq 0$ the isotropy subgroup is Z_1. Moreover, since all 3-stars are on the same orbit, the Z_3 orbits (although no Z_1 orbits) are isolated.

We now prove Theorem 1. Let n be odd and assume $\chi^{a\cdots c}_{(m)}$ is traceless for $m = 1, 3, \ldots, n$. Then $\chi^{a\cdots c}_{(m)}$ is an element of V_0^m and by transitivity every element of V_0^m can (to within sign) be written uniquely as a multiple of an m-star. Consider any $(n+2)$-star $\chi^{a\cdots cde}_{(n+2)}$. Its trace is a symmetric, nth rank tensor and therefore has the unique decomposition

$$\chi^{a\cdots cd}_{(n+2)\,d} = a_n \chi^{a\cdots c}_{(n)} + a_{n-2}\chi^{(a\cdots}_{(n-2)} g^{bc)} + \cdots + a_1 \chi^{(a}_{(1)} g^{fh}\cdots g^{bc)},$$

where the a_i's are constants. For $\chi^{a\cdots cd}_{(n+2)\,d}$ nonzero it follows that the isotropy subgroup is Z_q for $q \leq n$. But since every element of SO(2) leaves g_{ab} invariant, the isotropy subgroup of $\chi^{a\cdots cd}_{(n+2)d}$ must contain Z_{n+2}. Therefore, $\chi^{a\cdots cde}_{(n+2)}$ must be traceless. Since $\chi^a_{(1)}$ is traceless, it follows, by induction, that $\chi^{a\cdots c}_{(n)}$ is traceless when n is odd. Now, let n be even and assume $\chi^{a\cdots c}_{(m)} - c_m g^{(ab}\cdots g^{dc)}$ is traceless (for appropriate c_m) for $m=2,4,\ldots,n$. Consider any $(n+2)$-star $\chi^{a\cdots de}_{(n+2)}$. Its trace has the unique decomposition

$$\chi^{a\cdots cd}_{(n+2)d} = a_n \chi^{a\cdots c}_{(n)} + a_{n-2}\chi^{(a\cdots}_{(n-2)} g^{bc)} + \cdots + a_0 g^{(ad}\cdots g^{bc)},$$

where the a_i's are constants. Now, if $\chi^{a\cdots cd}_{(n+2)d} - a_0 g^{(ad}\cdots g^{bc)}$ is nonzero, its isotropy subgroup must be Z_q for $q \geq n$. But the isotropy subgroup must contain Z_{n+2}. Therefore,

$$\chi^{a\cdots cd}_{(n+2)d} = a_0 g^{(ad\cdots bc)}.$$

This implies that the traceless part of $\chi^{a\cdots de}_{(n+2)}$ is given by $\chi^{a\cdots de}_{(n+2)} - c_{n+2} g^{(ad}\cdots g^{de)}$. Since the traceless part of $\chi^{ab}_{(2)}$ is obviously $\chi^{ab}_{(2)} - (-\tfrac{1}{2})g^{ab}$, the second part of the theorem follows by induction.

4. ISOTROPY SUBGROUPS OF SO(3)

A. SO(3) and its closed subgroups

Let W be a real, three-dimensional vector space with positive definite metric g_{ab}. Denote by SO(3) the (compact group of all linear mappings $\Lambda^i{}_j$ from W to W which leave g_{ab} invariant and which, with respect to an orthonormal basis, have determinant 1. The closed subgroups of SO(3) are,[2,3] to within conjugacy, SO(3) itself, SO(2), the normalizer of SO(2) denoted N(SO(2)), for each $n \geq 1$ Z_n, for each $n \geq 2$ the dihedral group of order $2n$ denoted D_n, and the proper symmetry groups of the dodecahedron, the cube, and the tetrahedron denoted Y, O, and T respectively.

An SO(2) subgroup of SO(3) is completely specified by specifying a nonzero vector ξ^i that it leaves invariant. N(SO(2)) is precisely SO(2) itself along with all elements of SO(3) which are rotations by π around any nonzero vector orthogonal to ξ^i (and products of these elements).

B. Representations of SO(3) and the subduced representations of SO(2)

Denote by W^n, for $n \geq 1$, the vector space of all symmetric, traceless, nth rank tensors over W. It is easy to show that $\dim W^n = 2n+1$. To $\Lambda^i{}_j$ in SO(3) assign the linear mapping from W^n to W^n which sends $T^{a\cdots c}$ to $\Lambda^i{}_a\cdots\Lambda^j{}_c T^{a\cdots c}$. With these assignments W^n becomes the carrier space for an othogonal representation of SO(3). It is well known that these representations are irreducible and that, with the exception of the one-dimensional, trivial representation, all irreducible representations of SO(3) are in this class. We will give a method for determining the isotropy subgroups of these representations. To this end we first consider the subduced representation of an SO(2) subgroup.

Consider any SO(2) subgroup of SO(3). W^n carries a representation of SO(3) and therefore a representation of SO(2), which is said to be subduced from the SO(3) representation. For $n \geq 1$ the subduced representation is reducible, and, since SO(2) is compact, it is completely reducible. Therefore, any element of W^n can be written uniquely as a linear combination of tensors which are irreducible under SO(2). We want to determine this decomposition. As a first step we prove the following theorem.

Theorem 2: Let $T^{a\cdots c}$ be any symmetric, nth rank tensor over W and ξ^a be an arbitrary unit vector. Then

$$T^{ab\cdots c} = {}_{(n)}M^{ab\cdots c} + {}_{(n-1)}M^{(ab\cdots}\xi^{c)} + \cdots + {}_{(1)}M^{(a}\xi^b\cdots\xi^{c)} + {}_{(0)}M\xi^a\cdots\xi^c, \tag{3}$$

where $(p)M^{(ab\cdots x)} = (p)M^{ab\cdots x}$ and $(p)M^{ab\cdots x}\xi_x = 0$ for all $p = 1,\ldots,n$ and this decomposition is unique.

Proof: Denote by $g_{\perp ab}$ the induced metric on the two-dimensional vector space orthogonal to ξ^a. Note that $g_{ab} = g_{\perp ab} + \xi_a\xi_b$ and therefore $\delta^a{}_b = g_\perp{}^a{}_b + \xi^a\xi_b$. It follows that $g_\perp{}^a{}_b\xi_a = 0$. Now write

$$T^{ab\cdots c} = \delta^a{}_\alpha\delta^b{}_\beta\cdots\delta^c{}_\gamma T^{\alpha\beta\cdots\gamma}.$$

Substituting for all δ^a_α in this expression and expanding out, we find

$$T^{ab\cdots c} = g^a_{\perp\alpha} g^b_{\perp\beta}\cdots g^c_{\perp\gamma}T^{\alpha\beta\cdots\gamma} + (m_{(n-1)}g^{(a}_{\perp\alpha}g^b_{\perp\beta}\cdots\xi_\gamma T^{\alpha\beta\cdots\gamma})\xi^{c)} + \cdots + (m_{(1)}g_\perp{}^{(a}{}_\alpha\xi_\beta\cdots\xi_\gamma T^{\alpha\beta\cdots\gamma})\xi^b\cdots\xi^{c)} + (T^{\alpha\beta\cdots\gamma}\xi_\alpha\xi_\beta\cdots\xi_\gamma)\xi^a\cdots\xi^c,$$

where the $m_{(p)}$'s are nonzero rational numbers. We let

$${}_{(p)}M^{ab\cdots} = m_{(p)}g^a_{\perp\alpha}g^b_{\perp\beta}\cdots\xi_\alpha T^{\alpha\beta\cdots\alpha}$$

for all $p = 0,\ldots,n$. Tensor ${}_{(p)}M^{ab\cdots}$ is symmetric since $T^{\alpha\beta\cdots\gamma}$ is symmetric and ${}_{(p)}M^{ab\cdots x}\xi_x = 0$ follows from the fact that $g_\perp{}^a{}_b\xi_a = 0$. This decomposition is obviously unique, which proves the theorem.

We note that ${}_{(p)}M^{ab\cdots x}\xi_x = 0$ implies that ${}_{(p)}M^{ab\cdots x}$ is a tensor over the two-dimensional vector space orthogonal to ξ_a.

Consider the SO(2) subgroup of SO(3) that leaves some unit vector ξ_a invariant. By theorem 2 any element of W^n can be uniquely decomposed into a linear combination involving symmetric tensors over a two-dimensional vector space orthogonal to ξ_a. Using Eq. (2) we write any such two-dimensional tensor as a unique linear combination of n-stars. Substitute these linear combinations of n-stars into the ξ_a decomposition. Using the tracelessness of elements of W^n, solve for ${}_{(0)}M$. By rearranging these ${}_{(0)}M$ terms, we obtain a unique decomposition of elements of W^n into a linear combination of SO(2) irreducible tensors.

As an example consider W^2 and the SO(2) subgroup that leaves ξ_a invariant. Then by Theorem 2 any T^{ab} in W^2 can be written

$$T^{ab} = {}_{(2)}M^{ab} + {}_{(1)}M^{(a}\xi^{b)} + {}_{(0)}M\xi^a\xi^b. \tag{4}$$

From Eq. (2) we have

$$\begin{aligned} {}_{(2)}M^{ab} &= a_2\chi^{ab}_{(a)} + a_0 g^{ab}_\perp, \\ {}_{(1)}M^a &= a_1\chi^a_{(1)}. \end{aligned} \tag{5}$$

Substitute Eqs. (5) and (4). Take the trace, set it equal to zero, and solve for $M_{(0)}$. We find that $M_{(0)} = a_2 - 2a_0$. Rearranging so that each term is traceless, we have finally that

$$T^{ab}=\alpha(\xi^a\xi^b-\tfrac{1}{2}g^{ab})+\beta(\chi^{ab}_{(2)}+\tfrac{1}{2}g^{ab}_{1})+\gamma\chi_{(1)}{}^{(a}\xi^{b)},$$

where α, β, γ are constants.

Given an element $T^{a\cdots d}$ of W^n, we now show that a simplification in its tensor decomposition is obtained by decomposing with respect to a special unit vector.

Lemma 1: Given $T^{a\cdots d}$ in W^n there exists a unit vector η^a with the property that $T^{ab\cdots cd}\eta_b\cdots\eta_c\eta_d=\text{const}\times\eta^a$.

Proof: Consider the smooth mapping $f: S^2\to\mathbb{R}$ defined by $f(\xi)=T^{a\cdots d}\xi_a\cdots\xi_d$ for each unit vector ξ_a. S^2 is compact and, recalling that the continuous image of a compact set is compact, this implies that $f(S^2)$ is a compact subset of $\mathbb{R}$. But then $f(S^2)$ must be closed and bounded, and therefore f has a maximum value. Let η_a be any unit vector for which f is maximum. Let $\eta_a(\theta)$ be a one-parameter family of unit vector such that $\eta_a(0)=\eta_a$. Then

$$\left.\frac{df}{d\theta}\right|_0=\left.\frac{d}{d\theta}(T^{a\cdots d}\eta_a(\theta)\cdots\eta_d(\theta)\right|_0$$

$$=n(T^{ab\cdots d}\eta_b\cdots\eta_d)\left.\frac{d\eta_a}{d\theta}(\theta)\right|_0=0.$$

$d\eta_a/d\theta\,|_0$ is perpendicular to η_a but otherwise is arbitrary. Therefore,

$$T^{ab\cdots d}\eta_b\cdots\eta_d=\text{const}\times\eta^a,$$

which proves the lemma.

If we now decompose $T^{a\cdots c}$ with respect to this η^a and use Lemma 1, we find that the 1-star term must vanish. This simplification will be very useful when we discuss SO(3) invariant Higgs potentials.

Finally, we prove a lemma that we will need in Sec. 5.

Lemma 2: Let $T^{ab\cdots d}$ be a symmetric, nth rank tensor over W. If $T^{ab\cdots d}\xi_a\xi_b\cdots\xi_d=0$ for arbitrary ξ_a in W, then $T^{ab\cdots d}=0$.

Proof: For $n=1$ the result is immediate. Now consider $n\geq 2$. Let A_a and B_a be two, arbitrary, linearly independent vectors in W. Consider vectors of the form $n_a=A_a+\lambda B_a$ for arbitrary λ. Substituting for η_a in the equation $T^{ab\cdots d}\eta_a\eta_b\cdots\eta_d=0$, we find

$$(pT^{ab\cdots cd}A_aB_b\cdots B_d)\lambda^{n-1}+\cdots+(qT^{ab\cdots cd}A_a\cdots A_cB_d)\lambda=0,$$

where $p,\ldots,q$ are nonzero integers. But λ is arbitrary. Therefore, all the coefficients of the polynomial must vanish. In particular $T^{ab\cdots cd}A_a\cdots A_cB_d=0$. Now B_d is an arbitrary vector. Therefore, $T^{ab\cdots cd}A_a\cdots A_c=0$ for arbitrary A_a in W. Repeating the above process $n-1$ times, we have finally that $T^{ab\cdots d}\xi_a=0$ for arbitrary vector ξ_a. Therefore, $T^{ab\cdots d}=0$.

C. Isotropy subgroups

Recall that W^n carries an irreducible representation of SO(3). We can determine the isotropy subgroups using the SO(2) decomposition discussed in the preceeding section. The results for $n=1,2,3,4$ are tabulated in Table I.

Our method for determining isotropy subgroups is best illustrated by examples.

TABLE I. The isotropy subgroups of the $(2n+1)$-dimensional, irreducible representation of SO(3) for $n=1,2,3,4$. When the number m of isolated orbits (on a sphere) of an isotropy subgroup is nonzero, then the total number of orbits (on a sphere) with that isotropy subgroup is also m. For any n, an element of W^n has an SO(3) isotropy subgroup if and only if it is the zero element.

n	isotropy subgroups	number of isolated orbits (on a sphere of nonzero radius)
1	SO(3)	0
	SO(2)	1
2	SO(3)	0
	N(SO(2))	2
	D_2	0
3	SO(3)	0
	SO(2)	1
	T	1
	D_3	1
	Z_3	0
	Z_2	0
	Z_1	0
4	SO(3)	0
	N(SO(2))	2
	0	2
	D_4	0
	D_3	0
	D_2	0
	Z_2	0
	Z_1	0

1. n = 2

With respect to an arbitrary SO(2) subgroup (which leaves ξ^a invariant) we have the unique decomposition

$$T^{ab}=\alpha(\xi^a\xi^b-\tfrac{1}{3}g^{ab})+\beta(\chi^{ab}_{(2)}+\tfrac{1}{2}g^{ab}_{1})+\gamma\chi_{(1)}{}^{(a}\xi^{b)},\qquad(6)$$

where α,β,γ are arbitrary constants. In Appendix A we show that

$$\chi^{ab}_{(2)}+\tfrac{1}{2}g^{ab}_{1}=x^{(a}y^{b)},$$

where x^a, y^a, ξ^a are orthonormal vectors. The first term on the right-hand side of Eq. (6) has isotropy subgroup N(SO(2)). Each of the remaining two terms has isotropy subgroup D_2.

An element of W^n (for any $n\geq 1$) has isotropy subgroup SO(3) if and only if it is the zero element [since the SO(3) representation is irreducible]. O_0 is trivially an isolated orbit. It is clear that an element of W^2 has N(SO(2)) as an isotropy subgroup if and only if $T^{ab}=\alpha(\xi^a\xi^b-\frac{1}{3}g^{ab})$ for some unit vector ξ^a. Moreover, since all tensors of the form $\xi^a\xi^b-\frac{1}{3}g^{ab}$ are on the same orbit it follows that these are precisely two, isolated, N(SO(2)) orbits for tensors of fixed norm (one for $\alpha>0$ and the other for $\alpha<0$). Let $T^{ab}=\xi^a\xi^b-\frac{1}{3}g^{ab}$ for some ξ^a, and let δ^a be any unit vector such that $\delta^a\xi_a=0$. T^{ab} is invariant under rotations by π around δ^a. In Appendix A we show that

$$T^{ab}=-\tfrac{1}{2}(\delta^a\delta^b-\tfrac{1}{3}g^{ab})+x^{(a}y^{b)},\qquad(7)$$

where δ^a, x^a, y^a are orthonormal vectors. We can now determine the remaining isotropy subgroups.

The second and third terms in the decomposition involve a 2-star and a 1-star respectively. It is clear, therefore, that a tensor which does not have N(SO(2)) as an isotropy subgroup can be at most twofold invariant around an arbitrary unit vector ξ^a. We can immediately rule out Y, O, T and D_n, Z_n for $n>2$ as possible isotropy subgroups. Given any element T^{ab} in W^2 lemma 2 assures the existence of unit vector η^a with the property that when we decompose T^{ab} with respect to η^a we get

$$T^{ab}=\alpha(\eta^a\eta^b-\tfrac{1}{3}g^{ab})+\beta x^{(a}y^{b)}.$$

For $\beta\neq 0$ and $|\alpha/\beta|\neq\frac{1}{2}$, T^{ab} does not have isotropy subgroup $N(\mathrm{SO}(2))$. It is easy to see that such T^{ab} have D_2 as an isotropy subgroup (since the first term does not break the D_2 invariance of the second term). D_2 orbits are obviously never isolated orbits.

2. $n=3$

With respect to an arbitrary SO(2) subgroup (which leaves ξ^a invariant) we have the unique decomposition

$$T^{abc}=\alpha(\xi^a\xi^b\xi^c-\tfrac{3}{5}\xi^{(a}g^{bc)})+\beta\chi^{abc}_{(3)}+\gamma x^{(a}y^b\xi^{c)}+\delta\chi_{(1)}{}^{(a}(g^{bc)}-5\xi^b\xi^{c)}), \tag{8}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants and x^a, y^a, ξ^a are orthonormal vectors. The first and second terms on the right-hand side of Eq. (8) clearly have isotropy subgroups SO(2) and D_3 (see Fig. 2) respectively. The orthonormal vectors x^a, y^a, ξ^a lie along the twofold symmetry axes of a tetrahedron (see Fig. 3). Therefore, $x^{(a}y^b\xi^{c)}$ has isotropy subgroup T. The last term in Eq. (8) is invariant under rotations by π around $\chi^a_{(1)}$ and therefore its isotropy subgroup contains Z_2.

It is clear that an element of W^3 has SO(2) as an isotropy subgroup if and only if $T^{abc}=\alpha(\xi^a\xi^b\xi^c-\frac{3}{5}\xi^{(a}g^{bc)})$ for some unit vector ξ^a. Since all tensors of the form $\xi^a\xi^b\xi^c-\frac{3}{5}\xi^{(a}g^{bc)}$ are on the same orbit, it follows that there is precisely one, isolated, SO(2) orbit for tensors of fixed norm. The remaining terms in the decomposition involve only 3-, 2-, and 1-stars. It follows that a tensor that does not have SO(2) as an isotropy subgroup can be at most threefold invariant around an arbitrary unit vector ξ^a. We can immediately rule out Y, O and D_n, Z_n for $n>3$ as possible isotropy subgroups. The remaining possibilities are T, D_3, D_2, Z_3, Z_2, and Z_1.

We first consider the possibility of T being an isotropy subgroup. Assume that some element T^{abc} has T as an isotropy subgroup. Let α^a, β^a, ξ^a be orthonormal vectors along the three twofold axes of T. Then T^{abc} is invariant under rotation by π around ξ^a and the decomposition of T^{abc} with respect to ξ^a is given by

$$T^{abc}=\alpha(\xi^a\xi^b\xi^c-\tfrac{3}{5}\xi^{(a}g^{bc)})+\gamma x^{(a}y^b\xi^{c)},$$

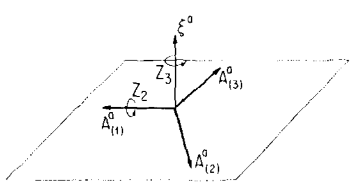

FIG. 2. $A^{(a}_{(1)}A^b_{(2)}A^{c)}_{(3)}$ is a 3-star. It is invariant under rotation by $2\pi/3$ around axis ξ^a and by π around axes $A^a_{(i)}$ for $i=1, 2, 3$. Therefore, its isotropy subgroup [with respect to SO(3)] is D_3.

where $\gamma\neq 0$. Contracting α^a twice into T^{abc}, we find that

$$T^{abc}\alpha_b\alpha_c=-\tfrac{1}{5}\alpha\xi^a.$$

But $T^{abc}\alpha_b\alpha_c$ is invariant under rotation by π around α_a and ξ^a is not. Therefore, $\alpha=0$. $T^{abc}=x^{(a}y^b\xi^{c)}$ is obviously twofold invariant around x^a and y^a. Therefore, $x^a=\alpha^a$ (or β^a) and $y^a=\beta^a$ (or α^a). Thus we have shown that T^{abc} has isotropy subgroup T if and only if $T^{abc}=\gamma\alpha^{(a}\beta^b\xi^{c)}$, where α^a, β^a, ξ^a are any orthonormal vectors. Since all tensors of the form $\alpha^{(a}\beta^b\xi^{c)}$ are on the same orbit, it follows that there is precisely one, isolated, T orbit for tensors of fixed norm. Let $T^{abc}=x^{(a}y^b\xi^{c)}$ for some x^a, y^a, ξ^a, and let δ^a be a unit vector along one of the threefold axes of T. In Appendix B we show that

$$T^{abc}=(5/6\sqrt{3})(\delta^a\delta^b\delta^c-\tfrac{3}{5}\delta^{(a}g^{bc)})+\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,\chi^{abc}_{(3)}, \tag{9}$$

where $\chi^{abc}_{(3)}$ is a 3-star orthogonal to δ^a. We can now determine the remaining isotropy subgroups.

Assume T^{abc} has D_3 or Z_3 isotropy subgroup. Let ξ^a be a unit vector along the threefold axis of $D_3(Z_3)$. Then T^{abc} is invariant under rotation by $2\pi/3$ around ξ^a and the decomposition of T^{abc} with respect to ξ^a is given by

$$T^{abc}=\alpha(\xi^a\xi^b\xi^c-\tfrac{3}{5}\xi^{(a}g^{bc)})+\beta\chi^{abc}_{(3)}, \tag{10}$$

where $\beta\neq 0$ [or T^{abc} would have isotropy subgroup SO(2)] and $|\alpha/\beta|\neq 5/4\sqrt{2}$ (or T^{abc} would have isotropy subgroup T). It is clear from the decomposition that T^{abc} has isotropy subgroup D_3 if and only if $T^{abc}=\beta\chi^{abc}_{(3)}$ are on the same orbit, it follows that there is precisely one, isolated, D_3 orbit for tensors of fixed norm. Let $T^{abc}=\chi^{abc}_{(3)}$ for some 3-star, and let δ^a be a unit vector along one of the three twofold axes of D_3. In Appendix B we show that

$$T^{abc}=\tfrac{5}{8}(\delta^a\delta^b\delta^c-\tfrac{3}{5}\delta^{(a}g^{bc)})+\tfrac{3}{4}x^{(a}y^b\delta^{c)}. \tag{11}$$

For $\alpha\neq 0$ in Eq. (10) it is clear that T^{abc} has isotropy subgroup Z_3 and that there are no isolated Z_3 orbits.

Now assume T^{abc} has D_2 or Z_2 as an isotropy subgroup. Let ξ^a be a unit vector along one of the twofold axes of $D_2(Z_2)$. Then the decomposition of T^{abc} with respect to ξ^a is given by

$$T^{abc}=\alpha(\xi^a\xi^b\xi^c-\tfrac{3}{5}\xi^{(a}g^{bc)})+\gamma x^{(a}y^b\delta^{c)},$$

where $\gamma\neq 0$ [or T^{abc} would have isotropy subgroup SO(2)], $\alpha\neq 0$ (or T^{abc} would have isotropy subgroup T) and $|\alpha/\gamma|=\frac{5}{6}$ (or T^{abc} would have isotropy subgroup D_3). Assume T^{abc} has isotropy subgroup D_2. Then there is an axis orthogonal to ξ^a such that T^{abc} is invariant under rotations by π around this axis. Let α_a be a unit vector along this axis. Contracting α_a twice into T^{abc}, we find that α must be zero. This is a contradiction, and therefore no element of W^3 has isotropy subgroup D_2. Every

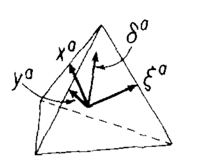

FIG. 3. The orthonormal vectors x^a, y^a, and ξ^a lie along the two fold symmetry axes of a tetrahedron. The unit vector δ^a lies along one of the tetrahedron's threefold symmetry axes.

TABLE II. The most general, renormalizable, SO(3) invariant Higgs potential for the $(2n+1)$-dimensional, irreducible representation of SO(3) ($n=2,3$). To insure that the VEV of the Higgs field in nonzero, we take $\mu^2>0$ and $\lambda>0$. The isotropy subgroups of the local minima of the potential are given.

n	Higgs potential	isotropy subgroups of the local minima	
2	$V=-(\mu^2/2)T^{ab}T_{ab}+(\lambda/4)(T^{ab}T_{ab})^2$	$\lvert c\rvert>0$	$N(\mathrm{SO}(2))$
	$+cT^{ab}T_{bc}T^c{}_a$	$c=0$	$N(\mathrm{SO}(2))$, D_2
3	$V=-(\mu^2/2)T^{abc}T_{abc}+(\lambda/4)(T^{abc}T_{abc})^2$	$c>0$	T
	$+c\,T^{abc}T_{bcd}T^{def}T_{efa}$	$c=0$	SO(2), T, D_3,
			Z_3, Z_2, Z_1
		$-\lambda/2<c<0$	D_3

tensor of the above form has isotropy subgroup Z_2 and there are no isolated Z_2 orbits.

Since Z_2 is not a subgroup of Z_3, Z_2 cannot be a principal isotropy subgroup. Since SO(3) is a compact Lie group, it follows[1] that there must exist elements of W^3 with isotropy subgroup Z_1 and that no Z_1 orbits are isolated. This same procedure can be applied for any n.

5. HIGGS POTENTIALS FOR SO(3)

In this section the most general, renormalizable, SO(3) invariant Higgs potential for the $(2n+1)$-dimensional irreducible representation of SO(3), where $n=2,3$. We then determine the local minima and find the associated isotropy subgroups. The results are tabulated in Table II. Renormalizability demands that the Higgs potential be at most fourth power in the fields. This requirement limits the form of the potential and has consequences for the direction of symmetry breaking.

A. $n=2$

To fourth power in the field there are precisely four SO(3) scalars[4] that can be formed from T^{ab}, namely $T^{ab}T_{ab}$, $T^{ab}T_{bc}T^c{}_a$, $(T^{ab}T_{ab})^2$, and $T^{ab}T_{bc}T^{cd}T_{da}$. However, the fourth term is a multiple of the third. Note that

$$T^a_{[a}T^b{}_bT^c{}_cT^d{}_{d]}=0.$$

Expanding this expression and remembering that T^{ab} is traceless, we find that

$$T^{ab}T_{bc}T^{cd}T_{da}=\tfrac{1}{2}(T^{ab}T_{ab})^2. \tag{12}$$

Therefore, the most general, renormalizable, SO(3) invariant Higgs potential is given by

$$V=-(\mu^2/2)T^{ab}T_{ab}+(\lambda/4)(T^{ab}T_{ab})^2+cT^{ab}T_{bc}T^c{}_a, \tag{13}$$

where μ^2, λ, and c are arbitrary constants. To insure that the VEV of T^{ab} is nonzero, we demand that $\mu^2>0$ and $\lambda>0$. For $c=0$ it is easy to see that V has extrema at $T^{ab}=0$ (local maximum) and $T^{ab}T_{ab}=\mu^2/\lambda$ (local minimum). Therefore, the VEV of T^{ab} is nonzero and has norm $\mu/\sqrt{\lambda}$. However, the direction of the VEV is arbitrary. Therefore, the isotropy subgroup will be either $N(\mathrm{SO}(2))$ or D_2 depending upon the direction we choose for the VEV of the Higgs field. Now assume $c\neq 0$. Recall that for any T^{ab} in W^2 there exists a unit vector η^a such that

$$T^{ab}=\alpha(\eta^a\eta^b-\tfrac{1}{3}g^{ab})+\beta x^{(a}y^{b)}, \tag{14}$$

where x^a, y^a, η^a are orthonormal vectors. Using Eq. (14), we find that $T^{ab}T_{ab}=\frac{2}{3}\alpha^2+\frac{1}{2}\beta^2$

and

$$T^{ab}T_{bc}T^c{}_a=\tfrac{2}{9}\alpha^3-\tfrac{1}{2}\alpha\beta^2.$$

Therefore, the Higgs potential is a function of α and β only. To find the local extrema we solve the equations $\partial V/\partial\alpha=\partial V/\partial\beta=0$. There are three sets of solutions.

1. $\alpha=\beta=0$

This corresponds to the zero vector. However, the matrix

$$\begin{pmatrix}\dfrac{\partial^2V}{\partial\alpha^2} & \dfrac{\partial^2V}{\partial\alpha\,\partial\beta}\\[2ex] \dfrac{\partial^2V}{\partial\beta\,\partial\alpha} & \dfrac{\partial^2V}{\partial\beta^2}\end{pmatrix} \tag{15}$$

evaluated at $\alpha=\beta=0$ has the form

$$\begin{pmatrix}- & \\ & -\end{pmatrix},$$

independent of the values of μ^2, λ, and c. Therefore, the potential always has a local maximum at $T^{ab}=0$. This implies that the VEV of T^{ab} is always nonzero.

2. $\alpha_{(\pm)}=[-3c\pm(9c^2+24\lambda\mu^2)^{1/2}]/4\lambda$, $\beta=0$

This corresponds to two orbits of tensors, each element of which has isotropy subgroups $N(\mathrm{SO}(2))$. For $c>0$ matrix (15) has the form

$$\begin{pmatrix}+ & \\ & -\end{pmatrix}$$

for the $\alpha_{(+)}$ orbit and

$$\begin{pmatrix}+ & \\ & +\end{pmatrix}$$

for the $\alpha_{(-)}$ orbit. Therefore, for $c>0$ the potential has a saddle point on the $\alpha_{(+)}$ orbit and a local minimum on the $\alpha_{(-)}$ orbit. For $c<0$ we find the reverse situation, i.e., the potential has a local minimum on the $\alpha_{(+)}$ orbit and a saddle point on the $\alpha_{(-)}$ orbit.

3. $\alpha_{(\pm)}=[3c\pm(9c^2+24\lambda\mu^2)^{1/2}]/8\lambda$, $\beta=\pm 2\alpha_{(\pm)}$

Note that $\lvert\alpha(\pm)/\beta\rvert=\frac{1}{2}$. Therefore, from Eq. (7) we know this solution corresponds to two orbits of tensors (one $\alpha_{(+)}$ and the other $\alpha_{(-)}$), each element of which has isotropy subgroup $N(SO(2))$. Furthermore, the norm of any tensor on the $\alpha_{(+)}$ ($\alpha_{(-)}$) orbit is easily shown to be the same as the norm of any tensor on the − (+) orbit of Sec. 2. Since $\alpha_{(+)} > 0$ and $\alpha_{(-)} < 0$, the two orbits of this section are precisely the same orbits as in Sec. 2.

Therefore, for any $|c| > 0$ the potential has one local minimum, and it occurs on an $N(SO(2))$ orbit. The VEV of the Higgs field must lie on this orbit and have $N(SO(2))$ isotropy subgroup.

B. $n=3$

To fourth power in the field there are precisely four SO(3) scalars that can be formed from T^{abc}, namely $T^{abc}T_{abc}$, $(T^{abc}T_{abc})^2$, $T^{abc}T_{bcd}T^{def}T_{efa}$, and $T^{abc}T_a{}^{dc}T_{bd}{}^fT_{ef}$. However, the last turn is a linear combination of the second and third. Let ξ^a be an arbitrary vector and consider any element T^{abc} of W^3. Then $L^{bc}=T^{abc}\xi_a$ is an element of W^2, and from Eq. (12) we know that

$$L^{ab}L_{bc}L^{cd}L_{da}=\tfrac{1}{2}(L^{ab}L_{ab})^2.$$

Therefore,

$$(2T^{\alpha ab}T^{\beta}{}_{bc}T^{\gamma cd}T^{f}{}_{da}-T^{\alpha ab}T^{\beta}{}_{ab}T^{\gamma cd}T^{f}{}_{cd})\xi_\alpha\xi_\beta\xi_\gamma\xi_f=0.$$

Symmetrize this expression over α, β, γ, and f. Then, by Lemma 2 we have

$$2T^{(\alpha|ab|}T^{\beta}{}_{bc}T^{\gamma|cd|}T^{f)}{}_{da}=T^{(\alpha|ab|}T^{\beta}{}_{ab}T^{\gamma|cd|}T^{f)}{}_{cd}.$$

Expanding these expressions and contracting α and γ with β and f respectively, we find that

$$T^{abc}T_a{}^{de}T_{bd}{}^{f}T_{cef}=\tfrac{1}{2}(T^{abc}T_{abc})^2-T^{abc}T_{bcd}T^{def}T_{efa}.$$

Therefore, the most general, renormalizable, SO(3) invariant Higgs potential is given by

$$V=-(\mu^2/2)(T^{abc}T_{abc})+(\lambda/4)(T^{abc}T_{abc})^2+c\,T^{abc}T_{bcd}T^{def}T_{efa}, \tag{16}$$

where μ^2, λ, and c are arbitrary constants. To insure that the VEV of T^{abc} is nonzero, we again demand $\mu^2>0$ and $\lambda>0$. For $c=0$, V has only one local minimum which occurs when $T^{abc}T_{abc}=\mu^2/\lambda$. Therefore, the VEV is nonzero and has norm $\mu/\sqrt{\lambda}$. Again, the direction of the VEV is arbitrary and may have six different isotropy subgroups depending upon the direction we choose. Now assume $c\neq 0$. Recall that for any T^{abc} in W^3 there exists a unit vector η^a such that

$$T^{abc}=\alpha(\eta^a\eta^b\eta^c-\tfrac{3}{5}\eta^{(a}g^{bc)})+\beta\chi^{abc}_{(3)}+\gamma x^{(a}y^{b}\eta^{c)}, \tag{17}$$

where x^a, y^a, η^a are orthonormal vectors. Using Eq. (17), we find that

$$T^{abc}T_{abc}=\tfrac{2}{5}\alpha^2+\tfrac{1}{4}\beta^2+\tfrac{1}{6}\gamma^2$$

and

$$T^{abc}T_{bcd}T^{def}T_{efa}=[44/(25)^2]\alpha^4+\tfrac{1}{25}\alpha^2\beta^2+\tfrac{2}{25}\alpha^2\gamma^2+\tfrac{1}{32}\beta^4+\tfrac{1}{24}\beta^2\gamma^2+[3/(18)^2]\gamma^4.$$

Therefore, the Higgs potential is a function of α, β, and γ only. On physical grounds we want V to approach $+\infty$ for large values of the parameters. It is not hard to show that this will be the case if and only if $c>-\lambda/2$. From now on we will assume this is the case. There are seven sets of local extrema solutions. However, only four of these correspond to possible local minima. Of these four sets, two represent the same T orbit and the other two represent the same D_3 orbit. The two different solutions are

1. $\beta=\pm 2\mu/(\lambda+2c)^{1/2}$, $\alpha=\gamma=0$

This corresponds to one orbit of tensors, each element of which has isotropy subgroup D_3. For $c>0$ the matrix of second partial derivatives of V (call it M) has the form

$$\begin{pmatrix} - & & \\ & + & \\ & & 0 \end{pmatrix}.$$

Therefore, for $c>0$ the potential has a saddle point on this orbit. For $-\lambda/2<c<0$, M has the form

$$\begin{pmatrix} + & & \\ & + & \\ & & 0 \end{pmatrix},$$

and therefore the potential has a local minimum on this orbit.

2. $\gamma=\pm\sqrt{6}\,\mu/[\lambda+(4/3)c]^{1/2}$, $\alpha=\beta=0$

This corresponds to an orbit of tensors, each element of which has isotropy subgroup T. For $c>0$, M has the form

$$\begin{pmatrix} + & & \\ & + & \\ & & + \end{pmatrix},$$

and, therefore, the potential has a local minimum on this orbit. For $-\lambda/2<c<0$, M has the form

$$\begin{pmatrix} - & & \\ & - & \\ & & + \end{pmatrix},$$

and therefore the potential has a saddle point on this orbit.

Therefore, for $c>0$ the potential has one local minimum, and it occurs on a T orbit. In this case the VEV of T^{abc} has isotropy subgroup T. For $-\lambda/2<c<0$ the potential has one local minimum, and it occurs on a D_3 orbit. Therefore, the VEV must have isotropy subgroup D_3. For $c\leq-\lambda/2$ the potential has no local minima. We note that for both the five- and seven-dimensional representations all local extrema occurred on isolated orbits (when $c\neq 0$).

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APPENDIX A

(1) Consider the tensor $T^{ab}=\chi^{ab}_{(2)}+\tfrac{1}{2}g^{ab}_1$ orthogonal to unit vector ξ^a. By definition $\chi^{ab}_{(2)}=-\alpha^a\alpha^b$ for some unit vector α^a. Let β^a be a unit vector orthogonal to α^a and

ξ^a and consider $x^a = (1/\sqrt{2})(\alpha^a + \beta^a)$ and $y^a = (1/\sqrt{2}) \times (-\alpha^a + \beta^a)$. The vectors x^a, y^a, ξ^a are orthonormal. Solving for α^a we have $\alpha^a = (1/\sqrt{2})(x^a - y^a)$. Substituting this expression for α^a in T^{ab} and noticing that $g_1^{ab} = x^a x^b + y^a y^b$ we find that $T^{ab} = x^{(a}y^{b)}$.

(2) Consider the tensor $T^{ab} = \xi^a \xi^b - \frac{1}{3} g^{ab}$, and let δ^a be any unit vector orthogonal to ξ^a. Let β^a be a unit vector such that $\beta^a \xi_a = \beta^a \delta_a = 0$ and consider $x^a = (1/\sqrt{2}) \times (\xi^a + \beta^a)$ and $y^a = (1/\sqrt{2})(\xi^a - \beta^a)$. The vectors x^a, y^a, δ^a are orthonormal and $\xi^a = (1/\sqrt{2})(x^a + y^a)$. Substituting this expression in T^{ab}, we find

$$T^{ab} = \tfrac{1}{2} g_1^{ab} + x^{(a}y^{b)} - \tfrac{1}{3} g^{ab}.$$

Remembering that $g_1^{ab} = g^{ab} - \delta^a \delta^b$, we have finally

$$T^{ab} = -\tfrac{1}{2}(\delta^a \delta^b - \tfrac{1}{3} g^{ab}) + x^{(a}y^{b)}.$$

APPENDIX B

(1) Let x^a, y^a, ξ^a, and δ^a be the unit vectors of Fig. 3 and consider the tensor $T^{abc} = x^{(a}y^b\xi^{c)}$. Now

$$x^a = (1/\sqrt{3})(\delta^a + \sqrt{2}\alpha^a), \quad y^a = (1/\sqrt{3})(\delta^a + \sqrt{2}\beta^a),$$

and

$$\xi^a = (1/\sqrt{3})(\delta^a + \sqrt{2}\gamma^a),$$

where $\alpha^{(a}\beta^b\gamma^{c)}$ is the 3-star $\chi_{(3)}^{abc}$ orthogonal to δ^a. Substituting these expressions into T^{abc}, we find

$$T^{abc} = (1/3\sqrt{3})(\delta^a\delta^b\delta^c + 2\delta^{(a}[\alpha^b\beta^{c)} + \alpha^b\gamma^{c)} + \beta^b\gamma^{c)}] + 2\sqrt{2}\chi_{(3)}^{abc}). \qquad \text{(B1)}$$

Now $\alpha^{(b}\beta^{c)} + \alpha^{(b}\gamma^{c)} + \beta^{(b}\gamma^{c)}$ is an element of V^2 the isotropy subgroup of which obviously contains Z_3. Therefore, it must be a multiple of g_1^{ab}, and, comparing the traces of these two tensors, we have

$$\alpha^{(b}\beta^{c)} + \alpha^{(b}\gamma^{c)} + \beta^{(b}\gamma^{c)} = -\tfrac{3}{4} g_1^{bc}. \qquad \text{(B2)}$$

Substituting Eq. (19) into Eq. (18), we have finally that

$$T^{abc} = (5/6\sqrt{3})(\delta^a\delta^b\delta^c - \tfrac{3}{5}\delta^{(a}g^{bc)}) + \tfrac{2}{3}\sqrt{\tfrac{2}{3}}\chi_{(3)}^{abc}.$$

(2) Consider the tensor $T^{abc} = \chi_{(3)}^{abc} = \alpha^{(a}\beta^b\delta^{c)}$ where, by definition of a 3-star, the angle between neighbors is $2\pi/3$. It is clear that α^a, β^a, and δ^a lie along the three twofold axes of D_3. Let z^a be a unit vector in the plane of the star such that $z^a\delta_a = 0$. Then

$$\alpha^a = -\tfrac{1}{2}(\delta^a - \sqrt{3}z^a) \quad \text{and} \quad \beta^a = -\tfrac{1}{2}(\delta^a + \sqrt{3}z^a).$$

Substituting these expressions into T^{abc} and using the results of Sec. 1 of Appendix A, we find that

$$T^{abc} = \tfrac{5}{8}(\delta^a\delta^b\delta^c - \tfrac{3}{5}\delta^{(a}g^{bc)}) + \tfrac{3}{4}x^{(a}y^b\delta^{c)},$$

where x^a, y^a, δ^a are orthonormal vectors.

[1]G. Bredon, *Introduction to Compact Transformation Groups* (Academic, New York, 1972), Theorem 3.1, pg. 179.
[2]W. Miller, *Symmetry Groups and Their Applications* (Academic, New York, 1972), pp. 27–32.
[3]J. A. Wolf, *Spaces of Constant Curvature* (McGraw-Hill, New York, 1968).

An alternative approach to the normal frequencies of a randomly disordered linear chain[a)]

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An attempt is made to gain some insight into the statistical character of the normal frequencies of a long chain of harmonic oscillators with randomly disordered masses. Instead of being concerned with the limit $L \to \infty$ of a chain whose random characteristics do not depend on its length L, the paper deals with the so-called diffusion limit where the "size" of the randomness in each of the masses tends to 0 like $L^{-1/2}$.

1. INTRODUCTION

Since the pioneering work of Dyson[1] and Schmidt,[2] there has been continued interest in the vibrational properties of disordered chains of harmonic oscillators.[3–5] Lieb and Mattis[6] gave a good introduction to the analytical aspects of this topic; the numerical work done in this field was reviewed by Dean.[7]

The literature dealing with the disordered linear chain is mainly concerned with the problem of calculating and describing the vibrational spectrum (distribution of the normal frequencies) in the limit of an infinitely long chain. In principle the problem of determining the spectrum was solved more than 20 years ago by Dyson[1] and later by Schmidt.[2] The analytical intractability of their solutions, however, makes a general examination of the qualitative behavior of the spectrum very difficult.

In this paper we pursue a different and, as it turns out, much simpler approach to the randomly disordered linear chain. Instead of dealing with the limit $L \to \infty$ of a chain whose random characteristics do not depend on its length L, we rather consider the so-called diffusion limit, where the "size" of the randomness in the masses tends to 0 like $L^{-1/2}$. For this limit we have powerful theorems at our disposal,[8–10] which enable us to gain some knowledge of the statistical character of the normal frequencies.

More specifically, we consider a chain (with fixed ends) of L masses, each coupled to its nearest neighbors by identical (nonrandom) linear springs. The masses are assumed to have the form

$$m_j(\epsilon) = m[1 + \epsilon\mu_j], \quad j = 1, 2, \ldots, L,$$

where $m > 0$ is a constant and the μ_j's are identically distributed (not necessarily independent) mean 0 random variables with range in $[-1, 1]$. Then the parameter $\epsilon \in [0, 1[$ is a measure of the randomness of the masses m_j. We are interested in the statistical properties of $N(\Omega; L, \epsilon)$, the number of normal frequencies of the chain in $[0, \Omega]$. Our main result, Theorem B in Sec. 2B, can be interpreted as follows. If the μ_j's constitute a stationary stochastic process with sufficiently strong mixing,[11] then, for large L and small ϵ, the random variable $L^{-1}[N(\Omega; L, \epsilon) - E\{N(\Omega; L, \epsilon)\}]$ is, with the exception of one single value of Ω, approximately normally distributed with mean 0, the variance being an explicit function of $\Omega, \epsilon^2/L$, and the covariances of the μ_j's. Moreover, an approximation to $L^{-1}E\{N(\Omega; L, \epsilon)\}$ is produced which involves only Ω, ϵ^2, and the covariances of the μ_j's. For the exceptional value of Ω a similar interpretation is possible. Our result is incomplete in as far as no explicit error estimate is given.

The proof (in Sec. 4B) is based on the observation that the function N can be obtained by solving a simple first-order initial-value problem which is of such a nature that a powerful diffusion limit theorem of Papanicolaou and Kohler[10] (stated in Sec. 3) can be applied. In order to show this, a slight modification of the phase-space argument, due to Prüfer and commonly used in the classical Sturm–Liouville theory, is adapted to the discrete case of coupled harmonic oscillators. This approach suggests itself when the continuous analog of the random chain is considered. For this reason the eigenfrequency problem for the reduced wave equation in one dimension with random index of refraction is briefly treated in Secs. 2A and 4A. It is interesting to note that a certain anomaly occurring in the discrete case [cf. (2.15)] is absent in the continuous analog.

This paper owes much to the work of Kohler and Papanicolaou[12]; in particular, the crucial transformation (4.8)–(4.9) is merely a modification of (2.15) in Ref. 12.

Throughout this paper $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of reals, nonnegative reals, nonnegative integers, and positive integers, respectively. The integer and fractional parts of $x \in \mathbb{R}$ are denoted by $\hat{x}$ and $\check{x}$ respectively; thus $x = \hat{x} + \check{x}$, $|\hat{x}| \in \mathbb{N}$, $\check{x} \in [0, 1[$. For a nonempty real interval I the symbols $L^1(I)$ and $AC(I)$ denote respectively the sets of all functions $I \to \mathbb{R}$ which are Lebesgue-integrable on I and those which are absolutely continuous on each compact subinterval of I. The space of absolutely convergent series $\mathbb{N}_0 \to \mathbb{R}$ is denoted by $l^1(\mathbb{N}_0)$. Phrases such as "almost all" (a.a.) and "almost everywhere" (a.e.) always refer to the Lebesgue measure on $\mathbb{R}$.

Some standard terminology of probability and stochastic processes is used. The symbol $E\{\cdot\}$ denotes expectation with respect to the measure of the underlying probability space. As is customary, the argument ranging in a probability space of a stochastic process

[a)]The bulk of this work was done during the author's 1974/75 visit to the Courant Institute of Mathematical Sciences, New York University.

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is usually suppressed; thus we speak, for instance, of a stochastic process $\mathbb{N}_0 \to \mathbb{R}$. The symbol w denotes the standard process of Brownian motion (Wiener process) $\mathbb{R}^+ \to \mathbb{R}$ with $w(0)=0$, $E\{w(\tau)\}=0$ and $E\{[w(\tau)]^2\}=\tau$, $\tau \in \mathbb{R}^+$; cf. Ref. 13, p. 7.

2. FORMULATION OF PROBLEMS AND MAIN RESULTS

A. Continuous case: Wave propagation in a random medium

For each $\epsilon \in [0,1[$ we consider a one-dimensional random medium, occupying the semi-infinite interval $\mathbb{R}^+$, which is characterized by its index of refraction $n(\cdot,\epsilon)$ relative to a homogeneous medium (corresponding to $\epsilon=0$). We assume that $n(\cdot,\epsilon)$ has the form

$$n(x,\epsilon)=[1+\epsilon\nu(x)]^{1/2}, \quad x\in \mathbb{R}^+,$$

where ν: $\mathbb{R}^+ \to [-1,1]$ is a mean 0 stationary stochastic process.

For $\omega>0$, $L>0$, and $\epsilon\in[0,1[$ let

$$U(t,x;\omega,L,\epsilon)=e^{-i\omega t}u(x;\omega,L,\epsilon), \quad t,x\in \mathbb{R}^+,$$

be a wavefield in the random medium represented by $n(\cdot,\epsilon)$, which satisfies the boundary conditions

$$U(t,0;\omega,L,\epsilon)=U(t,L;\omega,L,\epsilon)=0, \quad t\in \mathbb{R}^+.$$

Then $u(\cdot;\omega,L,\epsilon)$ satisfies the reduced wave equation in one dimension,

$$c^2u''(x)+\omega^2[1+\epsilon\nu(x)]u(x)=0, \quad \text{a.a. } x\in\mathbb{R}^+, \tag{2.1}$$

and the boundary conditions

$$u(0)=u(L)=0. \tag{2.2}$$

In (2.1) $c>0$ denotes the phase speed in the unperturbed homogeneous medium ($\epsilon=0$) and the primes denote differentiation with respect to $x\in\mathbb{R}^+$.

We are interested in the asymptotic behavior, for large L and small ϵ, of the number $N(\Omega;L,\epsilon)$ of eigenvalues (eigenfrequencies) ω^2 of (2.1) and (2.2) with $\omega\in[0,\Omega]$, $\Omega\in\mathbb{R}^+$. For the unperturbed homogeneous medium $N(\cdot;\cdot,0)$ is a deterministic function, namely [cf. (2.3) and (2.4) below]

$$N(\Omega;L,0)=[\Omega L/c\pi]^\wedge, \quad \Omega,L\in\mathbb{R}^+,$$

whereas for $\epsilon>0$, $N(\cdot;\cdot,\epsilon)$ is an $\mathbb{N}$-valued stochastic process defined on $\mathbb{R}^+\times\mathbb{R}^+$. In the following theorem k plays the role of the wavenumber associated with the undisturbed homogeneous medium ($\epsilon=0$), i.e., $k=\omega/c$.

THEOREM A: Let ν: $\mathbb{R}^+\to[-1,1]$ be a strict sense mean 0 stationary measurable stochastic process which satisfies the regularity and mixing conditions stated in Sec. 3. Then the stochastic initial-value problem

$$\psi'(x)=\epsilon k\pi^{-1}\nu(x)\sin^2(\pi\psi(x)+kx), \quad \text{a.a. } x\in\mathbb{R}^+,$$

$$\psi(0)=0, \tag{2.3}$$

$k\in\mathbb{R}^+$, $\epsilon\in[0,1[$, has a unique solution $\psi(\cdot,k,\epsilon)$: $\mathbb{R}^+\to\mathbb{R}$ with all its sample functions in $\mathrm{AC}(\mathbb{R}^+)$. This solution has the following two properties:

(i) For $\Omega>0$, $L>0$, and $\epsilon\in[0,1[$,

$$N(\Omega;L,\epsilon)=[\Omega L/c\pi+\psi(L,\Omega/c,\epsilon)]^\wedge. \tag{2.4}$$

(ii) For $k\in\mathbb{R}^+$ and $\epsilon\in]0,1[$ let $\Psi_\epsilon(\tau,k)=\psi(\tau/\epsilon^2,k,\epsilon)$, $\tau\in\mathbb{R}^+$. Then, as $\epsilon\searrow 0$, the process $\Psi_\epsilon(\cdot,k)$ converges weakly to the diffusion process[13] $\Psi(\cdot,k)$ given by

$$\Psi(\tau,k)=\frac{k}{4\pi}\left\{-\tau k\int_0^\infty R(x)\sin(2kx)\,dx + 2\left[\int_0^\infty R(x)[2+\cos(2kx)]\,dx\right]^{1/2} w(\tau)\right\}, \quad \tau\in\mathbb{R}, \tag{2.5}$$

where R denotes the covariance function of ν, $R(x)=\mathrm{Cov}(\nu(x),\nu(0))$, $x\in\mathbb{R}^+$. Moreover, all moments of $\Psi_\epsilon(\tau,k)$ converge uniformly on compact τ intervals to the corresponding moments of $\Psi(\tau,k)$ and the rate of convergence is of order ϵ.

In particular, for each $(\tau,k)\in\mathbb{R}^+\times\mathbb{R}^+$ the probability distribution of

$$\psi(L,k,\epsilon)+\frac{\tau k^2}{4\pi}\int_0^\infty R(x)\sin(2kx)dx$$

converges weakly, as $L\nearrow\infty$ and $\epsilon\searrow 0$ such that $L\epsilon^2=\tau>0$, to the normal distribution with mean 0 and variance

$$\frac{\tau k^2}{4\pi^2}\int_0^\infty R(x)[2+\cos(2kx)]\,dx.$$

B. Discrete case: Chain of random masses

For each $\epsilon\in[0,1[$ we consider a semi-infinite chain of masses $m_j(\epsilon)>0$, $j\in\mathbb{N}_0$, each coupled to its nearest neighbors by identical elastic springs obeying Hooke's law (spring constant $f>0$). By $U_j(t;\epsilon)$ we denote the displacement of mass $m_j(\epsilon)$ from its rest position j at time $t\in\mathbb{R}^+$. Then the equations of motion (with resting zeroth particle located at 0) are

$$m_j(\epsilon)\ddot{U}_j(t;\epsilon)=f[U_{j-1}(t;\epsilon)-2U_j(t;\epsilon)+U_{j+1}(t;\epsilon)], \quad j\in\mathbb{N}_0, \tag{2.6}$$

and $U_0(t;\epsilon)=0$ for $t\in\mathbb{R}^+$ and $\epsilon\in[0,1[$; the dots denote differentiation with respect to $t\in\mathbb{R}^+$.

For $\omega>0$, $L\in\mathbb{N}_0$, and $\epsilon\in[0,1[$ let

$$U_j(t;\omega,L,\epsilon)=u_j(\omega,L,\epsilon)\cos(\omega t), \quad j\in\mathbb{N}, \ t\in\mathbb{R},$$

be a solution of (2.6) which satisfies the boundary conditions

$$U_0(t;\omega,L,\epsilon)=U_{L+1}(t;\omega,L,\epsilon)=0, \quad t\in\mathbb{R}^+.$$

Then the real numbers $u_j(\omega,L,\epsilon)$, $j\in\mathbb{N}$, satisfy the reduced equations

$$u_{j-1}-[2-\omega^2 m_j(\epsilon)/f]u_j+u_{j+1}=0, \quad j\in\mathbb{N}_0. \tag{2.7}$$

and the boundary conditions

$$u_0=u_{L+1}=0. \tag{2.8}$$

It is well known (and it will follow from the proof of Theorem B below, cf. Sec. 5) that the boundary-value problem given by (2.7) and (2.8) has exactly L linearly independent solutions (normal mode vibrations) which correspond to L different normal frequencies

$$0<\omega_1(L,\epsilon)<\cdots<\omega_L(L,\epsilon)<2[f/\min\{m_j(\epsilon): j=1,\ldots,L\}]^{1/2}. \tag{2.9}$$

We assume that $m_j(\epsilon)$ has the form

$$m_j(\epsilon)=m[1+\epsilon\mu_j],\quad j\in\mathbb{N}_0,\quad \epsilon\in[0,1[, \tag{2.10}$$

where $m>0$ is a constant and $\mu:\mathbb{N}_0\to[-1,1]$ is a mean 0 stationary stochastic process. Then, in view of (2.9), the normal frequencies $\omega_j(L,\epsilon)$ associated with (2.7) and (2.8) are random variables with

$$0<\omega_1(L,\epsilon)<\cdots<\omega_L(L,\epsilon)<2[f^{-1}m(1-\epsilon)]^{-1/2}.$$

We are interested in the asymptotic behavior, for large L and small ϵ, of the number $N(\Omega;L,\epsilon)$ of eigenvalues (normal frequencies) ω^2 of (2.7) and (2.8) with $\omega\in[0,\Omega]$, $0\le\Omega<2(f/m)^{1/2}$, when the masses $m_j(\epsilon)$ are given by (2.10). For the perfect harmonic chain $N(\cdot;\cdot,0)$ is a deterministic function, namely [cf. (2.11) and (2.12) below]

$$N(\Omega;L,0)=[2\pi^{-1}(L+1)\arcsin(\Omega[m/4f]^{1/2})]^{\wedge},$$
$$\Omega\in[0,2(f/m)^{1/2}[,\quad L\in\mathbb{N}_0,$$

whereas for $\epsilon>0$, $N(\cdot;\cdot,\epsilon)$ is an $\mathbb{N}$-valued stochastic process defined on $[0,2(f/m)^{1/2}[\times\mathbb{N}_0$. In the following theorem k plays the role of the wavenumber associated with the perfect chain $(\epsilon=0)$, i.e., $k=2\arcsin(\omega[m/4f]^{1/2})$.

THEOREM B: Let $(\Lambda,\mathcal{J},P)$ be a probability space and let $\mu:\mathbb{N}_0\times\Lambda\to[1,1]$ be a mean 0 stationary stochastic process which satisfies the following mixing condition[11]: the (decreasing) function $\rho:\mathbb{N}_0\to[0,1]$, defined for $i\in\mathbb{N}_0$ by

$$\rho(i)=\sup\{|P(A|B)-P(A)|:j\in\mathbb{N}_0,\ A\in\mathcal{J}_{i+j}^{\infty},\ B\in\mathcal{J}_0^j,\ P(B)>0\},$$

has the property that $\rho^{1/2}\in l^1(\mathbb{N}_0)$; here $\mathcal{J}_i^k$ denotes the σ-subalgebra of $\mathcal{J}$ generated by the random variables $\mu_i,\mu_{i+1},\ldots,\mu_k$, $1\le i\le k\le\infty$. Then the stochastic initial-value problem

$$\psi'(x)=2\epsilon\pi^{-1}\tan(k/2)\mu_{\hat{x}}\sin^2(\pi\psi(x)+k\hat{x}),\quad 1<x\neq\hat{x},$$
$$\psi(1)=0, \tag{2.11}$$

$k\in[0,\pi[$, $\epsilon\in[0,1[$, has a unique solution $\psi(\cdot,k,\epsilon)$: $[1,\infty[\to\mathbb{R}$ with all its sample functions in $AC([1,\infty[)$. This solution has the following two properties:

(i) For $0\le\Omega=2(f/m)^{1/2}\sin(K/2)<2(f/m)^{1/2}$, $L\in\mathbb{N}_0$ and $\epsilon\in[0,1[$,

$$N(\Omega;L,\epsilon)=[K(L+1)/\pi+\psi(L+1,K,\epsilon)]^{\wedge}, \tag{2.12}$$

provided that $K\le k_\epsilon$, where $k_\epsilon\in]0,\pi]$ is the unique solution of the equation

$$\log(1+\sin k)=2\epsilon\tan(k/2). \tag{2.13}$$

(ii) For $k\in[0,\pi[$ and $\epsilon\in]0,1[$ let $\Psi_\epsilon(\tau,k)=\psi(\tau/\epsilon^2,k,\epsilon)$, $\tau\in\mathbb{R}^+$. Then, as $\epsilon\searrow0$, the process $\Psi_\epsilon(\cdot,k)$ converges weakly to the diffusion process[13] $\Psi(\cdot,k)$, defined for $\tau\in\mathbb{R}^+$ by

$$\Psi(\tau,k)=\frac{\tan(k/2)}{\pi}\Bigg[-\tau\tan\left(\frac{k}{2}\right)\sum_{n=1}^{\infty}R_n\sin(2kn)+\left(\tfrac{1}{2}\sum_{n=-\infty}^{\infty}R_n[2+\cos(2kn)]\right)^{1/2}w(\tau)\Bigg],\quad k\neq\pi/2, \tag{2.14}$$

$$\Psi(\tau,\pi/2)=\begin{cases}\dfrac{1}{2\pi}\operatorname{am}([8S_e]^{1/2}w(\tau)|S_a/2S_e), & S_e>0,\\ 0, & S_e=0.\end{cases} \tag{2.15}$$

Moreover, all moments of $\Psi_\epsilon(\tau,k)$ converge uniformly on compact τ intervals to the corresponding moments of $\Psi(\tau,k)$ and the rate of convergence is of order ϵ. In (2.14) R denotes the covariance function of μ, i.e.,

$$R_n=R_{-n}=\operatorname{Cov}(\mu_j,\mu_{j+n})=E\{\mu_j\mu_{j+n}\},\quad j,n\in\mathbb{N};$$

the sums S_e and S_a appearing in (2.15) are defined by (2.17) and (2.18) below and $\operatorname{am}(\cdot|p)$, $p\in[0,1]$, denotes the Jacobian amplitude, i.e., the inverse function of the elliptic integral of the first kind $F(\cdot|p)$.[14]

Remarks:

1. The result formulated in part (i) of the theorem is incomplete in as far as (2.12) gives $N(\cdot;L,\epsilon)$ only on the interval $[0,2(f/m)^{1/2}\sin(k_\epsilon/2)]$. It could in fact be improved, but since $\lim_{\epsilon\to0}k_\epsilon=\pi$, it is sufficient for our needs. (See also Sec. 5.)

2. It follows from the mixing conditions imposed on μ, (3.6) and (3.7), that

$$S=\sum_{n=-\infty}^{\infty}R_n=\lim_{N\to\infty}N^{-1}E\{[\sum_{j=1}^{N}\mu_j]^2\}\in\mathbb{R}^+, \tag{2.16}$$

$$S_e=\sum_{n=-\infty}^{\infty}R_{2n}=\lim_{N\to\infty}N^{-1}E\{[\sum_{j=1}^{N}\mu_{2j}]^2\}\in\mathbb{R}^+, \tag{2.17}$$

$$S_a=\sum_{n=-\infty}^{\infty}(-1)^nR_n=\lim_{N\to\infty}N^{-1}E\{[\sum_{j=1}^{N}(-1)^j\mu_j]^2\}\in\mathbb{R}^+, \tag{2.18}$$

and therefore, since $S+S_a=2S_e$, either $S_a/2S_e\in[0,1]$ or $S=S_e=S_a=0$. Thus, for $S_e>0$, the function $\operatorname{am}(\cdot|S_a/2S_e)$: $\mathbb{R}\to\mathbb{R}$ is strictly increasing with range $\mathbb{R}$ if $S>0$ (i.e., $S_a/2S_e<1$) and range $]-\pi/2,\ \pi/2[$ if $S=0$ (i.e., $S_a/2S_e=1$). In particular, for $S=0$ and $S_e>0$ the limiting diffusion process $\Psi(\cdot,\pi/2)$ is with probability 1 confined to the interval $]-\frac{1}{4},\frac{1}{4}[$.

3. It follows from Theorem B that for $\Omega=2(f/m)^{1/2}\times\sin(K/2)$, $K\in[0,\pi[$, large $L\in\mathbb{N}_0$, and small $\epsilon>0$,

$$E\{L^{-1}N(\Omega;L,\epsilon)\}\approx\pi^{-1}K-\epsilon^2\pi^{-1}\tan^2(K/2)\sum_{n=1}R_n\sin(2Kn),$$

and that an approximation to the distribution function of $L^{-1}[N(\Omega;L,\epsilon)-E\{N(\Omega;L,\epsilon)\}]$ is given by

$$\mathcal{N}\left[\frac{\epsilon^2}{2\pi^2L}\tan^2\left(\frac{K}{2}\right)\sum_{n=-\infty}^{\infty}R_n\{2+\cos(2Kn)\}\right],\quad K\neq\pi/2,$$

$$\mathcal{N}[8S_e\epsilon^2/L]\circ L^{-1}F(2\pi L\cdot|S_a/2S_e),\quad S_e>0,\quad K=\pi/2,$$

$$\mathcal{N}[0],\quad S_a=0,\quad K=\pi/2.$$

(Here $\mathcal{N}[\sigma^2]$ denotes the normal distribution with mean 0 and variance σ^2, the symbol $\circ$ denotes composition of functions and $F(\cdot|p)$ denotes the elliptic integral of the first kind[14] with domain $\mathbb{R}$ for $p\in[0,1[$ and domain $]-\pi/2,\ \pi/2[$ for $p=1$.) This result is incomplete in as far as no explicit error estimate is given. A further deficiency is the nonuniformity of the limit with respect to $K\in[0,\pi[$.

3. A THEOREM OF PAPANICOLAOU AND KOHLER

To prove parts (ii) of our theorems we shall use a diffusion limit theorem of Papanicolaou and Kohler[10] which improves earlier results obtained by Khas'minskii[8] and Papanicolaou and Varadhan.[9] The following simplified one-dimensional version of this theorem is sufficient for our needs.

Let $(\Lambda, \mathcal{J}, P)$ be a probability space and let ν: $\mathbb{R}^+ \times \Lambda \rightarrow [-1,1]$ be a measurable stochastic process with mean 0 and covariance function r:

$$E\{\nu(x)\} = \int_\Lambda \nu(x)\, dP = 0, \quad x \in \mathbb{R}^+,$$
$$r(x,y) = E\{\nu(x)\nu(y)\} = \int_\Lambda \nu(x)\nu(y)\, dP, \quad x,y \in \mathbb{R}^+.$$

For $0 \le x \le y \le \infty$ let $\mathcal{J}_x^y$ denote the σ-subalgebra of $\mathcal{J}$ generated by $\{\nu(z): x \le z \le y\}$ and assume that the conditional probabilities relative to $\mathcal{J}_0^y$, $0 \le y \le \infty$, have a regular version,[15] i.e., that there exist functions P_y: $\Lambda \times \mathcal{J} \rightarrow [0,1]$, $0 \le y \le \infty$, with the properties that $P_y(\lambda, \cdot)$ is a probability measure on $\mathcal{J}$ for each $\lambda \in \Lambda$ and that $P_y(\cdot, A)$ is $\mathcal{J}_0^y$-measurable with $P_y(\cdot, A) = P(A \mid \mathcal{J}_0^y)$ P-almost surely for each $A \in \mathcal{J}$. Assume furthermore that ν satisfies the following mixing condition[11]: the (decreasing) function $\rho: \mathbb{R}^+ \rightarrow [0,1]$, defined for $x \in \mathbb{R}^+$ by

$$\rho(x) = \sup\{|P(A|B) - P(A)| : y \in \mathbb{R}^+, \quad A \in \mathcal{J}_{x+y}^\infty,\ B \in \mathcal{J}_0^y,\ P(B) > 0\},$$

has the property that $\rho^{1/2} \in L^1(\mathbb{R}^+)$.

Let $F: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Lebesgue-measurable function with bounded and continuous partial derivatives up to order four with respect to the second variable, and assume that there are two bounded functions a: $\mathbb{R}^+ \rightarrow \mathbb{R}$ and σ: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, which have four bounded continous derivatives on $\mathbb{R}^+$, such that

$$\sup\left\{\left|Xa(\psi) - \int_x^{x+X} \frac{\partial F}{\partial \psi}(y,\psi) \int_x^y r(y,z) \times F(z,\psi)dzdy\right| : x, X \in \mathbb{R}^+,\ \psi \in \mathbb{R}\right\} < \infty \tag{3.1}$$

and

$$\sup\left\{\left|X\sigma^2(\psi) - 2\int_x^{x+X} F(y,\psi)\int_x^y r(y,z) \times F(z,\psi)\,dzdy\right| : x, X \in \mathbb{R}^+,\ \psi \in \mathbb{R}\right\} < \infty. \tag{3.2}$$

Then, for $\epsilon \in [0,1[$, the stochastic initial-value problem

$$\frac{d\psi}{dx}(x) = \epsilon\nu(x)F(x,\psi(x)), \quad \text{a.a. } x \in \mathbb{R}^+, \qquad \psi(0) = 0 \tag{3.3}$$

has a unique solution $\psi_\epsilon: \mathbb{R}^+ \rightarrow \mathbb{R}$ with sample paths in $AC(\mathbb{R}^+)$. Moreover, the stochastic process $\Psi_\epsilon: \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by

$$\Psi_\epsilon(\tau) = \psi_\epsilon(\tau/\epsilon^2), \quad \tau \in \mathbb{R}^+,\ \epsilon \in]0,1[,$$

converges weakly as $\epsilon \searrow 0$ to the diffusion process[13] $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}$ which is the (unique) solution of the Itô equation[13]

$$\Psi(\tau) = \int_0^\tau a(\Psi(t))dt + \int_0^\tau \sigma(\Psi(t))dw(t), \quad \tau \in \mathbb{R}^+. \tag{3.4}$$

In addition, there exists a function C: $\mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is continuous with respect to its second argument and is such that

$$|E\{[\Psi_\epsilon(\tau)]^n\} - E\{[\Psi(\tau)]^n\}| \le \epsilon C(n,\tau), \tag{3.5}$$

for $n \in \mathbb{N}$, $\tau \in \mathbb{R}^+$, and $\epsilon \in]0,1[$.

Remarks:

1. It follows from the definition of r and of ρ (cf. Ref. 11, Lemma 2, p. 171) that

$$|r(x,y)| \le 2\rho(|x-y|), \quad x,y \in \mathbb{R}^+. \tag{3.6}$$

Furthermore, since $\rho^{1/2} \in L^1(\mathbb{R}^+)$ is bounded, nonnegative, and decreasing, it follows that $\lim_{x\to\infty} x\rho^{1/2}(x) = 0$ and thus that

$$\int_0^\infty x\rho(x)dx \le \sup\{x\rho^{1/2}(x): x \in \mathbb{R}^+\}\int_0^\infty \rho^{1/2}(x)dx < \infty. \tag{3.7}$$

2. If the hypotheses (3.1) and (3.2) are satisfied, then we have necessarily

$$a(\psi) = \lim_{X\to\infty} X^{-1}\int_x^{x+X} \frac{\partial F}{\partial \psi}(y,\psi)\int_x^y r(y,z)F(z,\psi)dzdy \tag{3.8}$$

and

$$\sigma^2(\psi) = \lim_{X\to\infty} 2X^{-1}\int_x^{x+X} F(y,\psi)\int_x^y r(y,z)F(z,\psi)dzdy = \lim_{X\to\infty} X^{-1}E\{[\int_x^{x+X}\nu(y)F(y,\psi)dy]^2\} \ge 0, \tag{3.9}$$

uniformly in $(x,\psi) \in \mathbb{R}^+ \times \mathbb{R}$. The last identity follows from Fubini's theorem which is applicable under the stated hypotheses.

4. PROOFS

A. Derivation of Theorem A

By hypothesis all sample functions of the process ν are Lebesgue-measurable functions $\mathbb{R}^+ \rightarrow [-1,1]$ and therefore, for the derivation of the first half of Theorem A, it suffices to assume that ν: $\mathbb{R}^+ \rightarrow [-1,1]$ is a (deterministic) Lebesgue-measurable function. Thus (2.1) and (2.2) reduce to an ordinary Sturm—Liouville eigenvalue problem which can be handled by means of the usual phase space argument. For the sake of completeness, and also for motivating the derivation of Theorem B, we briefly outline this argument.

Equation (2.1) has a one-dimensional space of everywhere differentiable solutions u: $\mathbb{R}^+ \rightarrow \mathbb{R}$ with $u' \in AC(\mathbb{R}^+)$ and $u(0) = 0$. For any such nontrivial solution we have $u^2(x) + u'^2(x) > 0$ for all $x \in \mathbb{R}^+$ and thus, setting $k = \omega/c \in \mathbb{R}^+$, we can write

$$u(x) = r(x)\sin\phi(x), \quad u'(x) = kr(x)\cos\phi(x), \quad x \in \mathbb{R}^+, \tag{4.1}$$

where r: $\mathbb{R}^+ \rightarrow]0,\infty[$ and $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}$ are both in $AC(\mathbb{R}^+)$. [Except for the factor k, (4.1) is the transformation due to Prüfer, which is commonly used in the classical Sturm—Liouville theory.] The function ϕ is not uniquely determined by (4.1); for the sake of convenience we choose the ϕ with $\phi(0) = 0$. Then, by a straightforward calculation, it follows from (2.1) and (4.1) that ϕ satisfies the initial-value problem

$$\phi'(x) = k[1 + \epsilon\nu(x)\sin^2\phi(x)], \quad \text{a.a. } x \in \mathbb{R}^+, \qquad \phi(0) = 0, \tag{4.2}$$

and that r satisfies a similar first-order equation which involves ϕ.

Let $\phi(\cdot, k, \epsilon): \mathbb{R}^+ \rightarrow \mathbb{R}$ denote the unique solution in $AC(\mathbb{R}^+)$ of (4.2), $k \in \mathbb{R}^+$, $\epsilon \in [0,1[$. Differentiating (4.2) with respect to k and interchanging the order of differentiation of the left-hand side (which is legitimate in this case), we see that $\phi_2(\cdot, k, \epsilon)$, the partial derivative of ϕ with respect to its second argument k, is the unique solution in $AC(\mathbb{R}^+)$ of the linear initial-value problem

$$\phi_2'(x) = 1 + \epsilon\nu(x)\{\sin^2\phi(x,k,\epsilon) + k\sin[2\phi(x,k,\epsilon)]\phi_2(x)\}, \quad \text{a.a. } x \in \mathbb{R}^+,$$

$$\phi_2(0) = 0.$$

Consequently, for $x \in \mathbb{R}^+$, $k > 0$ and $\epsilon \in [0,1[$,

$$\phi_2(x,k,\epsilon) = \int_0^x [1 + \epsilon\nu(t)\sin^2\phi(t,k,\epsilon)] \times \exp\{\epsilon k \int_t^x \nu(s)\sin[2\phi(s,k,\epsilon)]ds\}dt$$

$$\geq \int_0^x (1-\epsilon)e^{\epsilon k(t-x)}dt = \frac{1-\epsilon}{\epsilon k}(1 - e^{-\epsilon k x}),$$

i.e., for $x > 0$ and $\epsilon \in [0,1[$ the function $\phi_2(x, \cdot, \epsilon)$ is bounded from below by a function which is positive and nonintegrable on $]0,\infty[$. Hence, for $x > 0$ and $\epsilon \in [0,1[$, the function $\phi(x, \cdot, \epsilon) : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and its range is $\mathbb{R}^+$. In view of (4.1) this implies that $u(\cdot;\omega, L, \epsilon)$ is a nontrivial solution of (2.1) and (2.2) iff $\phi(L, \omega/c, \epsilon) = n\pi$ for some $n \in \mathbb{N}$, and thus that

$$N(\Omega; L, \epsilon) = [\pi^{-1}\phi(L, \Omega/c, \epsilon)]^\wedge \tag{4.3}$$

for $\Omega > 0$, $L > 0$, and $\epsilon \in [0,1[$.

We now introduce a new dependent variable ψ by setting

$$\psi(x,k,\epsilon) = \pi^{-1}[\phi(x,k,\epsilon) - kx], \quad x \in \mathbb{R}^+, \quad k \in \mathbb{R}^+, \quad \epsilon \in [0,1[. \tag{4.4}$$

In view of (4.2) the function $\psi(\cdot, k, \epsilon)$ is the unique solution in $AC(\mathbb{R}^+)$ of (2.3) and (4.3) implies (2.4).

We now establish assertion (ii) by showing that the theorem stated in the previous section applies to (2.3) with

$$r(x,y) = R(|x-y|), \quad x, y \in \mathbb{R}^+, \tag{4.5}$$

and

$$F(x,\psi) = \pi^{-1}k\sin^2(\pi\psi + kx), \quad x \in \mathbb{R}^+, \quad \psi \in \mathbb{R}, \tag{4.6}$$

for $k \in \mathbb{R}^+$. With the exception of (3.1) and (3.2) all hypotheses of the diffusion limit theorem are evidently satisfied. To show that (3.1) and (3.2) also hold, we must first of all calculate the limits (3.8) and (3.9) with the data (4.5) and (4.6). The case $k = 0$ is trivial and we assume that $k > 0$.

Introducing new variables s and t by means of the formulas

$$s = y + z + 2\pi\psi/k, \quad t = y - z,$$

using some simple trigonometric identities and applying Fubini's theorem, we obtain from (3.8), (3.9) (4.5), and (4.6) that

$$a(\psi;k) = \lim_{X\to\infty} \frac{-k^2}{8\pi X}\int_0^X R(t)\int_{2x+2\pi\psi/k+t}^{2x+2\pi\psi/k+2X-t}\{\sin(2kt) - 2\sin(k[s+t]) + \sin(2ks)\}\,ds\,dt$$

$$= \frac{-k^2}{4\pi}\int_0^\infty R(t)\sin(2kt)dt$$

and

$$\sigma^2(\psi;k) = \lim_{X\to\infty}\frac{k^2}{8\pi^2 X}\int_0^X R(t)\int_{2x+2\pi\psi/k+t}^{2x+2\pi\psi/k+2X-t}[2 + \cos(2kt) + \cos(2ks) - 4\cos(kt)\cos(ks)]\,ds\,dt$$

$$= \frac{k^2}{4\pi^2}\int_0^\infty R(t)[2 + \cos(2kt)]dt$$

for $\psi \in \mathbb{R}$, $k > 0$, and $x \in \mathbb{R}^+$. The limits exist by virture of Lebesgue's dominated convergence theorem. In fact we have

$$\left| Xa(\psi;k) - \frac{k^2}{\pi}\int_x^{x+X}\sin(2\pi\psi + 2ky)\int_x^y R(y-z)\sin^2(\pi\psi + kz) \times dz\,dy \right|$$

$$= \frac{k^2}{4\pi}\left| -X\int_x^\infty R(t)\sin(2kt)\,dt - \int_0^X R(t)\{t\sin(2kt) - \int_{2x+2\pi\psi/k+t}^{2x+2\pi\psi/k+2X-t}[\sin(k[s+t]) - \tfrac{1}{2}\sin(2ks)]ds\}dt \right|$$

$$\leq \frac{k^2}{4\pi}\int_0^\infty |R(t)|(t + 5/2k)\,dt < \infty, \quad x, X \in \mathbb{R}^+, \quad \psi \in \mathbb{R}, \quad k > 0;$$

the last integral is bounded as a consequence of (4.5), (3.6), and (3.7). This shows that hypothesis (3.1) is satisfied. A similar estimate verifies (3.2), and thus assertion (ii) follows from the diffusion limit theorem.

B. Derivation of Theorem B

Our starting point is Eq. (2.7) with initial condition $u_0 = 0$ and with $m_j(\epsilon)$ given by (2.10). For the derivation of the first half of Theorem B it suffices to assume that $\mu: \mathbb{N}_0 \to [-1,1]$ is a numerical sequence; for the sake of convenience we extend the domain of μ to $\mathbb{N}$ by setting $\mu_0 = 0$.

We try to imitate the derivation of Theorem A, and as a first step in this direction we replace the angular frequency ω appearing in (2.7) by the wavenumber k via the dispersion relation associated with the perfect chain of harmonic oscillators ($\epsilon = 0$):

$$\omega = 2(f/m)^{1/2}\sin(k/2), \quad k \in [0,\pi[. \tag{4.7}$$

Next, adapting a variable transform used by Kohler and Papanicolaou,[12] we replace the u_j's by new variables A_j, B_j by means of the equations

$$u_j = A_j\cos(kj) + B_j\sin(kj), \quad j \in \mathbb{N}, \tag{4.8}$$

$$0 = [A_{j+1} - A_j]\cos(kj) + [B_{j+1} - B_j]\sin(kj), \quad j \in \mathbb{N}, \tag{4.9}$$

$$B_0 = B_1. \tag{4.10}$$

A simple induction argument shows that for $k \in]0,\pi[$ the u_j's uniquely determine the A_j's and B_j's for $j \in \mathbb{N}$. Moreover, it follows from (2.7), (2.10), and (4.7)–(4.10) that the A_j's and B_j's satisfy the difference equations

$$A_{j+1} = A_j + \epsilon\tan(k/2)\mu_j[A_j\sin(2kj) + 2B_j\sin^2(kj)],$$
$$B_{j+1} = B_j - \epsilon\tan(k/2)\mu_j[2A_j\cos^2(kj) + B_j\sin(2kj)], \tag{4.11}$$

for $j \in \mathbb{N}$; we recall that $u_0 = \mu_0 = 0$.

We now define continuous piecewise linear functions $A, B: \mathbb{R}^+ \to \mathbb{R}$ by setting

$$A(x)=A_j+\check{x}\epsilon\tan(k/2)\mu_j[A_j\sin(2kj)+2B_j\sin^2(kj)],$$
$$B(x)=B_j-\check{x}\epsilon\tan(k/2)\mu_j[2A_j\cos^2(kj)+B_j\sin(2kj)], \tag{4.12}$$

for $x\in[j,j+1[$, $j\in\mathbb{N}$. It follows from (4.11) and (4.12) that if $A(x)=B(x)=0$ for some $x\in\mathbb{R}^+$, then $A_j=B_j=0$ for all $j\in\mathbb{N}$. Thus for any nontrivial solution $\{(A_j, B_j): j\in\mathbb{N}\}$ of (4.11) we have

$$[A(x)\cos(kx)+B(x)\sin(kx)]^2 + [B(x)\cos(kx)-A(x)\sin(kx)]^2>0,$$

for all $x\in\mathbb{R}^+$, and therefore it is legitimate to introduce new dependent variables $r: \mathbb{R}^+\to]0,\infty[$ and $\phi: \mathbb{R}^+\to\mathbb{R}$ by means of the equations

$$r(x)\sin\phi(x)=A(x)\cos(kx)+B(x)\sin(kx),\quad x\in\mathbb{R}^+,$$
$$r(x)\cos\phi(x)=-A(x)\sin(kx)+B(x)\cos(kx),\quad x\in\mathbb{R}^+; \tag{4.13}$$

clearly, r, $\phi\in AC(\mathbb{R}^+)$. The function ϕ is not uniquely determined by (4.13); for the sake of convenience we choose the ϕ with $\phi(0)=0$. Then, by a straightforward calculation, it follows from (4.11)–(4.13) that ϕ satisfies the initial-value problem

$$\phi'(x)=k+2\epsilon\tan(k/2)\mu_{\hat{x}}\sin^2(\phi(x)-k\check{x}),\quad 0<x\neq\hat{x}$$
$$\phi(0)=0. \tag{4.14}$$

This initial-value problem corresponds to (4.2) and we can now proceed as we did in Sec. 4A.

Let $\phi(\cdot,k,\epsilon): \mathbb{R}^+\to\mathbb{R}$ denote the unique solution in $AC(\mathbb{R}^+)$ of (4.14), $k\in[0,\pi[$, $\epsilon\in[0,1[$, and denote by ϕ_2 its derivative with respect to k. Then, differentiating (4.14) with respect to k, interchanging the order of differentiation on the left-hand side (which is legitimate) and solving the resulting linear first-order initial-value problem for $\phi_2(\cdot,k,\epsilon)$, we arrive at the identity

$$\phi_2(x,k,\epsilon)=\int_0^x\left(1+\epsilon\cos^{-2}(k/2)\mu_{\hat{t}}\sin^2(\phi(t,k,\epsilon)-k\check{t})+\check{t}\frac{d}{dt}\right)e^{f(t,x)}\,dt,$$

for $x\in\mathbb{R}^+$, $k\in[0,\pi[$, and $\epsilon\in[0,1[$. Here, suppressing the variables k and ϵ, we have used the abbreviation

$$f(t,x)=2\epsilon\tan(k/2)\int_t^x\mu_{\hat{s}}\sin(2\phi(s,k,\epsilon)-2k\check{s})\,ds$$
$$0\le t\le x<\infty.$$

Integrating by parts and recalling the fact that $\mu_0=0$, we conclude that for $L\in\mathbb{N}_0$,

$$\begin{aligned}\phi_2(L+1,k,\epsilon)&=1+\sum_{j=1}^{L}e^{f(j,L+1)}[1+\epsilon\cos^{-2}(k/2)\mu_j\\&\quad\times\int_j^{j+1}\sin^2(\phi(t,k,\epsilon)-k\check{t})e^{-f(j,t)}dt]\\&\ge 1+\sum_{j=1}^{L}e^{f(j,L+1)}[1-\epsilon\cos^{-2}(k/2)\\&\quad\times\int_j^{j+1}\exp\{2\epsilon\tan(k/2)(t-j)\}dt]\\&=1+\left(1-\frac{\exp[2\epsilon\tan(k/2)]-1}{\sin k}\right)\sum_{j=1}^{L}e^{f(j,L+1)}\ge 1,\end{aligned}$$

provided that $k\in]0,k_\epsilon]$. Hence, for $L\in\mathbb{N}_0$ and $\epsilon\in[0,1[$, the function $\phi(L+1,\cdot,\epsilon): [0,\pi[\to\mathbb{R}$ is strictly increasing on $[0,k_\epsilon]$. In view of (4.8) and (4.11)–(4.13) the sequence $\{u_j(\omega,L,\epsilon): j\in\mathbb{N}\}$ is a nontrivial solution of (2.7) and (2.8) iff $\phi(L+1,k,\epsilon)=n\pi$ for some $n\in\mathbb{N}$. Consequently,

$$N(\Omega;L,\epsilon)=[\pi^{-1}\phi(L+1,K,\epsilon)]^{\wedge}, \tag{4.15}$$

provided that $K=2\arcsin(\Omega[m/4f]^{1/2})\in]0,k_\epsilon]$.

Again we introduce a new variable ψ by (4.4). Now it follows from (4.14) and $\mu_0=0$ that $\psi(\cdot,k,\epsilon)$ is the unique solution in $AC(\mathbb{R}^+)$ of (2.11), and (4.15) implies (2.12).

We now proceed to establish assertion (ii) by showing that the diffusion limit theorem applies to (2.11) with $\nu(x)=\mu_{\hat{x}}$, $x\ge1$,

$$r(x,y)=R_{\hat{x}-\hat{y}},\quad x\ge1,\quad y\ge1, \tag{4.16}$$

and

$$F(x,\psi)=2\pi^{-1}\tan(k/2)\sin^2(\pi\psi+k\hat{x}),\quad x\ge1,\quad \psi\in\mathbb{R}. \tag{4.17}$$

Without loss of generality we can assume that $\Lambda=[-1,1]^{\mathbb{N}_0}$ and thus ν has the required regularity property (Ref. 15, p. 363); the process ν obviously also satisfies the mixing condition. Moreover, F given by (4.17) has the stipulated differentiability and boundedness properties, so that we only have to show that the limits (3.8) and (3.9) exist and that they satisfy all requirements stated in Sec. 3. Again, the case $k=0$ is trivial and we assume that $k\in]0,\pi[$.

In order to calculate the limit (3.8), we first observe that a straightforward calculation yields the estimate

$$\begin{aligned}&\Bigg|\int_x^{x+X}\sin(2\pi\psi+2k\hat{y})\int_x^y R_{\hat{y}-\hat{z}}\sin^2(\pi\psi+k\hat{z})\,dz\,dy\\&\quad-\sum_{i=\hat{x}}^{\hat{x}+\hat{X}}\sin(2\pi\psi+2ki)\left(\frac{R_0}{2}\sin^2(\pi\psi+ki)+\sum_{j=\hat{x}}^{i-1}R_{i-j}\right.\\&\quad\left.\times\sin^2(\psi\pi+kj)\right)\Bigg|\le 2\sum_{i=0}^{\infty}|R_i|<\infty,\quad x\ge1,\quad X\in\mathbb{R}^+,\quad\psi\in\mathbb{R},\\&\qquad k\in]0,\pi[;\end{aligned} \tag{4.18}$$

the convergence of the series follows from (4.16), (3.6), and the hypothesis that $\rho^{1/2}\in l^1(\mathbb{N}_0)$. Next, using some well-known trigonometric identities,[16] we see that for $m, M\in\mathbb{N}_0$, $\psi\in\mathbb{R}$, and $k\in]0,\pi[$,

$$\begin{aligned}&\sum_{i=m}^{m+M}\sin(2\pi\psi+2ki)\left(\frac{R_0}{2}\sin^2(\pi\psi+ki)+\sum_{j=m}^{i-1}R_{i-j}\sin^2(\psi\pi+kj)\right)\\&=\tfrac{1}{8}\sum_{n=-M}^{M}R_nS(|n|,M;m,\psi,k),\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}S(|n|,M;m,\psi,k)&=4\sum_{j=0}^{M-|n|}\sin(2\pi\psi+2k\{m+|n|+j\})\\&\quad\times\sin^2(\pi\psi+k\{m+j\})\\&=\sum_{j=0}^{M-|n|}[2\sin(2\pi\psi+2k\{m+|n|+j\})-\sin(2k|n|)\\&\quad-\sin(4\pi\psi+2k\{2m+|n|+2j\})]\\&=\begin{cases}-(M-|n|)\sin(2k|n|)+E(|n|,M;m,\psi,k), & k\neq\pi/2,\\-(M-|n|)(-1)^n\sin(4\pi\psi)+E(|n|,M;m,\psi,\pi/2), & k=\pi/2,\end{cases}\end{aligned} \tag{4.20}$$

with

$$C_k=\sup\{|E(|n|,M;m,\psi,k)|:n\in\{0,\ldots,M\},\quad M,m\in\mathbb{N}_0,\ \psi\in\mathbb{R}\}<\infty. \tag{4.21}$$

It now follows from (3.8), (4.16)–(4.21), and Lebesgue's dominated convergence theorem that for $\psi\in\mathbb{R}$,

$$a(\psi;k)=\begin{cases}-\pi^{-1}\tan^2(k/2)\sum_{n=1}^{\infty}R_n\sin(2kn), & k\neq\pi/2,\\ -\frac{S_a}{2\pi}\sin(4\pi\psi), & k=\pi/2.\end{cases} \tag{4.22}$$

Moreover, it follows from (4.20)–(4.22) that for M, $m\in\mathbb{N}_0$ and $\psi\in\mathbb{R}$,

$$\left|Ma(\psi;k)-\frac{\tan^2(k/2)}{2\pi}\sum_{n=-M}^{M}R_mS(|n|,M;m,\psi,k)\right|$$
$$\leqslant\frac{\tan^2(k/2)}{2\pi}\sum_{n=-\infty}^{\infty}|R_n|(n+C_k)<\infty,\quad k\in]0,\pi[, \tag{4.23}$$

the convergence of the series being a consequence of (4.16), (3.6), and (3.7). Estimates (4.18) and (4.23) imply (3.1).

The limit (3.9) is calculated in the same way. Estimate (4.18) and identity (4.19) are still valid when $\sin(2\pi\psi+2k\hat{y})$ and $\sin(2\pi\psi+2ki)$ are replaced by $\sin^2(\pi\psi+k\hat{y})$ and $\sin^2(\pi\psi+ki)$ respectively, where now[16]

$$\begin{aligned}S(|n|,M;m,\psi,k)&=4\sum_{j=0}^{M-|n|}\sin^2(\pi\psi+k\{m+|n|+j\})\sin^2(\pi\psi+k\{m+j\})\\&=\tfrac{1}{2}\sum_{j=0}^{M-|n|}[2-2\cos(2\pi\psi+2k\{m+|n|+j\})\\&\quad-2\cos(2\pi\psi+2k\{m+j\})+\cos(2k|n|)\\&\quad+\cos(4\pi\psi+2k\{2m+|n|+2j\})]\\&=\begin{cases}\frac{M-|n|}{2}[2+\cos(2k|n|)]+E(|n|,M;m,\psi,k), & k\neq\pi/2,\\ \frac{M-|n|}{2}[2+(-1)^n\{1+\cos(4\pi\psi)\}]+E(|n|,M;m,\psi,\pi/2), & k=\pi/2,\end{cases}\end{aligned}$$

and (4.21) still holds. We thus infer from Lebesgue's dominated convergence theorem that the limit (3.9) exists and that for $\psi\in\mathbb{R}$,

$$\sigma^2(\psi;k)=\begin{cases}\frac{\tan^2(k/2)}{2\pi^2}\sum_{n=-\infty}^{\infty}R_n[2+\cos(2kn)], & k\neq\pi/2,\\ \frac{1}{\pi^2}[2S_e-S_a\sin^2(2\pi\psi)], & k=\pi/2.\end{cases} \tag{4.24}$$

An estimate similar to (4.23) finally shows that (3.2) holds.

The functions $a(\cdot;k)$ and $\sigma(\cdot;k)$ given by (4.22) and (4.24) respectively have bounded continuous derivatives of all orders on $\mathbb{R}$, which establishes the applicability of the diffusion limit theorem. Assertion (ii) thus follows, $\Psi(\cdot,k)$ being defined as the solution of (3.4) with coefficients $a(\cdot;k)$ and $\sigma(\cdot;k)$ given by (4.22) and (4.24) respectively. For $k\neq\pi/2$ and $(k,S_e)=(\pi/2,0)$ these coefficients are constant (0 in the second case, cf. Remark 2 in Sec. 2B), and (2.14) and the second half of (2.15) follow. In the case $k=\pi/2$ and $S_e>0$ the transformation technique a) on p. 34 of Ref. 13 yields the first half of (2.15). [To apply this technique in the case $S=0$, observe that then the Itô equation has stationary points at $-\frac{1}{4}$ and $\frac{1}{4}$ and that therefore, as a consequence of the uniqueness of solutions (Ref. 13, p. 40), $|\Psi(\cdot,\pi/2)|<\frac{1}{4}$ with probability 1.] Of course, (2.15) can also be verified *a posteriori* by calculating the stochastic differential of $\Psi(\cdot,\pi/2)$ with the aid of Itô's formula (Ref. 13, p. 24).

5. CONCLUDING REMARKS

The initial value problem (4.14), or the equivalent problem (2.11), can be solved analytically for each sequence $\{\mu_j:j\in\mathbb{N}_0\}$ in $[-1,1]$. Recalling that we have to choose $\mu_0=0$, we see that the solution $\phi(\cdot,k,\epsilon)$ in $AC(\mathbb{R}^+)$ of (4.14) is given by

$$\phi(x,k,\epsilon)=k(x-j)+\pi[\pi^{-1}\phi(j,k,\epsilon)]^{\wedge}+\operatorname{arc\,cot}[\cot\phi(j,k,\epsilon)-2\epsilon\mu_j(x-j)\tan(k/2)],$$

for $x\in[j,j+1]$ and $j\in\mathbb{N}$, where $\phi(0,k,\epsilon)=0$ for $k\in[0,\pi[$ and $\epsilon\in[0,1[$. Here, arc cot denotes the principal branch of the inverse cotangent, which is defined on the extended real line and whose range is the interval $[0,\pi]$. Moreover, the convention $\cot(n\pi)=+\infty$, $n\in\mathbb{N}$, is used.

For integer values of x we obtain the recursion formula

$$\begin{aligned}\phi(1,k,\epsilon)&=k,\\ \phi(L+1,k,\epsilon)&=k+\pi[\pi^{-1}\phi(L,k,\epsilon)]^{\wedge}\\&\quad+\operatorname{arc\,cot}[\cot\phi(L,k,\epsilon)-2\epsilon\mu_L\tan(k/2)]\\&=k+\phi(L,k,\epsilon)\\&\quad+\int_{\cot\phi(L,k,\epsilon)-2\epsilon\mu_L\tan(k/2)}^{\cot\phi(L,k,\epsilon)}[1+t^2]^{-1}dt,\end{aligned} \tag{5.1}$$

valid for $L\in\mathbb{N}_0$, $k\in[0,\pi[$, and $\epsilon\in[0,1[$. The identities (5.1) yield the crude estimates

$$|\phi(L+1,k,\epsilon)-\phi(L,k,\epsilon)-k|<\min\{k,\pi(1-[\pi^{-1}\phi(L,k,\epsilon)]^{\vee})\}$$

and (inductively)

$$k<\phi(L+1,k,\epsilon)<k+L\pi \tag{5.2}$$

for $L\in\mathbb{N}_0$, $k\in]0,\pi[$, and $\epsilon\in[0,1[$.

In deriving (4.14), (5.1) and (5.2) we did not use the fact that $\mu_j\in[-1,1]$. Thus we can choose all $\mu_j\geqslant0$ and $\epsilon=1$, which corresponds to a chain of harmonic oscillators with (deterministic) masses $m_j=m(1+\mu_j)\geqslant m>0$, $j\in\mathbb{N}_0$. Setting $\phi_L(k)=\phi(L+1,k,1)$, we can rewrite (5.1) as

$$\begin{aligned}\phi_0(k)&=k,\\ \phi_L(k)&=k+\pi[\pi^{-1}\phi_{L-1}(k)]^{\wedge}\\&\quad+\operatorname{arc\,cot}[\cot\phi_{L-1}(k)-2\mu_L\tan(k/2)]\\&=k+\phi_{L-1}(k)+\int_{\cot\phi_{L-1}(k)-2\mu_L\tan(k/2)}^{\cot\phi_{L-1}(k)}[1+t^2]^{-1}dt,\\ &\quad L\in\mathbb{N}_0,\end{aligned} \tag{5.3}$$

for $k\in[0,\pi[$, where we adopt the same conventions as above. It now follows from (4.14) and (5.2) that ϕ_L: $[0,\pi[\to[0,(L+1)\pi[$ is strictly increasing and that

$$(L+1)k\leqslant\phi_L(k)<k+L\pi,\quad k\in[0,\pi[, \tag{5.4}$$

for each $L \in \mathbb{N}_0$; in particular, $\lim_{k\to\pi}\phi_L(k) = (L+1)\pi$. In view of (4.8) and (4.11)–(4.13) this proves (2.9). More specifically, the normal frequencies $\omega_j(L)$ of (2.7) and (2.8) [with $\epsilon = 1$ and $m_j = m(1+\mu_j) \geq m > 0$] are given by

$$\omega_j(L) = 2(f/m)^{1/2}\sin(\phi_L^{-1}(j\pi)/2), \quad j = 1, 2 \ldots, L, \quad L \in \mathbb{N}_0, \tag{5.5}$$

and $[\pi^{-1}\phi_L(k)]^\wedge$ is the number of $\omega_j(L)$'s in $[0, 2(f/m)^{1/2} \times \sin(k/2)]$.

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Explicit results for the quantum-mechanical energy states basic to a finite square-well potential

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The theory of complex variables is used to establish explicit expressions for the discrete energy states relevant to a square-well potential.

INTRODUCTION

As one of the first examples of the principles of quantum mechanics, Schiff[1] solves the Schrödinger equation for a square-well potential,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}U(x)+V(x)U(x)=EU(x), \tag{1}$$

where

$$V(x)=0, \quad x\in(-a,a), \tag{2a}$$

and

$$V(x)=V_0, \quad |x|>a, \tag{2b}$$

to find the discrete energy levels. Thus, on establishing the solution to Eq. (1), subject to $U(x)$ and $U'(x)$ being continuous at $x=\pm a$, Schiff[1] finds that the bound states $(E<V_0)$ can be expressed as

$$E_j=\frac{\hbar^2}{2ma^2}\xi_j^2, \quad j=1,2,3,\ldots,n, \tag{3}$$

where ξ_j denotes one of the n positive solutions of

$$\xi\tan\xi=(A^2-\xi^2)^{1/2}, \quad j \text{ odd}, \tag{4a}$$

$$\xi\cot\xi=-(A^2-\xi^2)^{1/2}, \quad j \text{ even}, \tag{4b}$$

where $A\in(0,n\pi/2)$ is given by

$$A=\frac{a}{\hbar}\sqrt{2mV_0}. \tag{5}$$

Here we wish to report explicit solutions of Eqs. (4) that yield exact closed-form results for the discrete energy levels.

ANALYSIS

In order to relate the roots of Eqs. (4) to the zeros of a sectionally analytic function, we wish to consider

$$\Lambda_k(z)=-iAz+D(z)\left(k\pi i-\frac{1}{2}\int_{-1}^{1}\frac{d\mu}{\mu-z}\right), \tag{6}$$

where k is a constant and

$$D(z)=(z^2-1)^{1/2}. \tag{7}$$

Here we use a branch of the square root function such that $D(z)=-D(-z)$ is analytic in the complex plane cut from -1 to 1 along the real axis and $\arg D(z)\in(-\pi,\pi)$. We conclude that $\Lambda_k(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis. Further, we can use the argument principle[2] to deduce that $\Lambda_k(z)$ has one zero in the finite cut plane for $k\in(-\frac{1}{2},\frac{1}{2})$, that $\Lambda_k(z)$ has two zeros for $k>\frac{1}{2}$ and that $\Lambda_k(z)$ has no zeros for $k<-\frac{1}{2}$.

We first consider $k=0$ and note[3] that

$$\frac{\Lambda_0(k)}{z-z_0}=K_0X_0(z), \tag{8}$$

where $X_0(z)$ is a canonical solution of the Riemann problem[4] defined by

$$X_0^+(\tau)=G_0(\tau)X_0^-(\tau), \quad \tau\in(-1,1). \tag{9}$$

Here K_0 is a constant to be determined and

$$G_0(\tau)=\frac{\Lambda_0^+(\tau)}{\Lambda_0^-(\tau)}, \tag{10}$$

where the $\pm$ superscripts are used to denote the limiting values as z approaches the branch cut $[-1,1]$ from above and below. The Riemann problem defined by Eq. (9) can be solved, as discussed by Muskhelishvili,[4] to yield

$$X_0(z)=\exp\left[\frac{1}{2\pi i}\int_{-1}^{1}\log G_0(\tau)\frac{d\tau}{\tau-z}\right], \tag{11}$$

where we use the log function such that $\arg\log G_0(\tau)$ varies continuously from 0 at $\tau=-1$. If we now investigate Eq. (8) as $|z|\to\infty$ we find that $K_0=-iA$ and that $z_0=-iy_0$, where

$$y_0=\frac{1}{A}-\frac{1}{2\pi}L_0, \tag{12}$$

with, in general,

$$L_k=\int_0^1\ln(\{(\tfrac{1}{2}-k)^2\pi^2(1-t^2)+[(1-t^2)^{1/2}\tanh^{-1}(t)-At]^2\}/\{(\tfrac{1}{2}+k)^2\pi^2(1-t^2)+[(1-t^2)^{1/2}\tanh^{-1}(t)+At]^2\})\,dt. \tag{13}$$

It is now apparent that

$$\xi_1=\operatorname{Tan}^{-1}\left(\frac{1}{y_0}\right) \tag{14}$$

is the first of the desired solutions of Eqs. (4).

For $k>\frac{1}{2}$, we can readily generalize Eq. (8) to obtain

$$\frac{\Lambda_k(z)}{(z-z_{k,1})(z-z_{k,2})}=i(k\pi-A)X_k(z), \tag{15}$$

where $z_{k,1}$ and $z_{k,2}$ are the two zeros of $\Lambda_k(z)$. Here we write a canonical solution of the Riemann problem defined by the $k>\frac{1}{2}$ generalization of Eq. (9) as

$$X_k(z)=\frac{1}{z-1}\exp\left[\frac{1}{2\pi i}\int_{-1}^{1}\log G_k(\tau)\frac{d\tau}{\tau-z}\right]. \tag{16}$$

Here

$$G_k(\tau)=\frac{\Lambda_k^+(\tau)}{\Lambda_k^-(\tau)}, \tag{17}$$

and again we use $\log G_k(\tau)$ such that $\arg\log G_k(\tau)$ varies continuously from 0 at $\tau=-1$. With $\log G_k(\tau)$ so defined, we deduce, for $k>\frac{1}{2}$, that $\log G_k(1)=2\pi i$; and thus, as discussed by Muskhelishvili,[4] the factor $(z-1)$ appears explicitly in Eq. (16) to insure that $X_k(z)$ does not vanish at $z=1$. We can now investigate Eq. (15) for large $|z|$ to deduce that $z_{k,1}=-iy_{k,1}$ and $z_{k,2}=-iy_{k,2}$, where

$$y_{k,1}=B_k+(B_k^2+W_k)^{1/2} \tag{18a}$$

and

$$y_{k,2}=B_k-(B_k^2+W_k)^{1/2}. \tag{18b}$$

Here

$$B_k=\frac{1}{2}\left(\frac{1}{A-k\pi}-\frac{1}{2\pi}L_k\right) \tag{19}$$

and

$$W_k=\left(\frac{1}{A-k\pi}\right)\left(\frac{1}{2\pi}L_k+\frac{k\pi}{2}\right)-\frac{1}{8\pi^2}L_k^2-1+M_k. \tag{20}$$

In addition,

$$M_k=\frac{1}{\pi}\int_0^1 t\Theta(t)\,dt, \tag{21}$$

where

$$\Theta(t)=\tan^{-1}([\pi(1-t^2)\tanh^{-1}(t)-2kAt\pi(1-t^2)^{1/2}]/\{(\tfrac{1}{4}-k^2)\pi^2(1-t^2)-(1-t^2)[\tanh^{-1}(t)]^2+A^2t^2\}), \tag{22}$$

with $\Theta(0)=\pi$. If we now let $j=2k+1$, then the last $n-2$ desired positive solutions of Eqs. (4) can be expressed as

$$\xi_j=k\pi+\mathrm{Tan}^{-1}\left(\frac{1}{y_{k,1}}\right),\quad j=3,4,5,\ldots,n. \tag{23}$$

The case $k=\frac{1}{2}$ requires special attention since the corresponding $G_k(\tau)$ vanishes on the cut. We thus find it convenient to introduce

$$\Omega(z)=\Lambda_{1/2}(z)\Lambda_{1/2}(-z) \tag{24}$$

and consider the Riemann problem defined by

$$Y^+(\tau)=\frac{\Omega^+(\tau)}{\Omega^-(\tau)}Y^-(\tau),\quad \tau\in(-1,1). \tag{25}$$

We find we can write a canonical solution here as

$$Y(z)=\frac{1}{z-1}\exp\left[\frac{1}{\pi}\int_{-1}^1\phi(t)\frac{dt}{t-z}\right], \tag{26}$$

where

$$\phi(t)=\tan^{-1}\left(\frac{\pi(1-t^2)^{1/2}}{-(1-t^2)^{1/2}\tanh^{-1}(t)-At}\right), \tag{27}$$

with $\phi(-1)=0$. Thus, since $\Omega(z)$ has a zero at the origin and two additional zeros, $\pm z_{1/2}$, we can write

$$\frac{\Omega(z)}{z(z^2-z_{1/2}^2)}=\left(\frac{\pi}{2}-A\right)^2Y(z), \tag{28}$$

and let $|z|\to\infty$ to deduce that $z_{1/2}=\pm iy_{1/2}$, where

$$y_{1/2}=\left(\frac{2(\pi-A)}{2A-\pi}+\frac{2}{\pi}\int_0^1 t\phi(t)\,dt+\frac{4}{(2A-\pi)^2}\right)^{1/2}. \tag{29}$$

The positive solution of Eqs. (4) corresponding to $j=2$ thus is given by

$$\xi_2=\frac{\pi}{2}+\mathrm{Tan}^{-1}\left(\frac{1}{y_{1/2}}\right). \tag{30}$$

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Symmetry conditions and non-Abelian gauge fields[a)]

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Classical gauge fields are envisaged in the context of fibre bundle theory. General symmetry conditions are found which lead to an Abelian holonomy group. This, in turn, has important consequences on the solutions of the gauge field equations: Symmetric solutions have unphysical properties. Non-Abelian holonomy groups are thus needed.

1. INTRODUCTION

Recently, one has witnessed a great interest in the classical solutions to the non-Abelian gauge field equations. Powerful methods for finding the solutions of such nonlinear field equations are still lacking. The purpose of this paper is an attempt to get information on the solutions without solving the equations.

We take the point of view recently advocated by Wu and Yang[1] (preceded by many others) and consider the gauge fields as connections on a fibre bundle. The field strength tensor operator $\Phi_{\mu\nu}$ is then an internal curvature tensor. As emphasized by Loos,[2,3] important information on the properties of the gauge fields can be obtained from an investigation of the holonomy group of the fibre bundle as opposed to considering the structural (or gauge) group alone: Loos[2] has shown that spherically symmetric analytic solutions of the point charge Yang–Mills[4] equations have an Abelian internal holonomy group and Uzes[5] proved that for gauge fields with an Abelian internal holonomy group, the Yang–Mills equations and the Bianchi identities reduce to Maxwell's equations (possibly with magnetic monopoles). These two results by themselves indicate the relevance of the question as to whether the internal holonomy group is Abelian or not. There is another reason for studying this point.

Recently, Eguchi[6] has presented a classification of the unquantized Yang–Mills fields for the SU(2) gauge group. The differential properties of the Yang–Mills fields are given by the holonomy group. In his Table I, one notices immediately that there are only two classes in the holonomy group column: an Abelian holonomy group and a three-dimensional one. Thus it becomes important to find criteria for an Abelian holonomy group. (His claim that the Yang–Mills fields produced by classical point charges lead necessarily to an Abelian holonomy group should be taken with caution: Loos[7] has explicitly given a solution for a point charge with a non-Abelian holonomy group.)

Our task will be to show that very general symmetry conditions imposed on the field strengths lead to an Abelian internal holonomy group. Our results are valid for three cases:

(a) the sourceless Yang–Mills field,

(b) the source is C^∞ and vanishes in some small domain,

(c) a special case when the current density is proportional to a given component of the field strength.

Furthermore, we explicitly show in the model used by 't Hooft[8] to obtain his monopole solution that for Wu–Yang's[9] solution, the holonomy group is Abelian. On the other hand, Prasad and Sommerfield's[10] solution leads to a non-Abelian holonomy group as it does not satisfy our symmetry conditions. The presence of scalar fields seems to be important in this regard.

The paper is organized as follows. In Sec. 2, we introduce some definitions and our notation. Section 3 analyzes the conditions to obtain an Abelian holonomy group; it is extended in Sec. 4 in which symmetry conditions are provided. Finally Sec. 5 is devoted to examples using 't Hooft's[8] Langrangian. It is followed by a short conclusion.

2. SOME DEFINITIONS AND NOTATION

The gauge group will be denoted by G and its elements by g. The generators L_a of the Lie algebra L of the group G allow an expansion of the field strength tensor operator $\Phi_{\mu\nu}$ as follows ($a=1,\ldots,n$, where n is the dimension of the algebra):

$$\Phi_{\mu\nu}=F_{\mu\nu}^{a}L_a. \tag{1}$$

One has a similar expansion for the gauge fields,

$$\Gamma_\mu=B_\mu^a L_a. \tag{2}$$

The parameters Γ_μ determine an internal linear connection and we obtain

$$\Phi_{\mu\nu}=\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu-ie[\Gamma_\mu,\Gamma_\nu]. \tag{3}$$

The covariant derivative of a covariant internal vector A^a is given by

$$\nabla_\mu A^a=\partial_\mu A^a-ie(\Gamma_\mu)_{ac}A_c. \tag{4}$$

The covariant derivative of an internal linear operator Ω is given by

$$\nabla_\mu\Omega=\partial_\mu\Omega-ie[\Gamma_\mu,\Omega]. \tag{5}$$

After these preliminaries, let us introduce the internal holonomy group.[11] We first construct a fibre bundle E such that the base space will be the (flat) Minkowski space M, or some submanifold in that space. We can think of the fibre F as a vector space of the internal degree of freedom. It is helpful to have Fig. 1 in mind. Furthermore, the fibre F and the group G are related

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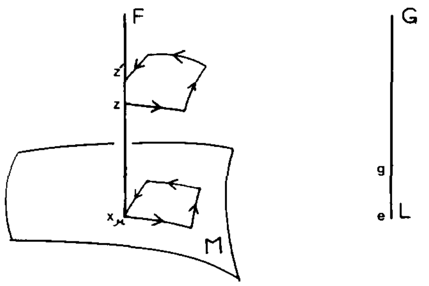


FIG. 1. The fibre bundle $E(M, G, F)$. L is the Lie algebra and e is the identity element.

by a set of differentiable homeomorphisms. Any path in the fibre bundle E can be projected onto the Minkowski space. A vertical displacement (along the fibre) is achieved by an element of the gauge group,

$$z' = zg^{-1} \tag{6}$$

where z and z' are points in the space E belonging to the same fibre F. Given a point z of E, the holonomy group for a given connection Γ_μ at x_β is the set of elements g of the gauge group G such that internal vectors at z can be transported to the point zg^{-1} in a parallel transfer. If we choose the transfer route to be an infinitesimal parallelogram, it is simple to show that locally, the corresponding element h of the holonomy group will reduce to

$$h = 1 + \Phi_{\mu\nu}\, dS^{\mu\nu}, \tag{7}$$

where $dS^{\mu\nu}$ is the area of the parallelogram in Minkowski space. One notices immediately that h is nothing else than Wu–Yang's[1] phase factor for an infinitesimal parallelogram. We thus see that at least some generators of the internal holonomy group are given by the components of the curvature tensor.

We take the basis manifold to be simply connected and our approach becomes natural if we consider the restricted holonomy group $H^0(x_\lambda)$ at x_λ. It is defined as the set of elements g of the structure group such that z and zg^{-1} are connected by a horizontal path whose projection on the basis manifold is a closed loop homotopic to zero. Introduce a sequence of shrinking Lie groups $H^0(U_i, x_\lambda)$, $i = 1, 2, \cdots$, where U_i is a family of connected open sets such that $U_1 \supset U_2 \supset \cdots$ and $\cap_i^\infty U_i = x_\lambda$. The intersection of these Lie groups is itself a Lie group $H^*(x_\lambda)$ called the local holonomy group at x_λ. The infinitesimal holonomy group $H'(x_\lambda)$ is defined in the case the basis manifold and the connection are of class C^∞. It is obtained by giving its Lie algebra and it corresponds to the connected piece containing the identity. We have

$$H'(x_\lambda) \subset H^*(x_\lambda) \subset H^0(M, x_\lambda).$$

We base our investigation on the fundamental result (proven by Nijenhuis[12]) that $H'(x_\lambda)$ is completely determined by the curvature tensor and its covariant derivatives. We seek information on $H^0(M, x_\lambda)$ by looking at $H'(x_\lambda)$ alone. Some results are available; although most of them have been proved in detail for affine connections, they can be extended to general linear fibre bundles without any serious modification. Our first requirement will be that for each point in Minkowski space, the local and infinitesimal holonomy groups coincide. Theorem 6 of Nijenhuis[12] then implies

$$H^0(M, x_\mu) = H^*(x_\mu) = H'(x_\mu).$$

It is therefore equivalent to study the infinitesimal holonomy group alone. Symmetry conditions will constrain the elements of the infinitesimal holonomy group algebra and the restricted holonomy group H^0 is determined in one point. One then uses the fact that the restricted holonomy groups for different points are isomorphic.

In non-Abelian gauge field theories, semisimple or direct products of semisimple gauge groups are favored in the construction of models in regard with the "naturalness" of Cabibbo universality and the quantization of charge.[13] Holonomy groups are subgroups of the structure group and we want to consider Abelian holonomy groups. This makes the study of holonomy groups more difficult: Their Killing form is nonregular and the adjoint representation of the Lie algebra is not faithful. Treat[14] obtains a short-range pointlike solution with the help of a nonsemisimple holonomy group which is also non-Abelian. In the conclusion of the same paper he is led to assert that, as a result of the nonsemisimple character of the holonomy group, there are nonvanishing components of the field which do not contribute to the energy density. This ghostlike behavior is, in our case, a direct consequence of the Abelian property of H^0. This comes about because, for non-Abelian gauge fields, the energy density involves the Killing form. (We exclude pure electromagnetism from our considerations.) This emphasizes another purpose of this investigation: Symmetric solutions for the curvature tensor should be avoided because they have components which do not contribute to the energy density.

3. CONDITIONS

We want to know under which conditions one has an Abelian holonomy group. A first step in that direction is to analyze the commutator of the field strengths

$$[\Phi_{\mu\nu}, \Phi_{\lambda\rho}] = 0. \tag{8}$$

The meaning of relation (8) is clear: It is the integrability condition for the quantities $\Phi_{\mu\nu}$. Consider formula (5) and take the covariant derivative on both sides,

$$\nabla_\lambda \nabla_\sigma \Omega = \partial_\lambda \partial_\sigma \Omega - [\partial_\lambda \Gamma_\sigma, \Omega] - [\Gamma_\sigma, \partial_\lambda \Omega] - [\Gamma_\lambda, \partial_\sigma \Omega] + [\Gamma_\lambda, [\Gamma_\sigma, \Omega]].$$

Alternate,

$$(\nabla_\lambda \nabla_\sigma - \nabla_\sigma \nabla_\lambda)\Omega = [\Omega, \Phi_{\lambda\sigma}]. \tag{9}$$

Choose Ω to be the quantity $\Phi_{\mu\nu}$ and we obtain a general condition for the commutator,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\Phi_{\lambda\rho} = [\Phi_{\lambda\rho}, \Phi_{\mu\nu}] = 0. \tag{10}$$

The first derivatives of the field strengths are not independent but satisfy the Bianchi identities

$$\nabla_\mu \Phi_{\nu\lambda} + \nabla_\nu \Phi_{\lambda\mu} + \nabla_\lambda \Phi_{\mu\nu} = 0. \tag{11}$$

Furthermore, for a flat Minkowski space, one can easily show that

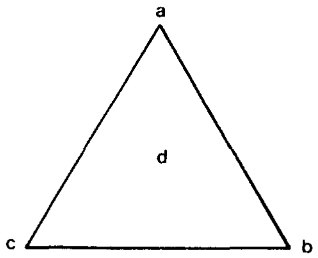


FIG. 2. The Bianchi identities.

$$\nabla_\mu \nabla_\nu \Phi_{\mu\nu} = 0. \tag{12}$$

We will make use of the Bianchi identities but for our purpose they are redundant. For instance, if two indices are the same in Eq. (11), we get a trivial identity,

$$\nabla_\mu \Phi_{\mu\lambda} = \nabla_\mu \Phi_{\mu\lambda}. \tag{13}$$

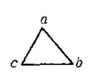
Simple combinatorial considerations show that one obtains only four nontrivial independent identities. We found it useful to express them by four triangles as follows (see Fig. 2):

(1) the indices $a, b, c, d = 1, 2, 3, 4$ are all different.

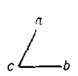
(2) the indices are to be read clockwise. The index in the center of the triangle is just a label and should only be considered to identify one of the four nontrivial identities.

(3) each vertex with index a is the covariant derivative ∇_a.

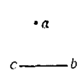
(4) each line joining the vertices a and b is Φ_{ab} (a missing line between vertices a and b means $\Phi_{ab} = 0$),



$$: \nabla_a \Phi_{bc} + \nabla_b \Phi_{ca} + \nabla_c \Phi_{ab} = 0, \tag{14}$$



$$: \nabla_b \Phi_{ca} + \nabla_a \Phi_{bc} = 0, \tag{15}$$



$$: \nabla_a \Phi_{bc} = 0. \tag{16}$$

Let us concentrate on the commutator (8). The quantities $\Phi_{\mu\nu}$ are antisymmetric as well as the commutator of any two such quantities. It is easy to see that one can form 15 independent commutators. Since each component of $\Phi_{\mu\nu}$ enters in five commutators (there are six independent components), each time one component vanishes, the number of surviving commutators will obey the following rule: If the number of vanishing $\Phi_{\mu\nu}$'s is n $(0 \leq n \leq 5)$, the number of surviving commutators is

$$\tfrac{1}{2}(5 - n)(6 - n). \tag{17}$$

We also know that the commutators are not yet fully independent because they are related via the Bianchi identities as follows: The integrability condition (10) tells us that conditions on the commutators are conditions on the covariant derivatives of the field strengths. On the other hand, the vanishing of a given component of $\Phi_{\mu\nu}$ will bear on the Bianchi identities. If we write out the four possible triangles, we see that each line appears twice. A given component will thus vanish in two Bianchi identities. This shows an important relationship between the field strengths and their first covariant derivatives. In physical cases, we will always have the following form for nonvanishing covariant derivatives envisaged from symmetry considerations:

$$\nabla_a \Phi_{bc} = M^a_{bc} \Phi_{bc}, \tag{18}$$

where M^a_{bc} is some function of the coordinates in Minkowski space (no summation over repeated indices is intended). This will be shown in the next section. It is then sufficient to consider only the field strengths and their first covariant derivatives to obtain the complete Lie algebra. It may happen that M^a_{bc} vanishes and formula (18) reduces to formula (16).

4. SYMMETRY

We shall assume that an event in Minkowski space is localized by four curvilinear orthogonal coordinates: ξ_0, ξ_1, ξ_2, ξ_3. These coordinates will be accompanied by scale factors y_n such that a line element will be given by

$$dl^2 = y_0^2 d\xi_0^2 - \sum_{n=1}^{3} y_n^2 d\xi_n^2. \tag{19}$$

Let us be more precise as to what we mean by symmetry. We want to consider symmetries which give us conditions on the components of $\Phi_{\mu\nu}$ and their first covariant derivatives such that the commutators vanish. We define symmetry for two-dimensional C^∞ manifolds in Minkowski space. These surfaces are to be covered by two orthogonal coordinates, for each set of values of the other two coordinates. We need a two-dimensional shell to define nontrivial field strengths,

$$\Phi_{\xi_a \xi_b} = \Phi_{ab} \quad (a, b = 0, 1, 2, 3). \tag{20}$$

We introduce some auxiliary symmetry parameters α, β, γ, $\cdots$ corresponding to symmetry operations on a two-dimensional shell. One may distinguish two types of symmetries:

(a) *symmetries about a fixed point* (*isotropy*): one is at a point on the surface and the symmetry parameter is an angle α specifying the orientation of a tangent vector with respect to the local frame. For instance, the surface element $|d\xi_a \wedge d\xi_b|$ is independent of α provided the vectors $d\xi_i$ do not change their length. If we want an element h of the holonomy group to have the same symmetry property, it must satisfy

$$\frac{\partial}{\partial \alpha} h = 0. \tag{21}$$

We will soon see that this leads to formula (18) with $M^a_{bc} = 0$ for four components of the curvature tensor.

(b) *displacement symmetries* (*homogeneity*): in this case, the symmetry parameters are to be identified with the coordinates themselves. The condition that the holonomy group element does not change along the coordinate ξ_c is

$$\nabla_c(\Phi_{ab} dS^{ab}) = 0. \tag{22}$$

Consider a small piece of the two-dimensional shell,

$$\Delta S^{ab} = y_a y_b \Delta\xi_a \wedge \Delta\xi_b. \tag{23}$$

One finds that

$$\partial_c \Delta S^{ab} = [y_a^{-1} y_b^{-1} \partial_c (y_a y_b)] \Delta S^{ab}. \tag{24}$$

Condition (22) gives us

$$\nabla_c \Phi_{ab} = -\Phi_{ab} [y_a^{-1} y_b^{-1} \partial_c (y_a y_b)]. \tag{25}$$

This is exactly the form presented in formula (18). Actually one should consider the effects of symmetry on higher covariant derivatives of the field strength as well but condition (25) is sufficient to close the Lie algebra and it has direct geometrical significance. Let us define

$$M_{ab}^c \equiv -y_a^{-1} y_b^{-1} \partial_c (y_a y_b), \tag{26}$$

such that formula (25) becomes

$$\nabla_c \Phi_{ab} = M_{ab}^c \Phi_{ab}. \tag{27}$$

Substitute (27) into the integrability condition (10). This gives us

$$(\partial_d M_{ab}^c - \partial_c M_{ab}^d) \Phi_{ab} = 0. \tag{28}$$

This equation is always satisifed if (27) is true. We thus get a sufficient condition in the form of a theorem.

Theorem: If all the covariant derivatives $\nabla_c \Phi_{ab}$ are of the form $M_{ab}^c \Phi_{ab}$ with

$$M_{ab}^c = -y_a^{-1} y_b^{-1} \partial_c (y_a y_b)$$

the holonomy group is Abelian and perfect.[15]

We remark that higher covariant derivatives applied to the form (27) give us

$$\nabla_d \nabla_c \Phi_{ab} = (\nabla_d M_{ab}^c) \Phi_{ab} + M_{ab}^c \nabla_d \Phi_{ab}. \tag{29}$$

Since $\nabla_d M_{ab}^c$ is a commuting number one needs only to consider the commutators of the field strengths and their first covariant derivatives; we have shown that those quantities commute. The argument can be easily generalized to the covariant derivative of any order by induction. Thus the whole algebra is commutative and the theorem is proved. It requires all the components of Φ_{ab} to satisfy equations (26) and (27). In that case, all the holonomy group elements are invariant under parallel transfer along any of the coordinates ξ_i.

Can we obtain a weaker condition? Consider a symmetry shell covered by a mesh of curvilinear coordinates ξ_2 and ξ_3. Define two vectors, one of which dx_1^μ, is tangent to the shell. This is a trivial generalization of Loos[2] ansatz,

$$\begin{aligned} dx_1^\mu &= (0, 0, y_2 d\xi_2, y_3 d\xi_3), \\ dx_2^\mu &= (y_0 d\xi_0, y_1 d\xi_1, 0, 0). \end{aligned} \tag{30}$$

Introduce a symmetry parameter α with the help of a tangent vector as follows:

$$dz^\mu = (0, 0, \cos\alpha\, dz, \sin\alpha\, dz), \tag{31}$$

where α is the angle between dz^μ and the local frame. Comparing (30) and (31) we have

$$y_2 d\xi_2 = \cos\alpha\, dz, \quad y_3 d\xi_3 = \sin\alpha\, dz. \tag{32}$$

The holonomy group element generated by $\Phi_{\mu\nu}$ is

$$\begin{aligned} \Phi_{\mu\nu} dS^{\mu\nu} = {} & (\Phi_{20} y_0 d\xi_0 + \Phi_{21} y_1 d\xi_1) \cos\alpha\, dz \\ & + (\Phi_{30} y_0 d\xi_0 + \Phi_{31} y_1 d\xi_1) \sin\alpha\, dz. \end{aligned} \tag{33}$$

Symmetry condition (21) and the fact that $d\xi_1$ and $d\xi_0$ are linearly independent implies that

$$\Phi_{20} = \Phi_{21} = \Phi_{30} = \Phi_{31} = 0. \tag{34}$$

This also satisfies equation (18) trivially and using (15) we find that only one independent commutator involving the field strengths alone survives. It is

$$[\Phi_{01}, \Phi_{23}] = (\nabla_2 \nabla_3 - \nabla_3 \nabla_2) \Phi_{01}. \tag{35}$$

Two of the four nontrivial Bianchi identities reduce to the form (16),

$$\nabla_3 \Phi_{01} = \nabla_2 \Phi_{01} = 0. \tag{36}$$

The commutator (35) vanishes. Using symmetry condition (a) only, [Eq. (21)] we are able to conclude that the internal curvature tensor operator spans an Abelian subalgebra of the holonomy group algebra. The algebra is perfect. Only Φ_{01} and Φ_{32} are nonzero among the six independent $\Phi_{\mu\nu}$'s; we have just shown they commute. Symmetry condition (21) is very stringent indeed. Let us use symmetry condition (b) for the coordinates ξ_2 and ξ_3

$$\nabla_2 \Phi_{23} = M_{23}^2 \Phi_{23}, \quad \nabla_3 \Phi_{23} = M_{23}^3 \Phi_{23}. \tag{37}$$

We realize that those first derivatives are also members of the perfect Abelian subalgebra. We are thus left with two elements of the holonomy group, $\nabla_0 \Phi_{10}$ and $\nabla_1 \Phi_{10}$. Taking higher covariant derivatives of these elements will generate many terms. Using the equations of motion for the Yang–Mills fields one defines currents as follows:

$$\nabla_\mu \Phi_{\mu\nu} = J_\nu. \tag{38}$$

We end up with

$$\nabla_0 \Phi_{01} = J_1, \quad \nabla_1 \Phi_{01} = -J_0. \tag{39}$$

At this point, one has to make some assumptions for J_0 and J_1. We want to consider three cases:

(1) $J_0 = J_1 = 0$ everywhere (sourceless Yang–Mills field).

(2) J_0 and J_1 are C^∞ and vanish in some small region.

(3) $J_i = M_i \Phi_{01}$ with $i = 0, 1$. (40)

Consider that the surface is chosen in the region where the sources vanish or satisfy Eq. (40). The holonomy group is then Abelian and perfect at least at one point. Using the isomorphism of holonomy groups at different points in the fibre bundle, we are allowed to conclude that the holonomy group is Abelian and perfect everywhere. Let us summarize this in a theorem which is a generalization of Loos'[2] result.

Theorem: If there exists a two-dimensional symmetry manifold in Minkowski space on which the internal curvature satisfies the symmetry conditions (21) and (37), and if there is a region where the current densities J_0 and J_1 satisfy conditions (40), the internal (restricted) holonomy group algebra is Abelian and perfect.

5. EXAMPLES

Let us consider the following Lagrangian:

$$L=-\tfrac{1}{4}F_{\mu\nu}^{a}F_{\mu\nu}^{a}-\tfrac{1}{2}\nabla_\mu\varphi^a\nabla_\mu\varphi^a+\tfrac{1}{4}\mu^2\varphi_a^2-\tfrac{1}{8}\lambda\varphi_a^4. \tag{41}$$

It leads to 't Hooft's[8] monopole solution. Symmetry condition (21) implies that four among the six independent components of $\Phi_{\mu\nu}$ must vanish. We now want to investigate two types of solutions when only three components are nonvanishing.

(i) Let us take Wu–Yang's solution[9] as it is presented in Hsu.[16] The gauge group is SU(2). We want to show that the holonomy group is still Abelian and perfect. The equations of motion are given by:

$$\nabla_\mu\Phi_{\mu\nu}=-2e[\varphi,\nabla_\nu\varphi], \tag{42}$$

$$\nabla_\mu(\nabla_\mu\varphi)+(\tfrac{1}{2}\mu^2-\tfrac{1}{2}\lambda\varphi^2)\varphi=0, \tag{43}$$

$$\Phi_{\mu\nu}=\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu-ie[\Gamma_\mu,\Gamma_\nu]. \tag{44}$$

The solution we consider is given by

$$\Gamma_a=i[R,L_a]\frac{A}{er^2} \tag{45}$$

with

$$R=r_aL_a,\quad \Gamma_0=0,\quad A=1 \text{ or } 2,$$
$$[L_a,L_b]=i\epsilon_{abc}L_c,\quad a,b,c=1,2,3. \tag{46}$$

This solution satisfies the equations of motion except at the origin. With this ansatz it is easy to show that

$$\Phi_{ab}=\frac{2iA}{er^2}\left([L_a,L_b]+\frac{r_a}{r^2}[L_b,R]-\frac{r_b}{r^2}[L_a,R]-\frac{A}{2r^2}[[L_a,R],[L_b,R]]\right). \tag{47}$$

Furthermore one has,

$$[[L_1,R],[L_2,R]]=ir_3R,$$
$$[[L_2,R],[L_3,R]]=ir_1R, \tag{48}$$
$$[[L_3,R],[L_1,R]]=ir_2R,$$

and finally,

$$\Phi_{ab}=-\frac{2A}{er^4}(1-A/2)R\epsilon_{abc}r_c. \tag{49}$$

Obviously the holonomy group is Abelian if it is perfect. The algebra is reduced to a one-dimensional vector space pointing in the direction of R. Let us see that it is perfect indeed. An explicit calculation (with $A=1$, the other case being trivial), shows that

$$\nabla_\mu\Phi_{ab}=-\frac{\epsilon_{abc}}{er^4}\left(\delta_{\mu c}-\frac{3r_\mu r_c}{r^2}\right)R.$$

Higher covariant derivatives will also point in the direction of R as one can convince oneself that $\nabla_\mu R$ also points in the same direction. An inductive argument can be used to make the final steps rigorous.

(ii) Let us now consider[17] the solution of Prasad and Sommerfield.[9] Here also the solution does not satisfy our symmetry condition (21) because three of the $\Phi_{\mu\nu}$'s are nonvanishing. We want to show that the presence of scalar fields plays a crucial role in this case, in determining whether the holonomy group is Abelian or not.

The solution we envisage is

$$\Gamma_\mu=i[L_\mu,R]\frac{K(r)-1}{er^2}, \tag{50}$$

where

$$K(r)=Cr\,\mathrm{csch}\,Cr, \tag{51}$$

$$H(r)=Cr\coth Cr-1. \tag{52}$$

Following essentially the same steps as in our analysis of the Wu–Yang's solution, we obtain an expression for three nonvanishing components of the field strength,

$$\Phi_{ab}=\frac{\epsilon_{abc}r_c}{er^4}R(K(r)H(r)+(K^2(r)-1))-\frac{\epsilon_{abc}L_c}{er^2}K(r)H(r). \tag{53}$$

The scalar field solution of Prasad and Sommerfield is given by

$$\varphi^a=\hat{r}^a\frac{H(r)}{er}, \tag{54}$$

where $\hat{r}^a$ is a unit vector. The solution (53) is non-Abelian due only to the presence of the second term in the right-hand side. Comparing with Eq. (54) we notice that the holonomy group would be Abelian if the scalar field vanished. The function $r^{-1}H(r)$ vanishes for $r\to 0$ but that limit is not defined for the unit vector $\hat{r}^a$. Similarly, the scalar field is responsible for the source term [see conditions (40)],

$$J_\nu=\nabla_a\Phi_{a\nu}=\frac{\epsilon_{a\nu c}r_aL_c}{er^4}K(r)H^2(r). \tag{55}$$

It would be interesting to investigate a case in which the scalar field vanishes outside $r=0$, to cast some light on the problem, but exact solutions to nonlinear field equations are scarce. We plan to consider the SU(3) solution of Marciano and Pagels[18] without scalar fields.

6. CONCLUSION

Our conception of symmetry is more concerned with the geometrical properties of the field strengths than with their actual form, and to whether they are separable into angular and radial variables, for instance. As a consequence of our definition of symmetry, four independent $\Phi_{\mu\nu}$ vanish. This in turn leads to an Abelian holonomy group for appropriate sources. It may happen that the holonomy group is still Abelian otherwise, as in the Wu–Yang solution (one is in an Abelian gauge). This shows that our conditions are sufficient but, possibly, not necessary. The main result of this investigation is a generalization of Loos'[2] result. He showed that every spherically symmetric internal holonomy group with at least one source-free region is Abelian. His proof can be extended and we are able to conclude that similar results hold if the sphere is deformed into a two-dimensional surface as long as it is a simply connected C^∞ manifold, admitting orthogonal coordinates. Uzes'[5] theorem then implies that short-range components in Yang–Mills fields can only appear if it is impossible to find a two-dimensional symmetry as defined in the text. This means that spherical, cylindrical, ellipsoidal symmetric solutions, plane waves and the like should not be considered for that purpose. Plane symmetry has also been considered by Uzes. His results agree with our general conclusion.

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Some static and nonstatic solutions of Brans–Dicke theory of gravitation

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Static and nonstatic vacuum solutions of Brans–Dicke field equations are derived. For this purpose, a new and convenient technique is proposed. Results are applied to some known solutions.

1. INTRODUCTION

Consideration of the gravitation field via the general theory of relativity proposed by Einstein has been customary almost from the dawn of the twentieth century. In this theory Einstein has introduced the principle of geometrization in physics. The general theory, in fact, succeeds in geometrizing the phenomenon of gravitation and in identifying the metric tensor of a Riemanian space—time with the gravitational potential but this theory lacks explanation of all aspects of Mach's principle, therefore, we start our study with the Brans—Dicke[1] theory of gravitation which incorporates the idea of Mack's principle to some extent. Jordan[2] has also made attempts in this direction but his theory lacked physical validity owing to nonconservation of the energy—momentum tensor and the aspect of mass creation.

In this paper, we have obtained some solutions of Brans—Dicke field equations by transforming them into Einstein-like field equations. Recently, Singh[3] has obtained some static solutions of the scalar—tensor theory of Sen and Dunn[4] by a similar technique but this theory has a negative point that the principle of mass energy conservation is violated here also. Therefore, on physical grounds, the scalar—tensor theory also does not arouse our interest. The importance of this paper is that we have obtained both static and nonstatic solutions of the more generalized theory of gravitational fields (Brans—Dicke theory).

In Sec. 2, we have given the Brans—Dicke field equations. In Sec. 3, static solution of these field equations are derived. In Sec. 4, results corresponding to some well-known solutions of Einstein theory have been obtained for the Brans—Dicke theory. In Sec. 5, we have gotten a nonstatic solution of the field equations obtained in Sec. 2. The last section contains some concluding remarks.

2. BRANS-DICKE FIELD EQUATION

The field equations of the Brans—Dicke theory are

$$R_{ij}-\tfrac{1}{2}g_{ij}R=(\omega/\phi^2)(\phi_{,i}\,\phi_{,j}-\tfrac{1}{2}g_{ij}\,\phi_{,k}\,\phi^{,k}) + \phi^{-1}(\phi_{i;j}-g_{ij}\phi^{k}_{;k})+(8\pi/c^4)\phi^{-1}T_{ij}, \qquad (1)$$

and

$$(3+2\omega)\phi^{k}_{;k}=(8\pi/c^4)T, \qquad (2)$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, g_{ij} is the matric tensor, ϕ is the scalar field, T_{ij} is the energy—momentum tensor, and ω is the coupling constant.

For this purpose, we consider an empty space for which $T_{ij}=0$. Therefore, in the case of the vacuum field, the Brans—Dicke field equations take the form

$$R_{ij}=(\omega/\phi^2)\phi_{,i}\phi_{,j}+(\phi_{i,j}/\phi), \qquad (3)$$

and

$$\phi^{k}_{;k}=0 \quad (\omega\neq -\tfrac{3}{2}). \qquad (4)$$

The substitution of

$$\phi=\exp(\lambda), \qquad (5)$$

makes (3) more convenient and we obtain

$$R_{ij}=(\omega+1)\lambda_{,i}\,\lambda_{,j}+\lambda_{i;j}. \qquad (6)$$

Let us consider a nonstatic line element

$$ds^2=\exp(2U)\,dt^2+\exp[-2(1+E)U](\gamma_{ab}\,dx^a\,dx^b), \qquad (7)$$

where U is a function of all the four coordinates (x_1,x_2,x_3,t), where $a,b,c,\cdots$ run from 1 to 3. Here γ_{ab} plays the role of the metric tensor in three-dimensional space and satisfies $\gamma_{ab}\gamma^{ac}=\delta^c_b$. E is an arbitrary constant.

Computing the component of Ricci tensor, we have

$$R_{ab}=P_{ab}-(E+1)\gamma_{ab}U^{c}_{;c}-EU_{a;b}+(2-E^2)U_{,a}U_{,b} +E(E+1)\gamma_{ab}\gamma^{cd}U_{,c}U_{,d}+(3E+4)(E+1) \times\exp[-2(E+2)U]\,U_4^2-(E-1)\exp[-2(E+2)U]\gamma_{ab}U_{,44}, \qquad (8)$$

$$R_{44}=\exp[2(E+2)U](U^{c}_{;c}-E\gamma^{cd}U_{,c}U_{,d}) +3(E+1)(E+2)U^2_{,4}-3(E+1)U_{,44}, \qquad (9)$$

$$R_{4a}=2(E+1)(U_{,a}U_{,4}-U_{,4a}), \qquad (10)$$

where P_{ab} is the Ricci tensor formed by the metric tensor γ_{ab} and covariant derivatives are also formed with respect to γ_{ab}. R_{ab} is the Ricci tensor used earlier.

Hence the field equations (6) become

$$[P_{ab}-(E+1)\gamma_{ab}U^{c}_{;c}-EU_{a;b}+(2-E^2)U_{,a}U_{,b}+E(E+1)\gamma_{ab}\gamma^{cd}U_{,c}U_{,d} +(3E+4)(E+1)\exp[-2(E+2)U]\gamma_{ab}U^2_{,4}-(E+1)\exp[-2(E+2)U]\,\gamma_{ab}U_{,44} =(\omega+1)\lambda_{,a}\lambda_{,b}+\lambda_{a;b}+(E+1)(\lambda_{,a}U_{,b}+\lambda_{,b}U_{,a}-\nu_{ab}\gamma^{cd}U_{,c}\lambda_{,d})-(E+1)\exp[-2(E+2)U]\gamma_{ab}\lambda_{,4}U_{,4}, \qquad (11)$$

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$$\exp[2(E+2)U](U^c_{;c} - E\gamma^{cd}U_{,c}U_{,d}) + 3(E+1)(E+2)U^2_{,4} - 3(E+1)U_{,44}$$
$$= (\omega+1)\lambda^2_{,4} + \exp[2(E+2)]\gamma^{cd}\lambda_{,c}U_{,d} - U_{,4}\lambda_{,4} + \lambda_{,44}, \quad (12)$$
$$2(E+1)(U_{,a}U_{,4} - U_{,4a}) = (\omega+1)\lambda_{,4}\lambda_{,a} + \lambda_{,4a} + (E+1)U_{,4}\lambda_{,a} - U_{,a}\lambda_{,4}. \quad (13)$$

3. STATIC SOLUTIONS OF THE BRANS-DICKE FIELD EQUATIONS

For static solutions, the function U in the time element (7) will be independent of time coordinate t and hence field equations (11), (12), and (13) are reduced to

$$P_{ab} - (E+1)\gamma_{ab}U^c_{;c} - EU_{a;b} + (2-E^2)U_{,a}U_{,b} + E(E+1)\gamma_{ab}\gamma^{cd}U_{,c}U_{,d}$$
$$= (\omega+1)\lambda_{,a}\lambda_{,b} + \lambda_{a;b} + (E+1)(\lambda_{,a}U_{,b} + \lambda_{,b}U_{,a} - \gamma_{ab}\gamma^{cd}U_{,c}\lambda_{,d}, \quad (14)$$
$$U^c_{;c} - E\gamma^{cd}U_{,c}U_{,d} = \gamma^{cd}U_{,c}\lambda_{,d}. \quad (15)$$

Now we assume that λ and U are related functionally by

$$\lambda = -EU, \quad (16)$$

which transforms the field equations (14) and (15) into

$$P_{ab} + 2F^2U_{,a}U_{,b} = 0, \quad (17)$$

where

$$F^2 = 1 + E - E^2\omega/2, \quad (18)$$

and

$$U^c_{;c} = 0. \quad (19)$$

Again let us transform U into V via the transformation

$$V = FU. \quad (20)$$

As a result we obtain field equations in a simpler form,

$$P_{ab} + 2V_aV_b = 0, \quad (21)$$
$$V^c_{;c} = 0. \quad (22)$$

These are the field equations $R_{ij} = 0$ of the Einstein theory for the static line element

$$ds^2 = \exp(2V)\,dt^2 + \exp(-2V)(\gamma_{ab}\,dx^a dx^b). \quad (23)$$

Now applying the transformation given by Eqs. (16) and (20) we are in a position to reduce the Brans–Dicke field equations (14) and (15) for the line element (7) into Einstein field equations (21) and (22) for the line element (23).

Thus we have established the following results: Corresponding to every static solution V and γ_{ab} of the empty space field equations of Einstein theory, we can find a solution of the vacuum field equations of the Brans–Dicke theory with the same γ_{ab}, ϕ as derived from Eqs. (5) and (16) and the same U from Eqs. (18) and (20).

4. SOME PARTICULAR STATIC SOLUTIONS OF THE BRANS-DICKE FIELD EQUATIONS CORRESPONDING TO WELL KNOWN SOLUTIONS OF EINSTEIN THEORY

Now using the technique given in Sec. 3, we shall obtain some solutions of vacuum field equations of the Brans–Dicke theory from some well known solutions of Einstein theory.

A. Schwarzchild type solution in standard coordinate

The Schwarzchild solution in the standard coordinate is given by the line element

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{(1-2m/r)} \times [dr^2 + r^2\left(1 - \frac{2m}{r}\right)(d\theta^2 + \sin^2\theta\,d\phi^2)\;. \quad (24)$$

The corresponding solution of the Brans–Dicke theory will be given by the following line element

$$ds^2 = (1-2m/r)^{1/F}dt^2 - (1-2m/r)^{-(1+E)/F} \times [dt^2 + r^2(1-2m/r)(d\sigma^2 + \sin^2\sigma\,d\phi^2)], \quad (25)$$

with the scalar field ϕ given by $\phi = (1-2m/r)^{-E/2F}$.

B. Schwarzchild type solution in the isotropic coordinate

This solution is given by the metric

$$ds^2 = \left(\frac{1-m/r}{1+m/r}\right)^2 dt^2 - \left(\frac{1-m/r}{1+m/r}\right)^{-2}\left(1 - \frac{m^2}{r^2}\right)^2 \times [dr^2 + \gamma^2(d\sigma^2 + \sin^2\sigma\,d\phi^2)]. \quad (26)$$

The corresponding solution of the Brans–Dicke theory will be given by

$$ds^2 = \left(\frac{1-m/r}{1+m/r}\right)^{2/F} dt^2 - \left(\frac{1-m/r}{1+m/r}\right)^{-2(1+E)/F} \times \left(1 - \frac{m^2}{r^2}\right)^2[dr^2 + r^2(d\sigma^2 + \sin^2\sigma\,d\phi^2)], \quad (27)$$

with ϕ given by

$$\phi = \left(\frac{1-m/r}{1+m/r}\right)^{-E/F}.$$

C. Brans-Dicke static solution

If in Eq. (27) we put $m = B$, $E = C$, and $F = (1 + E - E^2\omega/2)^{1/2} = \lambda$, we obtain the line element (27) in the following form:

$$ds^2 = \left(\frac{1-B/r}{1+B/r}\right)^{2/\lambda} dt^2 - \left(\frac{1-B/r}{1+B/r}\right)^{-2(1+C)/\lambda} \times\; 1 - \frac{B^2}{r^2}\;^2\,[dr^2 + r^2(d\sigma^2 + \sin^2\sigma\,d\phi^2)]$$

or

$$ds^2 = \left(\frac{1-B/r}{1+B/r}\right)^{2/\lambda} dt^2 - \left(\frac{1-B/r}{1+B/r}\right)^{2(\lambda-C-1)/\lambda}\left(1 + \frac{B}{r}\right)^4 \times [dr^2 + r^2(d\sigma^2 + \sin^2\sigma\,d\phi^2)], \quad (28)$$

with ϕ given by

$$\phi=\left(\frac{1-B/r}{1+B/r}\right)^{-C/\lambda}.$$

B, C, and λ are arbitrary constants. The solutions given by Eq. (28) are similar to the solution obtained by Brans and Dicke[1] when we choose $\alpha_0=0$, $\beta_0=0$, and $\phi_0=1$.

D. A conformastat solution

Das[5] has obtained a conformastat solution which is given by the metric

$$ds^2=(1-mx)^{-2}dt^2-(1-mx)^4(dx^2+dy^2+dz^2), \qquad (29)$$

where $m=$ const.

The corresponding solution of the Brans–Dicke theory will be given by

$$ds^2=(1-mx)^{-2/F}dt^2-(1-mx)^{2+2(1+E)/F}\times(dx^2+dy^2+dz^2), \qquad (30)$$

with $\phi=(1-mx)^{E/F}$.

E. A static plane symmetric solution

The static plane symmetric solution of Taub[6] is given by the metric

$$ds^2=(k_1x+k_2)^{-1/2}(dt^2-dx^2)-(k_1x+k_2)(dy^2+dx^2), \qquad (31)$$

where k_1 and k_2 are constants.

The solution of the Brans–Dicke theory will be

$$ds^2=(k_1x+k_2)^{-1/2F}dt^2-(k_1x+k_2)^{(E+1/2F-1)}dx^2-(k_1x+k_2)^{(E+1/2F+1/2)}(dy^2+dz^2), \qquad (32)$$

together with

$$\phi=(k_1x+k_2)^{E/4F}.$$

F. Levi-Civita solution

Levi-Civita[7] has obtained a static symmetric solution given by the following metric

$$ds^2=(r/r_0)^{(q^2+2q)/2}(dt^2-dr^2)-(r/r_0)^q r^2 d\phi^2-(r/r_0)^{-q}dz^2, \qquad (33)$$

where r_0 and q are constants.

Use of transformation

$$z\to it \quad\text{and}\quad t\to iz \qquad (34)$$

changes the metric (34) into the convenient form

$$ds^2=\left(\frac{r}{r_0}\right)^{-q}dt^2-\left(\frac{r}{r_0}\right)^{q}\times\left[\left(\frac{r}{r_0}\right)^{q^2/2}(dr^2+dz^2)+r^2d\phi^2\right]. \qquad (35)$$

The Brans–Dicke solution corresponding to this metric is

$$ds^2=\left(\frac{r}{r_0}\right)^{-q/F}dt^2-\left(\frac{r}{r_0}\right)^{q(1+E)/F}\times\left[\left(\frac{r}{r_0}\right)^{q^2/2}(dr^2+dz^2)+r^2d\phi^2\right], \qquad (36)$$

with $\phi=(r/r_0)^{Eq/2F}$.

Again using the inverse of the transformation (35), i.e., $z\to -it$, $t\to -iz$ we get the metric (36) in the form

$$ds^2=(r/r_0)^{q(1+E)/F+q^2/2}(dt^2-dr^2)-(r/r_0)^{-q/F}dz^2-(r/r_0)^{q(1+E)/F}r^2d\phi^2, \qquad (37)$$

with $\phi=(r/r_0)^{Eq/2F}$.

G. "Curzon" particle solution

The static axially symmetric solution representing a Curzon[8] particle is given by

$$ds^2=\exp(-2m/\rho)dt^2-\exp(2m/\rho)\times[\exp(-m^2r^2/2\rho^4)(dr^2+dz^2)+d\phi^2], \qquad (38)$$

where $m=$ const and $\rho=(r^2+z^2)^{1/2}$.

The corresponding solution of the Brans–Dicke theory is given by the metric

$$ds^2=\exp(-2m/\rho F)dt^2-\exp(2m/\rho F)\times[\exp(-m^2r^2/2\rho^4)(dr^2+dz^2)+d\phi^2], \qquad (39)$$

with ϕ given by $\phi=\exp(mE/\rho F)$.

5. NONSTATIC SOLUTIONS

In this section, we consider the nonstatic Brans–Dicke vacuum field and propose a technique by which Brans–Dicke solutions analogous to nonstatic solutions of the Einstein vacuum field equations can be obtained.

For the purpose, we consider a nonstatic line element (7) taking $E=0$ without loss of generality because E is an arbitrary constant there. Here we also assume that U is a function of t only.

The field equations (11), (12), and (13) in this case, are reduced to

$$P_{ab}+\exp(-4U)(4U_{,4}^2-U_{,44})\gamma_{ab}=-\exp(-4U)\gamma_{ab}\lambda_{,4}U_{,4}, \qquad (40)$$

and

$$3(2U_{,4}^2-U_{,44})=(\omega+1)\lambda_{,4}^2-U_{,4}\lambda_{,4}+\lambda_{,44}. \qquad (41)$$

Assuming λ to be functionally related to U as

$$\lambda=-U/(\omega+1), \qquad (42)$$

the above equations are reduced to

$$P_{ab}+[2-1/(\omega+1)]U_{,4}^2\exp(-4U)\gamma_{ab}=0, \qquad (43)$$

$$2U_{,4}^2-U_{,44}=0. \qquad (44)$$

Equation (44) implies that

$$U_4=K_1\exp(2U), \qquad (45)$$

where K_1 is a arbitrary constant.

Therefore, Eq. (43) takes the form

$$P_{ab}+2\alpha_1^2\gamma_{ab}=0, \qquad (46)$$

where

$$=K^2(1-\tfrac{1}{2}(\omega+1)^{-1}).$$

Further, Eq. (44) yields on integration

$$U=\log(k_1 t+\alpha_2)$$

or

$$U=\log\{\alpha_1 t/[1-\tfrac{1}{2}(\omega+1)^{-1}]^{1/2}+\alpha_2\}, \quad (47)$$

where α_1 and α_2 are arbitrary constants.

For Einstein nonstatic vacuum field, the analog of field equation (40) is $P_{ab}+2\alpha_1^2\gamma_{ab}=0$, which is exactly the same as Eq. (46), but the analog of field equation (41) in this case yields

$$U=\log(\alpha_1 t+\alpha_2).$$

α_1 and α_2 are arbitrary constants.

Thus for ϕ given by Eqs. (7) and (42) the solutions are identical for the two theories for line element (7) with slight difference in constants. In fact, as Eq. (46) implies that the 3-geometry is a space of constant curvature, these solutions are just certain Robertson–Walker cosmologies without matter.

6. CONCLUSIONS

The immediate use of the results obtained in Secs. (3) and (5) is to find the solution of the Brans–Dicke field equations from known vacuum solutions of Einstein theory. Besides, these solutions furnish examples of singularities occurring in the Brans–Dicke theory.

We hope that this technique will provide an important tool to handle the sophisticated and intricate problems of Brans–Dicke fields with possible applications in cosmology.

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Quadratic Hamiltonians, quadratic invariants and the symmetry group SU(n)

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We show that any $2n$-dimensional quadratic Hamiltonian may be transformed by a (usually time-dependent) linear canonical transformation into any other $2n$-dimensional quadratic Hamiltonian, in particular that of the isotropic harmonic oscillator. This latter Hamiltonian possesses the symmetry group SU(n) and n^2-1 linearly independent quadratic invariants which provide a basis for the generators of the group. Every other quadratic Hamiltonian is shown to have a quadratic invariant possessing SU(n) symmetry. The free particle structure is given explicitly. The anisotropic oscillator is shown not to possess SU(3) symmetry based on quadratic invariants. However, its wavefunctions and energy levels may be obtained directly from those of the isotropic oscillator whether the frequencies are commensurable or not.

1. INTRODUCTION

Since the pioneering work of Fock[1] on the Coulomb problem and Bargmann[2] on the isotropic time-independent oscillator, the study of dynamical symmetry groups in quantum mechanics has been one of the dominant features of the subject. Not the least of the reasons for this has been the connection between the existence of a particular symmetry and the solution of the Schrödinger equation.[3]

The existence of a symmetry group requires the existence of a set of constants of the motion such that all elements of the set commute (classically have zero Poisson bracket) with one particular element. The other elements or suitable linear combinations of them are then required to have commutation relations appropriate to the generators of the particular symmetry group sought. For problems with time-independent Hamiltonians, the Hamiltonian itself was taken as the central invariant and the search for a symmetry group became the search for sufficient other constants to constitute a basis. For both the Coulomb and oscillator problems, the angular momentum provided some of the required constants. The remaining constants were found in the Runge–Lenz vector[4] for the former and in a symmetric matrix[5] for the latter. Physically these constants describe the shape of the classical orbit.[6] Each of these quantities has an unambiguous definition in terms of quantum mechanical operators and so the symmetry group was applicable to the quantum mechanical problem. This was not the case for the anisotropic oscillator with incommesurable frequencies.[6,7] It has been pointed out frequently[8] that the problem is in finding the appropriate operator expressions for classical expressions involving nonintegral powers.

The case of a time-dependent Hamiltonian produces the need to determine the basic constant of the motion, if any. A class of problems which has been of considerable interest is that which reduces to the time-dependent harmonic oscillator. This problem occurs in the motion of a charged particle in an electromagnetic field[9-11] or in the evolution of coherent states in lasers.[12] The existence of an exact invariant for the time-dependent oscillator was shown by Lewis[10] and Riesenfeld.[11] A simpler demonstration has been provided more recently.[13] A discussion of the symmetry group of the three-dimensional time-dependent harmonic oscillator was given by Günther and Leach[14] in which they showed that SU(3) was the symmetry group of the invariant.

The use of canonical transformations in the solution of quantum mechanical problems has received considerable attention in recent years.[15-19] The employment of time-dependent transformations[13,20,21] has broadened the range of problems which can be successfully tackled. Such transformations have been particularly fruitful when applied to time-dependent oscillator Hamiltonians, in establishing both an interpretation for the invariant associated with the motion and that the motion is characterized by the symmetry group SU(n) and also providing a relatively simple method for the solution of the Schrödinger equation.

In this paper we examine the general class of quadratic Hamiltonians describing some n-dimensional motion. We show that classically every such Hamiltonian has a quadratic constant of the motion which possesses SU(n) as its symmetry group. In general, this is not the symmetry group of the Hamiltonian, but we may say that the Hamiltonian is characterized by the noninvariance group SU(n). There is no problem in the transition to quantum mechanics. We give an explicit demonstration of the SU(3) structure for a three-dimensional free particle and discuss the problem of the anisotropic oscillator with noncommensurable frequencies. In particular we show that its nondegenerate energy levels may be obtained by transformation methods from those of the degenerate isotropic oscillator.

2. LINEAR CANONICAL TRANSFORMATIONS

Writing the conjugate canonical coordinates (q,p) as

$$q_i=\omega^\mu,\quad i=1,n,\quad \mu=1,n$$
$$p_i=\omega^\mu,\quad i=1,n,\quad \mu=n+1,2n, \tag{2.1}$$

Hamiltonian's equations of motion are

$$\dot{\omega}=\epsilon\,\frac{\partial H}{\partial\omega} \tag{2.2}$$

and the Poisson bracket of two scalars F and G is

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$$[F, G]_{PB\omega} = \left(\frac{\partial F}{\partial \omega}\right)^T \epsilon \left(\frac{\partial G}{\partial \omega}\right), \tag{2.3}$$

where ϵ is the $2n \times 2n$ sympletic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

The general linear transformation from coordinates ω to $\overline{\omega}$ is

$$\overline{\omega} = S\omega + \mathbf{r} \Longleftrightarrow \omega = \overline{S}\,\overline{\omega} + \overline{\mathbf{r}}, \tag{2.4}$$

where S is a $2n \times 2n$ (real) matrix and $\mathbf{r}$ a $2n \times 1$ (real) column matrix. The condition that (2.4) be canonical is

$$S\epsilon S^T = \epsilon. \tag{2.5}$$

In terms of ω, the general quadratic Hamiltonian is

$$H(\omega) = \tfrac{1}{2}\,\omega^T A \omega + \mathrm{B}^T \omega + C, \tag{2.6}$$

in which A is a $2n \times 2n$ symmetric matrix, B is a $2n \times 1$ column matrix, and C is a scalar. A, B, and C are coordinate free, but may be time-dependent. Under the transformation (2.4), $H(\omega)$ is transformed to $\overline{H}(\overline{\omega})$ where

$$\overline{H}(\overline{\omega}) = \tfrac{1}{2}\,\overline{\omega}^T \overline{A}\,\overline{\omega} + \overline{\mathrm{B}}^T \overline{\omega} + \overline{C} \tag{2.7}$$

provided[20]

$$\dot{S} = \epsilon \overline{A} S - S\epsilon A, \tag{2.8}$$

$$\dot{\mathbf{r}} = \epsilon \overline{A}\,\mathbf{r} + \epsilon\,\overline{\mathrm{B}} - S\epsilon \mathrm{B}. \tag{2.9}$$

Equations (2.8) and (2.9) are linear first order systems and so possess solutions[22] with continuous first derivatives provided the elements of the matrices $A, \overline{A}$, B, $\overline{\mathrm{B}}$ are continuous functions of time over the interval of interest. In general S will contain $(2n)^2$ and $\mathbf{r}$ $(2n)$ arbitrary constants. The condition (2.5) imposes some constraint on the number of arbitrary constants of S, but does not determine them uniquely.

We note that C and $\overline{C}$ do not appear in (2.8) and (2.9). In classical mechanics this reflects the invariance of Hamilton's equations of motion to the transition $H \to \overline{H}$: $\overline{H} = H - C$. Quantum mechanically, this invariance is expressed as an arbitrary time phase in the Schrödinger wave function.

3. THE FORM OF $\overline{H}$

In our previous applications of time-dependent linear canonical transformations,[13,14,20,21] the signature of $\omega^T A \omega$ and $\overline{\omega}^T \overline{A}\,\overline{\omega}$ has been $2n$, i.e. the transformations have been between attractive oscillator Hamiltonians. This restriction is not implicit in any of (2.5), (2.8), or (2.9). That it is unnecessary may be illustrated by using the simple example of the one-dimensional oscillator with Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2), \quad \omega \text{ constant}. \tag{3.1}$$

Some possible forms for $\overline{H}$ and the required transformation matrices are

(i) $\overline{H} = \frac{1}{2}(P^2 - \omega^2 Q^2)$.

$$S = \begin{bmatrix} -(e^{\omega t}\dot{\alpha} + e^{-\omega t}\dot{\beta}), & (e^{\omega t}\alpha + e^{-\omega t}\beta) \\ -\omega(e^{\omega t}\dot{\alpha} - e^{-\omega t}\dot{\beta}), & \omega(e^{\omega t}\alpha - e^{-\omega t}\beta) \end{bmatrix}, \tag{3.2}$$

where

$$\alpha = A\sin\omega t + B\cos\omega t, \quad \beta = C\sin\omega t + D\cos\omega t, \tag{3.3}$$

$$2\omega^2(AD - BC) = 1. \tag{3.4}$$

(ii) $\overline{H} = 0$.

$$S = \begin{bmatrix} -\dot{\alpha}, & \alpha \\ -\dot{\beta}, & \beta \end{bmatrix}, \tag{3.5}$$

where

$$\alpha = A\cos\omega t + B\sin\omega t, \quad \beta = C\cos\omega t + D\sin\omega t, \tag{3.6}$$

$$\omega(AD - BC) = 1. \tag{3.7}$$

(iii) $\overline{H} = \omega PQ$

$$S = \begin{bmatrix} -\dot{\alpha}e^{\omega t}, & \alpha e^{\omega t} \\ -\dot{\beta}e^{-\omega t}, & \beta e^{-\omega t} \end{bmatrix}, \tag{3.8}$$

where

$$\alpha = B\cos\omega t - A\sin\omega t, \quad \beta = D\cos\omega t - C\sin\omega t, \tag{3.9}$$

$$\omega(AD - BC) = 1. \tag{3.10}$$

(iv) $\overline{H} = P$.

$$S = \begin{bmatrix} -\dot{\alpha}, & \alpha \\ -\dot{\beta}, & \beta \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} t + c \\ k \end{bmatrix}, \tag{3.11}$$

where α and β are the same as in (ii), c and k are arbitrary constants and use has been made of the equivalence of H and $(H - C)$.

The particular case $\overline{H} = 0$ is the one used in the solution of the Hamilton–Jacobi equation. However, the solution of Hamilton's equations for (3.1) may be obtained from the solution of the Hamilton's equations for any $\overline{H}(\overline{\omega})$ obtained from (3.1) via a canonical transformation. Generally H and $\overline{H}$ are not numerically equal since

$$\overline{H} = H + \frac{\partial F}{\partial t}, \tag{3.12}$$

where F is the generating function of the transformation. For a linear transformation F is a quadratic form whose coefficients depend upon the elements of S (and $\mathbf{r}$ where applicable). If S is time-dependent, clearly $\overline{H} \neq H$.

4. THE ARCHTYPAL QUADRATIC HAMILTONIAN

From the foregoing, it is obvious that any quadratic Hamiltonian (2.6) is related to

$$H = \tfrac{1}{2}\,\omega^T \omega \tag{4.1}$$

by a linear canonical transformation. We term this the archtypal quadratic Hamiltonian because it possesses the dynamical symmetry group SU(n) as an *invariance* symmetry group. The generators of the group may be written down in terms of constants of the motion described by (4.1) which are quadratic in ω.

Suppose $\mathfrak{C}$ is a time-independent quadratic form given by

$$\mathfrak{C} = \tfrac{1}{2}\,\omega^T C \omega, \tag{4.2}$$

where C is a constant $2n \times 2n$ real symmetric matrix. $\mathfrak{C}$ is a constant of the motion provided

$$[\mathfrak{C}, H]_{PB\omega} = \omega^T C^T \epsilon\,\omega = 0, \tag{4.3}$$

i.e., C has the form

$$C=\begin{bmatrix} U, & W\\ -W, & U\end{bmatrix}, \tag{4.4}$$

where U is a symmetric and W a skew-symmetric $n\times n$ matrix. We define the set of matrices $(U_{ij}, W_{ij}; i=1,n; j=1,n)$ as

$$[U_{ij}]_{mn}=\delta_{im}\delta_{nj}+\delta_{jm}\delta_{ni}, \tag{4.5}$$

$$[W_{ij}]_{mn}=\mathrm{sgn}(j-i)(\delta_{im}\delta_{nj}-\delta_{jm}\delta_{ni}). \tag{4.6}$$

Any general C may be written as

$$C=\begin{bmatrix} \alpha_{ij}U_{ij}, & \beta_{ij}W_{ij}\\ -\beta_{ij}W_{ij}, & \alpha_{ij}U_{ij}\end{bmatrix}, \tag{4.7}$$

where the scalar coefficients α_{ij} and β_{ij} are symmetric in i and j.

Writing

$$\mathfrak{U}_{ij}=\frac{1}{2}\,\omega^T\begin{bmatrix} U_{ij}, & 0\\ 0, & U_{ij}\end{bmatrix}\omega, \tag{4.8}$$

$$\mathfrak{W}_{ij}=\frac{1}{2}\,\omega^T\begin{bmatrix} 0, & W_{ij}\\ -W_{ij}, & 0\end{bmatrix}\omega, \tag{4.9}$$

we have a set of n^2 linearly independent constants of the motion which have zero Poisson bracket with H (4.1). Since

$$H=\tfrac{1}{2}\textstyle\sum_i \mathfrak{U}_{ii}, \tag{4.10}$$

there are n^2-1 constants of the motion linearly independent of H. The $\mathfrak{U}_{ij}$ are the components of the n-dimensional counterpart to the Fradkin tensor.[5] The $\mathfrak{W}_{ij}$ are the components of the angular momentum tensor.

The above analysis applies equally well to quantum mechanics. From the n^2-1 constants of the motion we may obtain the standard generators of the Lie algebra su(n) by suitable linear combinations. Thus for $n=3$, the generators of SU(3) for the quantum mechanical mechanical problem are

$$\begin{aligned} &2\sqrt{3}\,H_1=\mathfrak{W}_{12},\\ &12\,H_2=\mathfrak{U}_{11}+\mathfrak{U}_{22}-2\,\mathfrak{U}_{33},\\ &4\sqrt{3}\,E_\epsilon^\lambda=\mathfrak{W}_{23}+i\epsilon\mathfrak{W}_{31}-\lambda(\mathfrak{U}_{13}+i\epsilon\mathfrak{U}_{23}),\\ &4\sqrt{6}\,E_{2\epsilon}=\mathfrak{U}_{11}-\mathfrak{U}_{22}+2i\epsilon\,\mathfrak{U}_{12},\end{aligned} \tag{4.11}$$

in which ϵ and λ take the values ± 1 independently. The H's and E's satisfy the usual commutation relations (c.f., Fradkin,[5] Günther and Leach,[14] and Sec. 6 for the discussion of the free particle).

5. BEHAVIOR UNDER TRANSFORMATION

Under the linear canonical transformation

$$\overline{\omega}=S\omega+\mathrm{r}\Longleftrightarrow\omega=\overline{S}\,\overline{\omega}+\overline{\mathrm{r}}, \tag{2.4}$$

the Hamiltonian $\overline{H}(\overline{\omega})$ which gives a description of the motion in $\overline{\omega}$ equivalent to that of $H(\omega)$,(4.1) in ω is given by

$$\overline{H}(\overline{\omega})=H(\omega)+\partial F/\partial t. \tag{3.12}$$

In $\overline{\omega}$ we also have an expression for H which we write as $H(\overline{\omega})$. Since

$$\dot{H}(\overline{\omega})=0, \tag{5.1}$$

$H(\overline{\omega})$ is a nontrivial constant of the motion described by $\overline{H}(\overline{\omega})$ because $H(\omega)$ is nonzero. In general $[H(\overline{\omega}),\overline{H}(\overline{\omega})]_{\mathrm{PB}\overline{\omega}}$ is nonzero unless $\partial F/\partial t$ has zero Poisson bracket with $\overline{H}(\overline{\omega})$.

As the transformation from ω to $\overline{\omega}$ is both canonical and nondegenerate, the symmetry group of $H(\omega)$ is preserved for $H(\overline{\omega})$. Suppose the generators of SU(n) for $H(\omega)$ are $\mathfrak{C}_N(\omega)$, $N=1,n^2-1$ with $\mathfrak{C}_N(\omega)$ having the form

$$\mathfrak{C}_N=\alpha_{ij}^N\mathfrak{U}_{ij}+\beta_{ij}^N\mathfrak{W}_{ij}. \tag{5.2}$$

Then

$$\begin{aligned} &\text{(i)}\ \ \mathfrak{C}_N(\omega)\neq 0,\\ &\text{(ii)}\ \ \mathfrak{C}_N(\omega)\neq\mathfrak{C}_M(\omega),\quad M\neq N,\\ &\text{(iii)}\ \ [\mathfrak{C}_N(\omega),H(\omega)]_{\mathrm{PB}\omega}=0,\\ &\text{(iv)}\ \ [\mathfrak{C}_M(\omega),\mathfrak{C}_N(\omega)]_{\mathrm{PB}\omega}=f_{MN}^K\mathfrak{C}_K(\omega),\end{aligned} \tag{5.3}$$

where K, M and N range over the values 1 to n^2-1 and the f_{MN}^K are the structure constants. These properties are invariant under a linear canonical tranformation, (5.4) (i) and (ii) due to the nondegeneracy and (5.4) (iii) and (iv) due to the canonicity of the transformation. For example,

$$\begin{aligned}[\mathfrak{C}_M(\overline{\omega}),\mathfrak{C}_N(\overline{\omega})]_{\mathrm{PB}\overline{\omega}}&=\left\{\frac{\partial}{\partial\overline{\omega}}\mathfrak{C}_M(\overline{\omega})\right\}^T\epsilon\left\{\frac{\partial}{\partial\overline{\omega}}\mathfrak{C}_N(\overline{\omega})\right\}\\ &=\left\{\frac{\partial}{\partial\omega}\mathfrak{C}_M(\omega)\right\}^T S\epsilon S^T\left\{\frac{\partial}{\partial\omega}\mathfrak{C}_N(\omega)\right\}\\ &=f_{MN}^K\mathfrak{C}_K(\overline{\omega})\end{aligned}$$

since $S\epsilon S^T=\epsilon$.

For every $\overline{H}$ obtainable from H (4.1) under a linear canonical transformation, there exists a constant of the motion $H(\overline{\omega})$ possessing the symmetry group SU(n) which is thereby a (usually noninvaraince) symmetry group characterizing $\overline{H}$. Alternatively, any given quadratic Hamiltonian may be transformed to the archtypal form (4.1) which possesses n^2-1 associated constants. In the original coordinates, (4.1) and the associated constants provide the (noninvariance) symmetry group for the Hamiltonian.

We emphasize that these results apply equally well to quantum mechanics when the usual conventions are observed. We note that the definitions of H and the $\mathfrak{C}_N$ are already symmetric in the products of ω^μ and ω^ν.

6. THE FREE PARTICLE AND SU(3)

In three dimensions, a free particle has the Hamiltonian

$$H=\tfrac{1}{2}\mathrm{p}^2, \tag{6.1}$$

where p is a three-vector. The archtypal form

$$\overline{H}=\tfrac{1}{2}(\mathbf{P}^2+\mathbf{Q}^2) \tag{6.2}$$

is obtained by the transformation

$$\begin{bmatrix}\mathbf{Q}\\ \mathbf{P}\end{bmatrix}=\begin{bmatrix}\alpha, & -\alpha t+\beta\\ \dot{\alpha}, & -\dot{\alpha}t+\dot{\beta}\end{bmatrix}\begin{bmatrix}\mathrm{q}\\ \mathrm{p}\end{bmatrix}, \tag{6.3}$$

in which

$$\alpha = A\cos t + B\sin t, \quad \beta = C\cos t + D\sin t. \tag{6.4}$$

The transformation (6.3) is canonical provided the constant 3×3 matrices A, B, C, and D satisfy

$$AC^T = CA^T, \quad BD^T = DB^T, \quad AD^T - CB^T = I. \tag{6.5}$$

In particular we may set

$$A = I, \quad D = I, \quad C = 0, \quad B = 0, \tag{6.6}$$

so that the transformation matrix is

$$S = \begin{bmatrix} I\cos t, & -It\cos t + I\sin t \\ -I\sin t, & It\sin t + I\cos t \end{bmatrix}. \tag{6.7}$$

Applying the transformation (6.3) to the generators of SU(3) given in (4.9), in the (q, p) coordinates we obtain the quantum mechanical generators

$$\begin{aligned} 2\sqrt{3}H_1 &= q_1p_2 - q_2p_1, \\ 12H_2 &= q_1^2 + q_2^2 - 2q_3^2 + (1+t^2)(p_1^2 + p_2^2 - 2p_3^2) \\ &\quad - 2t(q_1p_1 + q_2p_2 - 2q_3p_3), \\ 4\sqrt{3}E_\epsilon^\lambda &= q_2p_3 - q_3p_2 + i\epsilon(q_3p_1 - q_1p_3) \\ &\quad - \lambda\{q_1q_3 + (1+t^2)p_1p_3 - t(q_1p_3 + p_1q_3)\} \\ &\quad - i\lambda\epsilon\{q_2q_3 + (1+t^2)p_2p_3 - t(q_2p_3 + p_2q_3)\}, \\ 4\sqrt{6}E_{2\epsilon} &= q_1^2 - q_2^2 + (1+t^2)(p_1^2 - p_2^2) - 2t(q_1p_1 - q_2p_2) \\ &\quad + 2i\epsilon\{q_1q_2 + (1+t^2)p_1p_2 - t(q_1p_2 + p_1q_2)\}. \end{aligned} \tag{6.8}$$

(Note that there is no necessity to write H_2 or $E_{2\epsilon}$ in symmetric form since $i\hbar$ terms cancel.)

Using these generators we directly confirm the SU(3) commutation relations, viz.

$$\begin{aligned} &[H_1, H_2] = 0, \\ &[H_1, E_\epsilon^\lambda] = \epsilon\hbar(2\sqrt{3})^{-1}E_\epsilon^\lambda, \quad [H_2, E_\epsilon^\lambda] = \epsilon\lambda\hbar(2)^{-1}E_\epsilon^\lambda, \\ &[H_1, E_{2\epsilon}] = \epsilon\hbar(\sqrt{3})^{-1}E_{2\epsilon}, \quad [H_2, E_{2\epsilon}] = 0, \\ &[E_\epsilon^\lambda, E_{-\epsilon}^{-\lambda}] = 0, \quad [E_\epsilon^\lambda, E_{2\epsilon}] = 0, \\ &[E_\epsilon^\lambda, E_{-\epsilon}^\lambda] = \epsilon\hbar(2\sqrt{3})^{-1}H_1 + \epsilon\lambda\hbar(2)^{-1}H_2, \\ &[E_\epsilon^\lambda, E_\epsilon^{-\lambda}] = -\epsilon\lambda\hbar(\sqrt{6})^{-1}E_{2\epsilon}, \\ &[E_\epsilon^\lambda, E_{-2\epsilon}] = \epsilon\lambda\hbar(\sqrt{6})^{-1}E_{-\epsilon}^{-\lambda}, \quad [E_{2\epsilon}, E_{-2\epsilon}] = \epsilon\hbar(\sqrt{3})^{-1}H_1. \end{aligned} \tag{6.9}$$

The invariant for the motion is

$$\begin{aligned} I = \tfrac{1}{2}[&q_1^2 + q_2^2 + q_3^2 + (I + t^2)(p_1^2 + p_2^2 + p_3^2) \\ &- t(q_1p_1 + p_1q_1 + q_2p_2 + p_2q_2 + q_3p_3 + p_3q_3)] \end{aligned} \tag{6.10}$$

and, as has been directly demonstrated, I possesses the dynamical symmetry of SU(3). As $\partial I/\partial t$ is nonzero, I does not commute with H and so SU(3) is a noninvariance symmetry group for H (6.1).

7. THE ANISOTROPIC OSCILLATOR: NONINVARIANCE GROUP

The three-dimensional time-independent anisotropic oscillator has the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{3}(p_i^2 + \omega_i^2 q_i). \tag{7.1}$$

It is transformed to the archtypal oscillator Hamiltonian by the linear canonical transformation with coefficient matrix,

$$S = \left[\begin{array}{ccc|ccc} \omega_1^{1/2}\cos(\omega_1-1)t, & 0, & 0, & -\omega_1^{-1/2}\sin(\omega_1-1)t, & 0, & 0 \\ 0, & \omega_2^{1/2}\cos(\omega_2-1)t, & 0, & 0, & -\omega_2^{-1/2}\sin(\omega_2-1)t, & 0 \\ 0, & 0, & \omega_3^{1/2}\cos(\omega_3-1)t & 0, & 0, & -\omega_3^{-1/2}\sin(\omega_3-1)t \\ \hline \omega_1^{1/2}\sin(\omega_1-1)t, & 0, & 0, & \omega_1^{-1/2}\cos(\omega_1-1)t, & 0, & 0 \\ 0, & \omega_2^{1/2}\sin(\omega_2-1)t, & 0, & 0, & \omega_2^{-1/2}\cos(\omega_2-1)t, & 0 \\ 0, & 0, & \omega_3^{1/2}\sin(\omega_3-1)t & 0, & 0, & \omega_3^{-1/2}\cos(\omega_3-1)t \end{array}\right]. \tag{7.2}$$

The transformed Hamiltonian

$$\bar{H} = \frac{1}{2}\sum_{i=1}^{3}(P_i^2 + Q_i^2) \tag{7.3}$$

is isotropic and exhibits the full degeneracy associated with SU(3). This degeneracy is associated with the expression for $\bar{H}$ in the (q, p) coordinate system which is the quadratic invariant

$$I = \frac{1}{2}\sum_{i=1}^{3}(\omega_i q_i^2 + \omega_i^{-1}p_i^2). \tag{7.4}$$

I (7.4) commutes with H (7.1). The constants of the motion which provide a basis for the generators of the SU(3) group of I are

$$I_{ii} = \tfrac{1}{2}(\omega_i q_i^2 + \omega_i^{-2}p_i^2), \tag{7.5}$$

$$\begin{aligned} I_{ij} = &(\omega_i^{1/2}\omega_j^{1/2}q_iq_j + \omega_i^{-1/2}\omega_j^{-1/2}p_ip_j)\cos(\omega_i - \omega_j)t \\ &+ (\omega_i^{1/2}\omega_j^{-1/2}q_ip_j - \omega_i^{-1/2}\omega_j^{1/2}q_jp_i)\sin(\omega_i - \omega_j)t, \end{aligned} \tag{7.6}$$

$$\begin{aligned} L_{ij} = &-(\omega_i^{1/2}\omega_j^{1/2}q_iq_j + \omega_i^{-1/2}\omega_j^{-1/2}p_ip_j)\sin(\omega_i - \omega_j)t \\ &+ (\omega_i^{1/2}\omega_j^{-1/2}q_ip_j - \omega_i^{-1/2}\omega_j^{1/2}q_jp_i)\cos(\omega_i - \omega_j)t \end{aligned} \tag{7.7}$$

(no summation on either i or j in (7.5–7). Although $[I_{ii}, H]$ is zero,

$$[I_{ij}, H] = -i\hbar(\omega_i - \omega_j)L_{ij}, \tag{7.8}$$

$$[L_{ij}, H] = i\hbar(\omega_i - \omega_j)I_{ij}. \tag{7.9}$$

Thus the SU(3) basis for I does not provide a like basis for H.

We point out that this does not exclude SU(3) from being the symmetry group for the anisotropic oscillator, but it does exclude the possibility of a quadratic basis. When the ratios of frequencies are rational, the symmetry group exists both classically and quantum mechanically.[7,17] When the frequencies are incommensurable the classical generators involve irrational powers.[6] Although considerable progress has been made in constructing consistent quantum mechanical operators when the powers are rational,[18] irrational powers as yet defy description. In this case it is possible that a consistent quantum mechanical discussion of the operators is not possible.

8. THE ANISOTROPIC OSCILLATOR: ENERGY STATES

The effect of a linear canonical transformation on the Schrödinger wavefunction is well established.[19,23] Under the transformation

$$\begin{bmatrix} \mathbf{Q} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} \tag{8.1}$$

we have

$$\psi(\mathbf{q}, t) = \int_{-\infty}^{\infty} d\mathbf{Q}\, K_1(\mathbf{q}, \mathbf{Q}, t) \overline{\psi}(\mathbf{Q}, t), \tag{8.2}$$

where the kernel $K_1(\mathbf{q}, \mathbf{Q}, t)$ is given by

$$K_1(\mathbf{q}, \mathbf{Q}, t) = (2\pi)^{-N/2} |\det S_2|^{-1/2} \exp\{i F_1(\mathbf{q}, \mathbf{Q}, t)\}, \tag{8.3}$$

$$2F_1(\mathbf{q}, \mathbf{Q}, t) = -\mathbf{Q}^T S_2^T S_4^T \mathbf{Q} - \mathbf{q}^T S_1^T S_2^T \mathbf{q} + 2\mathbf{Q}^T S_2^T \mathbf{q}, \tag{8.4}$$

provided S_2 is nonsingular. If S_2 is singular, the expression for K_1 may be written in an alternate form. For our discussion this is not the case and (8.3) with (8.4) suffices.

The energy levels of the motion described by $\psi(\mathbf{q}, t)$ may be calculated without the form of $\psi(\mathbf{q}, t)$ being known since

$$\begin{aligned} \langle \psi_n | H | \psi_n \rangle = i\hbar \int_{-\infty}^{\infty} d\mathbf{q} \int_{-\infty}^{\infty} d\mathbf{Q} \int_{-\infty}^{\infty} d\mathbf{Q}'\, \overline{\psi}_n^*(\mathbf{Q}, t) K_1^*(\mathbf{q}, \mathbf{Q}, t) \\ \times \left\{ \overline{\psi}_n(\mathbf{Q}', t) \frac{\partial}{\partial t} K_1(\mathbf{q}, \mathbf{Q}', t) \right. \\ \left. + K_1(\mathbf{q}, \mathbf{Q}', t) \frac{\partial}{\partial t} \overline{\psi}_n(\mathbf{Q}', t) \right\}. \end{aligned} \tag{8.5}$$

For the anisotropic oscillator, $K_1(\mathbf{q}, \mathbf{Q}, t)$ may be constructed from (7.2) and $\overline{\psi}_n(\mathbf{Q}, t)$ is the Schrödinger wavefunction for the isotropic oscillator. It is merely a matter of persistent calculation to show that

$$\langle \psi_n | H | \psi_n \rangle = \hbar\omega_1(n_1 + \tfrac{1}{2}) + \hbar\omega_2(n_2 + \tfrac{1}{2}) + \hbar\omega_3(n_3 + \tfrac{1}{2}) \tag{8.6}$$

for

$$\langle \overline{\psi}_n | \overline{H} | \overline{\psi}_n \rangle = \hbar(n_1 + n_2 + n_3 + \tfrac{3}{2}). \tag{8.7}$$

Thus, when ω_1, ω_2, and ω_3 are noncommensurable, nondegenerate energy levels are obtained even though the energy levels of $\overline{\psi}$ are degenerate.

It may have been observed that we wrote the kernel of (8.2) as $K_1(\mathbf{q}, \mathbf{Q}, t)$ which was expressed in terms of an $F_1(\mathbf{q}, \mathbf{Q}, t)$. This is not usually done in the literature (cf. Boon and Seligman[23]). We do this to emphasize that classically the function $F_1(\mathbf{q}, \mathbf{Q}, t)$ is the generating function of the first type for the canonical transformation from $\overline{H}$ to H. Should we so desire, the other generating functions may be used to construct a kernel. Then we may transform wavefunctions from momentum to coordinate, coordinate to momentum, and momentum to momentum representations in $(\mathbf{Q}, \mathbf{P})$ and $(\mathbf{q}, \mathbf{p})$, respectively. Thus

$$\psi(\mathbf{q}, t) = \int_{-\infty}^{\infty} d\mathbf{P}\, K_3(\mathbf{q}, \mathbf{P}, t) \overline{\phi}(\mathbf{P}, t), \tag{8.8}$$

$$\phi(\mathbf{p}, t) = \int_{-\infty}^{\infty} d\mathbf{Q}\, K_2(\mathbf{p}, \mathbf{Q}, t) \overline{\psi}(\mathbf{Q}, t), \tag{8.9}$$

$$\phi(\mathbf{p}, t) = \int_{-\infty}^{\infty} d\mathbf{P}\, K_4(\mathbf{p}, \mathbf{P}, t) \overline{\phi}(\mathbf{P}, t). \tag{8.10}$$

9. COMMENT

We have restricted the present discussion to quadratic Hamiltonians, quadratic invariants, and linear transformations for two reasons. Firstly the theory has a particularly elegant form and secondly there is no difficulty in the transition to quantum mechanics. For every quantum mechanical problem described by a quadratic Hamiltonian, the group SU(n) may be associated in at least a noninvariant way.

It would appear that the wavefunction and energy levels for any such Hamiltonian may be obtained via (8.2) and (8.5) from the archtypal quadratic Hamiltonian. This is certainly the case when the signature of the quadratic part of the Hamiltonian is $2n$. However, when (8.5) is applied in the case of a free particle Hamiltonian a quadratic function of time is obtained. Evidently the transition from classical to quantum mechanics imposes some constraints on the validity of the application of (8.2) and (8.5), a feature which has been noted in a different context by Kennedy and Kerner.[24] The nature of these constraints will be the subject of further investigation.

Another worthy area of investigation is the applicability of the ideas used here to general Hamiltonian systems. It has been pointed out[16] that classically every $2n$-dimensional Hamiltonian may be transformed to any other $2n$-dimensional Hamiltonian by a suitable canonical transformation. If the canonical tranformation is a point transformation, the transition to quantum mechanics is always possible.[25] However, in the more general case, the extent to which such results may be applied in quantum mechanics will be determined by whether or not the quantum mechanical operators in the different coordinate systems are uniquely related.

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Splitting and representation groups for Polish groups[a)]

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It is shown that all continuous unitary/antiunitary projective representations of a Polish group G, with the same subgroup of elements represented into the projective unitary groups, arise from continuous unitary/antiunitary (ordinary) representations of a topological group (called a splitting group) obtained from an extension of G by an Abelian topological group. Moreover, there exists always a splitting group which is "minimal" in some well-defined sense (a representation group). A sufficient (resp. a necessary and sufficient) condition for the existence of a Polish (resp. of a second countable locally compact) representation group is given.

I. INTRODUCTION

The theory of continuous unitary/antiunitary projective representations (CUAP-reps) of topological groups on a separable complex Hilbert space $\mathfrak{H}$ has a deep physical motivation in Wigner's approach to symmetries of quantum mechanical systems.[1–3] In some sense, it is rooted into the foundations of quantum mechanics (Ref. 4, Chap. IV, Sec. 14). Nonlinearity is an essential feature of CUAP-reps: The group elements are represented by bijective mappings of $\mathbf{P}(\mathfrak{H})$ (the projective space deduced from $\mathfrak{H}$) into itself which belong to the projective unitary/antiunitary group $\mathbf{PU}_{\mathrm{UA}}(\mathfrak{H})$ (or, in physical terms, which preserve transition probabilities). However, it is possible to "linearize/antilinearize" CUAP-reps in many ways. Two methods were analyzed in Ref. 3 for the case where the group G considered is Polish, i.e., is a second countable topological group which is metrically topologically complete (or, what amounts to the same, has a Polish underlying topological space).

The first alternative is to lift the CUAP-reps of the Polish group G to Borel unitary/antiunitary multiplier representations (BUAM-reps) of the same group G. The multiplier representations are sometimes also called "projective representations"; they emphasize the more striking characteristic of the theory of projective representations, namely the occurrence of multipliers. The study of multipliers leads us, following Mackey,[5] to the second alternative: Every BUAM-rep of G such that the elements represented by unitary operators constitute a fixed subgroup N of G of index 1 or 2 can be derived from a continuous unitary/antiunitary (ordinary) representation (CUA-rep) of a Polish group G_N^{μ}, obtained from a topological extension of G by $\mathbf{U}(1)$, where μ is a Borel multiplier for (G,N). Actually, we must consider a collection of such extension groups, one for each equivalence class of multipliers.

In the present paper, we investigate, again for a Polish group G, a third alternative modeled on the one put forward originally by Schur in his fundamental works on (linear) projective representations of finite groups.[6,7] Explicitly, we look for a topological extension of G such that all CUAP-reps of G with the same subgroup N of elements represented into the projective unitary groups can be derived from CUA-reps of the group obtained from the extension. The difference with the second alternative recollected above is that now only one extension group is required in order to give all CUAP-reps of G with the same subgroup N. In Sec. II, we show that such a group [a "splitting group for (G,N)"] always exists and that, in addition, there is one which is in some sense "minimal" [a "representation group for (G,N)"]. The question (already delicate for finite groups) of the uniqueness of a representation group for (G,N), up to topological group isomorphisms, is not studied in this article. The representation group constructed in Sec. II is not, in general, Polish. Thus, we are faced with the problem if there exists a Polish (resp. second countable locally compact if so is G) representation group for (G,N). In Sec. III, we give a sufficient condition for the existence in the case where G is Polish and a necessary and sufficient condition in the particular case of G second countable locally compact. The results of the previous sections are applied in Sec. IV to some relevant types of Polish groups. Quasifibered extensions are briefly considered in Appendix A and the exactness of the inflation-restriction sequence, in a particular case, is proved in Appendix B.

The notations, definitions, and results of Ref. 3 (about unitary/antiunitary projective representations), of Ref. 8 (about cohomology of groups), and of Ref. 9 (about "locally continuous Borel cochain complexes") are used throughout the paper, with the convention that, unless otherwise specified, every group G considered is written multiplicatively and its neutral element is denoted by e_G (as in Refs. 3 and 9). In particular, $\mathfrak{H}$ (resp. $\mathfrak{H}'$) is always an arbitrary *separable complex Hilbert space*, Ω is the group homomorphism of $\mathbf{U}_{\mathrm{UA}}(\mathfrak{H})$ onto $\mathbf{PU}_{\mathrm{UA}}(\mathfrak{H})$ defined in Ref. 3, Subsec. II.1, and Φ_N is the operation of G on $\mathbf{U}(1)$ defined by Ref. 3, (II.8). Moreover, we denote by I the trivial operations and, if (E,ρ) is an extension of a group G by an Abelian group A and σ is a normalized section associated with ρ, f_σ is the factor set of (E,ρ) defined by σ. The symbol $A_\Psi fG$, where G is a group, A_Ψ is a G-module, and $f\in Z^2(G,A_\Psi)$, stands for the group of all ordered pairs (a,g) $(a\in A;\ g\in G)$ with the multiplication defined by

$$(a,g)(a',g')=(a(\Psi(g)a')f(g,g'),gg').$$

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By a *character* of an Abelian topological group A we mean a *continuous unitary character*, i.e., a continuous group homomorphism of A into $\mathbf{U}(1)$. If A is a group without topology (an "abstract group"), we consider it tacitly as topological in an obvious way, namely as a discrete group. The Abelian group of all characters of A (with the pointwise multiplication) is denoted by $\hat{A}$, and we say that $\hat{A}$ *separates points* if, for each pair a, a' of distinct elements of A, there exists $\chi \in \hat{A}$ such that $\chi(a) \neq \chi(a')$. The symbol τ is used, when a group topology is defined on $\hat{A}$, to denote the canonical mapping of A into $\hat{\hat{A}}$, i.e., the mapping of A into $\hat{\hat{A}}$ such that $\tau(a)(\chi) = \chi(a)$ for all $a \in A$ and all $\chi \in \hat{A}$.

II. SCHUR'S THEORY

In his first paper on projective representations,[6] Schur showed that if G is a finite group, there exists a central extension (E, ρ) of G by a finite Abelian group such that, for each projective representation $\tilde{v}$ of G on the projective space $\mathbf{P}(V)$ deduced from a finite-dimensional complex vector space V [i.e., for each group homomorphism $\tilde{v}: G \to \mathbf{PGL}(V)$], we can find a linear representation w of E on V [i.e., a group homomorphism $w: E \to \mathbf{GL}(V)$], making the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{w} & \mathbf{GL}(V) \\ \downarrow \rho & & \downarrow \Lambda \\ G & \xrightarrow{\tilde{v}} & \mathbf{PGL}(V). \end{array} \tag{II.1}$$

Here Λ is the canonical mapping corresponding to Ω. The group E (which is a "splitting group for G" in the terminology borrowed from Moore[10]) was called a "sufficiently supplemented group" ("hinreichend ergänzte Gruppe") by Schur, who showed in addition that there always exists a "representation group" ("Darstellungsgruppe") for G, namely an E of minimal order. The same result is valid with $\mathbf{PU}_{\mathrm{UA}}(\mathfrak{H})$ instead of $\mathbf{PGL}(V)$ and $\mathbf{U}_{\mathrm{UA}}(\mathfrak{H})$ instead of $\mathbf{GL}(V)$, but then the extension is not always central (Ref. 11, Theorem 4). Our goal is now to enlarge this to Polish groups and CUAP-reps.

Let G be a Polish group, let N be a closed normal subgroup of G of index 1 or 2, and let A be an Abelian Hausdorff topological group. The operation φ_N of G on A such that, for each $a \in A$,

$$\varphi_N(g)a = a \quad \text{if } g \in N$$

and

$$\varphi_N(g)a = a^{-1} \quad \text{if } g \in G - N \tag{II.2}$$

is topological, and so A_{φ_N} is a Hausdorff topological G-module. From now on, φ_N will stand for this operation, the groups G and A concerned being clear from the context. Notice that, if $\chi \in \hat{A}$, we have

$$\Phi_N(g)\chi(a) = \chi(\varphi_N(g)a) \tag{II.3}$$

for all $g \in G$ and all $a \in A$.

Now let (E, ρ) be a topological extension of G by A_{φ_N} and suppose that there exists a normalized section σ associated with ρ which satisfies the following conditions:

(SG 1) $\Omega \circ w \circ \sigma$ is continuous for every CUA-rep w of E on $\mathfrak{H}$ satisfying $w(a) = \chi(a)\mathrm{Id}_{\mathfrak{H}}$ for all $a \in A$, with some $\chi \in \hat{A}$, and $E_{\mathrm{U}}(w) = \rho^{-1}(N)$.

(SG 2) There exists a mapping $\upsilon: \chi \mapsto \upsilon_\chi$ of $\hat{A}$ into $C^1(G, \mathbf{U}(1)_{\Phi_N})$ such that, for each $\chi \in \hat{A}$, $(\chi \circ f_\sigma)\delta\upsilon_\chi \in Z^2_B(G, \mathbf{U}(1)_{\Phi_N})$ and, if v is a locally continuous BUAM-rep of G on $\mathfrak{H}$ with $G_{\mathrm{U}}(v) = N$ and multiplier $(\chi \circ f_\sigma)\delta\upsilon_\chi$, the mapping

$$a\sigma(g) \mapsto \chi(a)\overline{\upsilon_\chi(g)}v(g) \quad (a \in A;\, g \in G), \tag{II.4}$$

of E into $\mathbf{U}_{\mathrm{UA}}(\mathfrak{H})$ is continuous.

Notice that $\chi \circ f_\sigma \in Z^2(G, \mathbf{U}(1)_{\Phi_N})$ by virtue of (II.3). The mapping $\chi \mapsto (\chi \circ f_\sigma)\delta\upsilon_\chi$ of $\hat{A}$ into $Z^2_B(G, \mathbf{U}(1)_{\Phi_N})$ passes to the quotient to define a mapping

$$\chi \mapsto [(\chi \circ f_\sigma)\delta\upsilon_\chi](\mathrm{mod}\, B^2_B(G, \mathbf{U}(1)_{\Phi_N})) \tag{II.5}$$

of $\hat{A}$ into $H^2_B(G, \mathbf{U}(1)_{\Phi_N})$ which we denote by $\gamma_B^{\sigma,\upsilon}$. We tacitly understand that the groups $H^2_b(G, \mathbf{U}(1)_{\Phi_N})$ and $H^2_B(G, \mathbf{U}(1)_{\Phi_N})$ are identified by means of γ_* (Ref. 9, Proposition 2).

For each CUA-rep w of E on $\mathfrak{H}$ satisfying $E_{\mathrm{U}}(w) = \rho^{-1}(N)$ and $w(a) = \chi(a)\mathrm{Id}_{\mathfrak{H}}$ for all $a \in A$, with some $\chi \in \hat{A}$, we have

$$w(\sigma(g))w(\sigma(g')) = \chi(f_\sigma(g, g'))w(\sigma(gg')),$$

for all g, g' in G. Hence $\Omega \circ w \circ \sigma$ is a CUA$[(\chi \circ f_\sigma)\delta\upsilon_\chi]$-rep of G on $\mathbf{P}(\mathfrak{H})$ with $G_{\mathrm{U}}(\Omega \circ w \circ \sigma) = N$. Now we assume that

(SG3) The mapping $\gamma_B^{\sigma,\upsilon}$ of $\hat{A}$ into $H^2_B(G, \mathbf{U}(1)_{\Phi_N})$ defined by (II.5) is surjective (resp. bijective).

Thus, if $\mu \in Z^2_B(G, \mathbf{U}(1)_{\Phi_N})$ and $\tilde{v}$ is a CUA$[\mu]$-rep of G on $\mathbf{P}(\mathfrak{H})$ with $G_{\mathrm{U}}(\tilde{v}) = N$, we can choose $\chi \in \hat{A}$ such that $(\chi \circ f_\sigma)\delta\upsilon_\chi \in [\mu]$. By Ref. 9, Lemma 1, there exists a normalized locally continuous Borel section Σ associated with Ω such that $v = \Sigma \circ \tilde{v}$ is a (locally continuous Borel) lifting of $\tilde{v}$ with multiplier $(\chi \circ f_\sigma)\delta\upsilon_\chi$. Let w be the mapping defined by (II.4); it is a CUA-rep of E on $\mathfrak{H}$ with $E_{\mathrm{U}}(w) = \rho^{-1}(N)$ because

$$\begin{aligned} w(a\sigma(g))w(a'\sigma(g')) &= \chi(a)(\Phi_N(g)\chi(a'))\chi(f_\sigma(g, g'))\overline{\upsilon_\chi(gg')}v(gg') \\ &= w(a(\varphi_N(g)a')f_\sigma(g, g')\sigma(gg')) \end{aligned}$$

for all a, a' in A and all g, g' in G. So we are led to the following definition.

Definition 1: Let G be a Polish group, let N be a closed normal subgroup of G of index 1 or 2, let A_{φ_N} be a Hausdorff topological G-module such that $\hat{A}$ separates points, and let (E, ρ) be a topological extension of G by A_{φ_N}. Then E is said to be a *splitting* (resp. *representation*) *group for* (G, N) [or, alternatively, (G, N) is said to admit a splitting (resp. representation) group E] if there exists a normalized section σ associated with ρ such that, for each separable complex Hilbert space $\mathfrak{H}$, the conditions (SG1), (SG2), and (SG3) written above are satisfied.

The continuous open mapping ρ of Definition 1 is called the *splitting projection* of E (onto G), the mapping σ a *splitting section* of E (or, alternatively, a splitting section associated with ρ), and the group A the *splitting kernel* of E. If $N = G$, we say simply that E is a splitting (resp. representation) group for G. Notice that E is Hausdorff.

Let G, N, A_{φ_N} be as in Definition 1 and let (E, ρ) be a quasifibered extension of G by A_{φ_N} (cf. Appendix A). We can select a normalized section σ continuous at e_G associated with ρ; furthermore, there exists a mapping $\upsilon : \chi \mapsto \upsilon_\chi$ of $\hat{A}$ into $C^1(G, \mathbf{U}(1)_{\Phi_N})$ such that, for each $\chi \in \hat{A}$, υ_χ is continuous at e_G and $(\chi \circ f_\sigma)\delta\upsilon_\chi \in Z^2_B(G, \mathbf{U}(1)_{\Phi_N})$ (Remark of Appendix A), for $\chi \circ f_\sigma$ is regular at e_G because so is f_σ. The mapping $\gamma_B^{\sigma, \upsilon}$ of $\hat{A}$ into $H^2_B(G, \mathbf{U}(1)_{\Phi_N})$ defined by (II.5) is then a group homomorphism which is actually independent of σ and υ, provided they are chosen as above. In the following, we shall denote it simply by γ_B. Hence E is a splitting (resp. representation) group for (G, N) with splitting section σ if and only if the following condition is satisfied:

($\mathrm{SG}^{\mathrm{QF}}$) The mapping γ_B of $\hat{A}$ into $H^2_B(G, \mathbf{U}(1)_{\Phi_N})$ is a surjective (resp. bijective) group homomorphism.

Indeed, the continuity of $\Omega \circ w \circ \sigma$ and of the mapping (II.4) follows from the continuity of σ and υ_χ at e_G, and from the local continuity of v. The splitting group E just considered is said to be *quasifibered*, and in particular *fibered* if (E, ρ) is a fibered extension.

A Polish splitting group is quasifibered. On the other hand, let (E, ρ) be a topological extension of a Polish group G by a Polish G-module A_{φ_N}. Then any normalized Borel section σ associated with ρ satisfies (SG1). In addition, (SG2) is satisfied with $\upsilon_\chi(g) = 1$ for all $\chi \in \hat{A}$ and all $g \in G$, so that γ_B is simply the group homomorphism $\chi \mapsto [\chi \circ f_\sigma]$ (Proposition A.2). Therefore, the Polish group E is a splitting (resp. representation) group for (G, N) with splitting section σ if and only if $\hat{A}$ separates points and ($\mathrm{SG}^{\mathrm{QF}}$) is satisfied. Since $\gamma_B = \mathrm{trg}^1_B$ (cf. Appendix B), we have also that, if $\hat{A}$ separates points, E is a splitting group for (G, N) if and only if $\inf^2_B$ is a trivial group homomorphism and that E is a representation group for (G, N) if and only if $\inf^2_B$ is trivial and $\inf^1_B$ is bijective (Corollary to Proposition B.1).

Remark 1: If the groups G, A above are assumed to be second countable locally compact, then so is E, and $\hat{A}$ separates points. The foregoing discussion on the meaning of the inflation-restriction sequence in the theory of CUAP-reps makes the junction with the work of Moore,[10] who studied second countable locally compact representation groups for G (i.e., with $N = G$) and "splitting groups", these however in a more general setting.

Remark 2: If G is a finite group, every extension of G by a Hausdorff topological G-module is trivially fibered. Definition 1 is really a generalization of Schur's definition, for in the case of projective representations of G on $\mathbf{P}(V)$ considered above, condition ($\mathrm{SG}^{\mathrm{QF}}$) translates into the requirement that the group homomorphism $\chi \mapsto [\chi \circ f_\sigma]$ of $Z^1(A, \mathbf{C}^*_0)$ into $H^2(G, \mathbf{C}^*_0)$ is surjective (resp. bijective), where $\mathbf{C}^*$ is the multiplicative group of $\mathbf{C}$. This is necessary and sufficient in order that E be a splitting (resp. representation) group for G (Ref. 12, Proposition 1.4).

Definition 2: Let G be a Polish group, let N be a closed normal subgroup of G of index 1 or 2, and let E be a splitting group for (G, N) with splitting projection ρ. A CUA-rep w of E on $\mathfrak{H}$ is said to be *split* if there exists $\chi_w \in (\mathrm{Ker}\rho)^\wedge$ such that $w(a) = \chi_w(a)\mathrm{Id}_{\mathfrak{H}}$ for all $a \in \mathrm{Ker}\rho$.

The following result was already proven above, before Definition 1.

Proposition 1: Let G, N be as in Definition 1, let E be a splitting group for (G, N) with splitting projection ρ, and let σ be any splitting section associated with ρ.

(i) For each CUAP-rep $\tilde{v}$ of G on $\mathbf{P}(\mathfrak{H})$ with $G_{\mathrm{U}}(\tilde{v}) = N$, there exist a locally continuous Borel lifting v of $\tilde{v}$ and an element χ of $(\mathrm{Ker}\rho)^\wedge$ such that the mapping w defined by (II.4) is a split CUA-rep of E on $\mathfrak{H}$ with $E_{\mathrm{U}}(w) = \rho^{-1}(N)$.

(ii) For each split CUA-rep w of E on $\mathfrak{H}$ with $E_{\mathrm{U}}(w) = \rho^{-1}(N)$, the mapping $\tilde{v} = \Omega \circ w \circ \sigma$ is a CUAP-rep of G on $\mathbf{P}(\mathfrak{H})$ with $G_{\mathrm{U}}(\tilde{v}) = N$.

Remark 3: By Proposition 1, we have the commutativity of the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Ker}\rho & \xrightarrow{\ \imath\ } & E & \xrightarrow{\ \rho\ } & G & \longrightarrow & 1, \\ & & \downarrow{\scriptstyle \chi_w} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \tilde{v}} & & \\ 1 & \longrightarrow & \mathbf{U}(1) & \xrightarrow{\ \imath'\ } & \mathbf{U}_{\mathrm{UA}}(\mathfrak{H}) & \xrightarrow{\ \Omega\ } & \mathbf{PU}_{\mathrm{UA}}(\mathfrak{H}) & \longrightarrow & 1, \end{array} \qquad (\mathrm{II}.6)$$

where $\imath$ is the canonical injection, $\imath'$ is the injection $\zeta \mapsto \zeta \mathrm{Id}_{\mathfrak{H}}$, and χ_w is as in Definition 2. The right-hand side of this diagram is the pendant of (II.1).

Remark 4: From Proposition 1, we also infer the following analog of Ref. 3, Theorem 2′: A mapping $\mu : G \times G \to \mathbf{U}(1)$ is a $(G, N, \mathfrak{H})_b$-multiplier if and only if there exists a split CUA-rep w of E on $\mathfrak{H}$ such that

$$\mu \in [(\chi_w \circ f_\sigma)\delta\upsilon_{\chi_w}](\mathrm{mod} B^2_b(G, \mathbf{U}(1)_{\Phi_N})).$$

Proposition 1 affirms that, if $\mathfrak{U}^s_{\mathrm{CUA}}(E, \rho^{-1}(N)\mathfrak{H})$ is the set of all split CUA-reps w of E on $\mathfrak{H}$ with $E_{\mathrm{U}}(w) = \rho^{-1}(N)$ and $\mathfrak{P}_{\mathrm{CUA}}(G, N, \mathfrak{H})$ is the set of all CUAP-reps $\tilde{v}$ of G on $\mathbf{P}(\mathfrak{H})$ with $G_{\mathrm{U}}(\tilde{v}) = N$, there exists a surjection

$$\eta_c : \mathfrak{U}^s_{\mathrm{CUA}}(E, \rho^{-1}(N), \mathfrak{H}) \to \mathfrak{P}_{\mathrm{CUA}}(G, N, \mathfrak{H}),$$

defined by $\eta_c(w) = \Omega \circ w \circ \sigma$. As we did already for the mappings of Ref. 3, Theorem 3′, we do not specify the dependence of η_c on $\mathfrak{H}$. Actually, η_c is independent of the splitting section σ and is compatible with the equivalence relation R_{cs} defined in $\mathfrak{U}^s_{\mathrm{CUA}}(E, \rho^{-1}(N)\mathfrak{H})$ by similarity, when pseudoequivalence (and then equivalence and similarity) of split CUA-reps are defined as follows, in concordance with the corresponding definitions for UAM-reps (Ref. 3, Subsec. II.2):

Two split CUA-reps, w on $\mathfrak{H}$ and w' on $\mathfrak{H}'$, of a splitting group E for (G, N) are said to be *pseudoequivalent* if there exists a unitary or antiunitary mapping V: $\mathfrak{H} \to \mathfrak{H}'$ and a mapping $\nu : E \to \mathbf{U}(1)$ such that

$$\nu(x) V \circ w(x) = w'(x) \circ V \qquad (\mathrm{II}.7)$$

for all $x \in E$. They are said to be *equivalent* if $\nu(x) = 1$ for all $x \in E$ and *similar* if $\mathfrak{H}' = \mathfrak{H}$ and $V = \mathrm{Id}_{\mathfrak{H}}$.

Proposition 2: Let E, G, N, ρ be as in Proposition 1, and let w, w' be split CUA-reps of E on $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, such that $E_{\mathrm{U}}(w) = E_{\mathrm{U}}(w') = \rho^{-1}(N)$.

(i) The equivalence (resp. the equality) of the CUAP-reps $\eta_c(w)$ and $\eta_c(w')$ is a necessary and sufficient condition for the pseudoequivalence (resp. for the similarity) of w and w'. For w to be irreducible, it is necessary and sufficient that $\eta_c(w)$ be irreducible.

(ii) The mapping

$$\eta_{cq} : \mathfrak{U}^s_{\mathrm{CUA}}(E, \rho^{-1}(N), \mathfrak{H})/\mathrm{R}_{cs} \to \mathfrak{H}_{\mathrm{CUA}}(G, N, \mathfrak{H}),$$

deduced from η_c by passing to the quotient, is bijective.

Proof: Let σ be any splitting section associated with ρ.

(i) The necessity of the given condition is proven by the same argument used for $\vartheta(v)$ and $\vartheta(v')$ in Ref. 3, Theorem 3(i). To prove the sufficiency, suppose that $\eta_c(w)$ and $\eta_c(w')$ are equivalent (resp. equal) and satisfy Ref. 3, (II.4), where $\tilde{v} = \eta_c(w)$ and $\tilde{v}' = \eta_c(w')$; then there exists a mapping $\nu' : G \to \mathrm{U}(1)$ such that

$$\nu'(g) V \circ w(\sigma(g)) = w'(\sigma(g)) \circ V \tag{II.8}$$

for all $g \in G$. Now we define a mapping $\nu : E \to \mathrm{U}(1)$ by

$$\nu(a\sigma(g)) = \nu'(g) k_V(\overline{\chi_w(a)}) \chi_{w'}(a), \tag{II.9}$$

where $k_V(\overline{\chi_w(a)}) = \overline{\chi_w(a)}$ if V is unitary and $k_V(\overline{\chi_w(a)}) = \chi_w(a)$ if V is antiunitary. By virtue of (II.8) and (II.9), we have

$$\nu(a\sigma(g))\, V \circ w(a\sigma(g)) = w'(a\sigma(g)) \circ V \tag{II.10}$$

for all $a \in \mathrm{Ker}\rho$ and all $g \in G$.

Since w is irreducible if and only if $w|\sigma(G)$ is irreducible (in an obvious sense), the proof of the assertion about irreducibility is as in Ref. 3, Theorem 3(i).

$\mathfrak{H}$ (ii) We have only to prove that η_{cq} is injective; so we suppose there exist two split CUA-reps w, w' of E on such that $E_{\mathrm{U}}(w) = E_{\mathrm{U}}(w') = \rho^{-1}(N)$ and $\eta_c(w) = \eta_c(w')$. This implies (II.8), and consequently (II.10), with $\mathfrak{H}' = \mathfrak{H}$, $V = \mathrm{Id}_{\mathfrak{H}}$. ■

We shall now establish (by construction) the existence of a representation group (and hence, *a fortiori*, of a splitting group). We start by introducing some terminology.

Let G, N be as above and let p_B (resp. p_b) be the canonical mapping of $Z^2_B(G, \mathrm{U}(1)_{\Phi_N})$ [resp. of $Z^2_b(G, \mathrm{U}(1)_{\Phi_N})$] onto $H^2_B(G, \mathrm{U}(1)_{\Phi_N})$. By a $(G,N)_B$-*selector* [resp. by a $(G,N)_b$-*selector*] we shall mean a section associated with p_B (resp. with p_b) which in addition is a group homomorphism. Any such section s determines a subgroup $s(H^2_B(G, \mathrm{U}(1)_{\Phi_N}))$ of $Z^2_b(G, \mathrm{U}(1)_{\Phi_N})$ which we denote simply by $s(H^2_B)$. Obviously, every $(G,N)_B$-selector is a $(G,N)_b$-selector. Let $\mathrm{ev}^s_{(g,g')}$ be the *evaluation mapping* of $s(H^2_B)$ at $(g,g') \in G \times G$, i.e., the mapping of $s(H^2_B)$ into $\mathrm{U}(1)$ given by

$$\mathrm{ev}^s_{(g,g')}(s([\mu])) = s([\mu])(g,g') \quad [\mu \in Z^2_B(G, \mathrm{U}(1)_{\Phi_N})].$$

If $s(H^2_B)$ is endowed with a group topology such that $\mathrm{ev}^s_{(g,g')}$ is continuous for all g, g' in G, then $\mathrm{ev}^s_{(g,g')} \in s(H^2_B)\hat{}$ and we denote by ev^s the mapping of $G \times G$ into $s(H^2_B)\hat{}$ defined by $\mathrm{ev}^s(g,g') = \mathrm{ev}^s_{(g,g')}$. Obviously, $\mathrm{ev}^s_{(g,g')} \in s(H^2_B)\hat{}$ for all g, g' in G when $s(H^2_B)$ is a discrete group. It is easy to check that $\mathrm{ev}^s \in Z^2(G, s(H^2_B)\hat{}_{\varphi_N})$, where we have written $s(H^2_B)\hat{}_{\varphi_N}$ for $(s(H^2_B)\hat{})_{\varphi_N}$. Notice that, if $s(H^2_B)$ is a discrete group, the compact open topology on $s(H^2_B)\hat{}$ coincides with that of pointwise convergence and makes $s(H^2_B)\hat{}$ into a compact (Hausdorff) group.

Proposition 3: Let G be a Polish group, let N be a closed normal subgroup of G of index 1 or 2, and let s be an arbitrary $(G,N)_B$-selector. If $s(H^2_B)\hat{}$ is equipped with the topology of pointwise convergence, there exists a unique topology on $s(H^2_B)\hat{}_{\varphi_N} \mathrm{ev}^s G$ making it into a fibered representation group for (G,N) with splitting projection pr_2 and a locally continuous splitting section $\sigma : g \mapsto (e_{s(H^2_B)\hat{}}, g)$.

The assertion is meaningful by virtue of the following lemma.

Lemma: If G and N are as in Proposition 3, then there exists a $(G,N)_B$-selector.

Proof: It is enough to show that $B^2_B(G, \mathrm{U}(1)_{\Phi_N})$ is divisible: The existence of a $(G,N)_B$-selector follows then because the extension $(Z^2_B(G, \mathrm{U}(1)_{\Phi_N}), p_B)$ of $H^2_B(G, \mathrm{U}(1)_{\Phi_N})$ by $B^2_B(G, \mathrm{U}(1)_{\Phi_N})$ is inessential (Ref. 13, Corollary 11.4). *Mutatis mutandis*, the proof of the divisibility is the same as in Ref. 10, Lemma 2.1. ■

Proof of Proposition 3: We define a topology on $s(H^2_B)\hat{}_{\varphi_N} \mathrm{ev}^s G$ by taking the set $\{U_\iota \times V_\kappa\}$ as the nbd filter (neighborhood filter) of the neutral element, where $\{U_\iota\}$ and $\{V_\kappa\}$ are, respectively, the nbd filters of the neutral elements of $s(H^2_B)\hat{}$ and G. The nbd filter of any other element is obtained by translation. Since the operation φ_N is topological, this topology makes $(s(H^2_B)\hat{}_{\varphi_N} \mathrm{ev}^s G, \mathrm{pr}_2)$ into a fibered extension of G by $s(H^2_B)\hat{}_{\varphi_N}$, provided ev^s is a locally regular element of $Z^2_l(G, s(H^2_B)\hat{}_{\varphi_N})$ (Ref. 14, Sec. 4). This is the case, for the mapping

$$(g,g') \mapsto \mathrm{ev}^s(g,g')(s([\mu])) = s([\mu])(g,g')$$

of $G \times G$ into $\mathrm{U}(1)$ is locally continuous as well as locally regular (Ref. 9. Corollary 2 to Proposition 3) for all $\mu \in Z^2_B(G, \mathrm{U}(1)_{\Phi_N})$. Furthermore, ev^s is the factor set defined by σ, and σ is locally continuous. In fact, let V be a symmetric nbd of e_G such that ev^s is continuous in $V \times V$ and let (g_λ) be an arbitrary net of elements of G converging to $g \in V$. Then, since σ is continuous at e_G,

$$\lim_\lambda \sigma(g_\lambda) = \lim_\lambda (\mathrm{ev}^s(g_\lambda, g^{-1}) \sigma(g_\lambda g^{-1}) \sigma(g^{-1})^{-1}) = \sigma(g).$$

The topology is unique because the identity mapping of $s(H^2_B)\hat{}_{\varphi_N} \mathrm{ev}^s G$ is a homeomorphism for all group topologies on $s(H^2_B)\hat{}_{\varphi_N} \mathrm{ev}^s G$ making σ locally continuous. Since $s(H^2_B)\hat{}$ is compact, $s(H^2_B)\hat{\hat{}}$ separates points. It remains to show that condition $(\mathrm{SG}^{\mathrm{QF}})$ is satisfied. By Pontryagin duality, the elements of $s(H^2_B)\hat{\hat{}}$ are of the form $\tau(s([\mu]))$, where $\mu \in Z^2_B(G, \mathrm{U}(1)_{\Phi_N})$; hence γ_B is bijective, for

$$\gamma_B(\tau(s([\mu]))) = [\tau(s([\mu])) \circ \mathrm{ev}^s] = [\mu]. ■ \tag{II.11}$$

In the case, studied by Schur, of projective representations of a finite group G, there exist, in general, different nonisomorphic representation groups for G. However, the corresponding splitting kernels all are isomorphic (Ref. 6, Sec. 3). It is obvious that also in the case of CUAP-reps of Polish groups we are confronted with the existence of nonisomorphic representation groups. For what concerns the splitting kernels, we generalize Schur's result as follows.

Proposition 4: Let G,N be as in Proposition 3 and suppose (G,N) admits a quasifibered representation group with splitting projection ρ. For each $(G,N)_b$-selector s, the mapping $\imath_s$ of $\mathrm{Ker}\rho$ into $s(H_B^2)^\wedge$ such that

$$(\imath_s(a))(s([\mu]))=(\gamma_B^{-1}([\mu]))(a)\quad (a\in \mathrm{Ker}\rho) \qquad \text{(II.12)}$$

for all $\mu\in Z_B^2(G,\mathbf{U}(1)_{\Phi_N})$ is a continuous injective group homomorphism with dense image if $s(H_B^2)^\wedge$ is endowed with the topology of pointwise convergence.

Proof: One sees at once that $\imath_s$ is a group homomorphism which is continuous because so is the mapping $a\mapsto(\gamma_B^{-1}([\mu]))$ (a) of $\mathrm{Ker}\rho$ into $\mathbf{U}(1)$ for all $\mu \in Z_B^2(G,\mathbf{U}(1)_{\Phi_N})$. Since $(\mathrm{Ker}\rho)^\wedge$ separates points, it follows from (II.12) that $a=a'$ whenever a,a' are in $\mathrm{Ker}\rho$ and satisfy $\imath_s(a)=\imath_s(a')$. Now let $(\mathrm{Im}\imath_s)^\perp$ denote the orthogonal of $\mathrm{Im}\imath_s$, i.e., the subgroup of all elements χ of $s(H_B^2)^{\hat{}}$ such that $\chi\,|\,\mathrm{Im}\imath_s$ is the constant function with the value 1. If χ is an arbitrary element of $(\mathrm{Im}\imath_s)^\perp$, we have

$$\chi(\imath_s(a))=(\gamma_B^{-1}(p_b(\tau^{-1}(\chi))))(a)=1$$

for all $a\in\mathrm{Ker}\rho$; hence χ is the constant function with the value 1. By virtue of Ref. 15, Chap. II, Sec. 1, Corollaire 1 to Théorème 4),

$$\mathrm{cl}(\mathrm{Im}\imath_s)=(\tau^{-1}((\mathrm{Im}\imath_s)^\perp))^\perp=s(H_B^2)^\wedge,$$

where cl stands for "closure" and all character groups are equipped with the compact open topologies. ■

Remark 5: The ordered pair $(s(H_B^2)^\wedge,\imath_s)$ supplied by Proposition 4 is the *Bohr compactification* of $\mathrm{Ker}\rho$ (as solution of a universal mapping problem).[16] Indeed, let $_d(\mathrm{Ker}\rho)^\wedge$ denote the group $(\mathrm{Ker}\rho)^\wedge$ equipped with the discrete topology and put $(s\circ\gamma_B)^{-1}=\xi$. Then ξ is a topological group isomorphism of $s(H_B^2)$ onto $_d(\mathrm{Ker}\rho)^\wedge$ and so is its dual

$$\hat{\xi}:(_d(\mathrm{Ker}\rho)^\wedge)^\wedge\to s(H_B^2)^\wedge$$

(with the compact open topologies). As $\imath_s=\hat{\xi}\circ\tau_d$, where τ_d is the canonical mapping of $\mathrm{Ker}\rho$ into $(_d(\mathrm{Ker}\rho)^\wedge)^\wedge$, our assertion follows from Ref. 17, Theorem 2.

III. POLISH AND LOCALLY COMPACT REPRESENTATION GROUPS

In the study of CUAP-reps, the existence of Polish representation (or, at least, splitting) groups is obviously desirable. However, the (fibered) representation group for (G,N) constructed in Proposition 3 is, in general, not Polish and not even metrizable. Indeed, it is Polish (resp. metrizable) if and only if the compact group $s(H_B^2)^\wedge$ is metrizable, i.e., if and only if $H_B^2(G,\mathbf{U}(1)_{\Phi_N})$ is countable (Ref. 15, Chap. II, Sec. 2, Exercice 1). Furthermore, Moore has given an example of a second countable locally compact group G which does not admit a second countable locally compact splitting group for G (Ref. 10, Chap. III, 3). In this section, we shall find conditions for the existence of Polish (and, in particular, second countable locally compact) representation groups.

Proposition 5: Let G be a Polish group and let N be a closed normal subgroup of G of index 1 or 2. Suppose that we can find a $(G,N)_b$-selector s and a group topology on $s(H_B^2)$ satisfying the following conditions:

(1) $\mathrm{ev}^s{}_{(g,g')}$ is continuous for all g,g' in G;

(2) $s(H_B^2)^\wedge$ can be endowed with a topology finer than that of pointwise convergence and making it into a Polish group P;

(3) the canonical mapping τ of $s(H_B^2)$ into $\hat{P}$ is a group isomorphism.

Then there exists a unique topology such that $P_{\varphi_N}\mathrm{ev}^sG$, equipped with this topology, is a Polish representation group for (G,N) with splitting projection pr_2 and a Borel splitting section $\sigma: g\mapsto(e_P,g)$.

Proof: Let $s(H_B^2)^\wedge_p$ denote the group $s(H_B^2)^\wedge$ endowed with the topology of pointwise convergence. Then $s(H_B^2)^\wedge_p$ is a Lusin space by virtue of assumption (2) (Ref. 18, TG IX, Sec. 6, Prop. 11), hence fully Lindelöf (Ref. 19, Chap. III, Sec. 1, Theorem 2). It follows that the Borel structure generated by its closed sets is the coarsest Borel structure on $s(H_B^2)^\wedge$ making Borel all the mappings

$$\chi\mapsto\chi(s([\mu]))\quad(\mu\in Z_B^2(G,\mathbf{U}(1)_{\Phi_N}))$$

of $s(H_B^2)^\wedge$ into $\mathbf{U}(1)$ (Ref. 19, Chap. IV, Sec. 3, Theorem 4). Thus, taking account of (1), we have that ev^s is a Borel mapping of $G\times G$ into $s(H_B^2)^\wedge_p$. On the other hand, the identity mapping of $s(H_B^2)^\wedge_p$ onto P is Borel (Ref. 18, TG IX, Sec. 6, Prop. 14), hence ev^s is Borel as a mapping of $G\times G$ into P. It follows that the extension $(P_{\varphi_N}\mathrm{ev}^sG,\mathrm{pr}_2)$ of G by P_{φ_N} is topological, with a unique Polish group topology on $P_{\varphi_N}\mathrm{ev}^sG$ such that σ is Borel (Ref. 3, Remark 13). Condition $(\mathrm{SG}^{\mathrm{QF}})$ is checked as in Proposition 3 [using assumption (3)]. Furthermore, $\hat{P}$ separates points by virtue of (3). ■

Conditions (2) and (3) of Proposition 5 are always satisfied if $s(H_B^2)$ can be equipped with a second countable locally compact group topology and then $s(H_B^2)^\wedge$ with the corresponding compact open one (which is again second countable and locally compact).

Corollary: Let G,N be as in Proposition 5, let $H_B^2(G,\mathbf{R}_{\varphi_N})$ be isomorphic to the additive group of $\mathbf{R}^n$ $(n\geq 0)$, and let π be the covering projection of $\mathbf{R}$ onto $\mathbf{U}(1)$. If the group homomorphism $(\tilde{\pi}_B)^2_*$ of $H_B^2(G,\mathbf{R}_{\varphi_N})$ into $H_B^2(G,\mathbf{U}(1)_{\Phi_N})$ has closed kernel in the canonical topology and countable cokernel, then there exists a Polish representation group for (G,N).

Proof: We shall show the existence of a $(G,N)_B$-selector s and of a second countable locally compact group topology on $s(H_B^2)$ such that $\mathrm{ev}^s{}_{(g,g')}$ is continuous for all g,g' in G: $s(H_B^2)^\wedge_{\varphi_N}\mathrm{ev}^sG$ [topologized as in Proposition 5 with $s(H_B^2)^\wedge$ endowed with the compact open topology] is then a Polish representation group for (G,N) with splitting projection pr_2.

The existence of a group isomorphism of $H_B^2(G,\mathbf{R}_{\varphi_N})$ onto $\mathbf{R}^n$ $(n>0)$ being assured by assumption, we identify the two groups through it and topologize $H_B^2(G,\mathbf{R}_{\varphi_N})$ with the canonical topology of $\mathbf{R}^n$. The additive notation is used for $Z_B^2(G,\mathbf{R}_{\varphi_N})$ and for the additive group of $\mathbf{R}^n$. Then there exist n elements $\nu_1,\ldots,\nu_n$ of $Z_B^2(G,\mathbf{R}_{\varphi_N})$ such that $\{[\nu_1],\ldots,[\nu_n]\}$ is a (vector space) basis and thus, for each $[\nu]\in H_B^2(G,\mathrm{O}_{\varphi_N})$, we can write in a unique way

$$[\nu]=\sum_{i=1}^{n} r_i[\nu_i], \qquad \text{(III. 1)}$$

where $r_i \in \mathbf{R}$ $(1 \leq i \leq n)$. We note that, if $\nu \in Z_B^2(G, \mathbf{R}_{\varphi_N})$ and $r \in \mathbf{R}$, then $[r\nu]=r[\nu]$, with $(r\nu)(g,g')=r\nu(g,g')$ for all g,g' in G. So, we have an injective group homomorphism t of $H_B^2(G, \mathbf{R}_{\varphi_N})$ into $Z_B^2(G, \mathbf{R}_{\varphi_N})$ defined by

$$t([\nu])=\sum_{i=1}^{n} r_i\nu_i, \qquad \text{(III. 2)}$$

where $[\nu]$ is given by (III. 1). Since $\mathrm{Im}(\tilde{\pi}_B)_*^2$ is divisible [and is isomorphic to $\mathbf{R}^p \times \mathbf{T}^q$, with $0 \leq p+q \leq n$ (Ref. 18, TG VII, Sec. 1, Prop. 9)], we can find a group isomorphism of $H_B^2(G, \mathbf{U}(1)_{\Phi_N})$ onto $\mathrm{Im}(\tilde{\pi}_B)_*^2 \times D$, where D is some countable group. After obvious identifications by means of this group isomorphism, every element $[\mu]$ of $H_B^2(G, \mathbf{U}(1)_{\Phi_N})$ can be written as

$$[\mu]=((\tilde{\pi}_B)_*^2([\nu]))[\epsilon], \qquad \text{(III. 3)}$$

where $[\nu] \in H_B^2(G, \mathbf{R}_{\varphi_N})$ and $[\epsilon] \in D$. We topologize $H_B^2(G, \mathbf{U}(1)_{\Phi_N})$ with the product of the finest topology on $\mathrm{Im}(\tilde{\pi}_B)_*^2$ making $(\tilde{\pi}_B)_*^2$ continuous and of the discrete topology on D. There exists an injective group homomorphism s' of $\mathrm{Im}(\tilde{\pi}_B)_*^2$ into $Z_B^2(G, \mathbf{U}(1)_{\Phi_N})$ such that

$$s'((\tilde{\pi}_B)_*^2([\nu]))=\pi \circ t([\nu]),$$

where t is given by (III. 2). On the other hand, by the argument of the lemma of Sec. II, we can find a group homomorphism s'' of D into $Z_B^2(G, \mathbf{U}(1)_{\Phi_N})$ such that $p_B(s''([\epsilon]))=[\epsilon]$ for all $[\epsilon] \in D$. Now, we can define the $(G,N)_B$-selector s by

$$s([\mu])=s'((\tilde{\pi}_B)_*^2([\nu]))s''([\epsilon]),$$

where $[\mu]$ is given by (III. 3). If $s(H_B^2)$ is topologized with the (second countable locally compact) group topology transported from $H_B^2(G, \mathbf{U}(1)_{\Phi_N})$ via s, we have then that $\mathrm{ev}^s{}_{(g,g')}$ is continuous for all g,g' in G because, by virtue of (III. 2),

$$s([\mu])=\Big(\prod_{i=1}^{n} (\pi \circ r_i\nu_i)\Big)s''([\epsilon]).$$

The corollary is obviously true also when $n=0$. ∎

Let G,N be as above and let E be a Polish representation group for (G,N) with splitting projection ρ. Then, for each normalized Borel section σ associated with ρ, the mapping

$$s_\sigma : [\mu] \mapsto \gamma_B^{-1}([\mu]) \circ f_\sigma \qquad (\mu \in Z_B^2(G, \mathbf{U}(1)_{\Phi_N}))$$

is a $(G,N)_b$-selector that we say to be defined by σ.

Remark 6: The $(G,N)_b$-selector s and the Borel splitting section σ of Proposition 5 satisfy $\mathrm{ev}^s=f_\sigma$ and $s=s_\sigma$. The last relation is proved by noting that, for each $\mu \in Z_B^2(G, \mathbf{U}(1)_{\Phi_N})$,

$$s_\sigma([\mu])=\tau(s([\mu])) \circ \mathrm{ev}^s$$

because of (II. 11).

Remark 7: If, in Proposition 5 and its corollary, G is a second countable locally compact group, we can replace everywhere (in the statements and proofs) "Polish" by "second countable locally compact".

From the exactness of the inflation-restriction sequence (B. 4) (Appendix B), we have at once

Proposition 6: Let G,N be as in Proposition 5 and suppose that there exists a topological extension (E,ρ) of G by a Polish G-module A_{φ_N} such that $\hat{A}$ separates points. If $H_B^2(E, \mathbf{U}(1)_{\Phi_N \circ \rho})$ is trivial, then E is a Polish splitting group for (G,N).

Proposition 7: Let G be a second countable locally compact group and let N be a closed normal subgroup of G of index 1 or 2. The existence of a $(G,N)_b$-selector s and of a second countable locally compact group topology on $s(H_B^2)$ such that $\mathrm{ev}^s{}_{(g,g')}$ is continuous for all g,g' in G is a necessary and sufficient condition in order that (G,N) should admit a second countable locally compact representation group.

Proof: The sufficiency of the condition follows at once from Proposition 5 with the compact open topology on $s(H_B^2)\hat{}$. To prove the necessity, let E be a second countable locally compact representation group for (G,N) with splitting projection ρ. Take a normalized Borel section σ associated with ρ, the $(G,N)_b$-selector s_σ defined by σ, and transport on $s_\sigma(H_B^2)$ the compact open topology of $(\mathrm{Ker}\rho)\hat{}$ via $s_\sigma \circ \gamma_B$. Then the mapping

$$s_\sigma([\mu]) \mapsto \mathrm{ev}^{s_\sigma}{}_{(g,g')}(s_\sigma([\mu]))$$
$$=\gamma_B^{-1}([\mu])(f_\sigma(g,g'))$$

is continuous for all g,g' in G because the compact open topology is finer than that of pointwise convergence. The result follows with $s=s_\sigma$. ∎

We call *s-topology* the second countable locally compact group topology on $s(H_B^2)\hat{}_{\varphi_N}\mathrm{ev}^sG$ considered in Proposition 7 [and determined by the topologies of G and $s(H_B^2)$].

Proposition 8: Let G,N be as in Proposition 7, suppose that there exists a second countable locally compact representation group E for (G,N) with splitting projection ρ, let σ be any normalized Borel section associated with ρ, transport on $s_\sigma(H_B^2)$ the compact open topology of $(\mathrm{Ker}\rho)\hat{}$ via $s_\sigma \circ \gamma_B$, and put on $s_\sigma(H_B^2)\hat{}$ the compact open topology. Then

(i) There exists a topological group isomorphism λ_σ^E of E onto $s_\sigma(H_B^2)\hat{}_{\varphi_N}\mathrm{ev}^{s_\sigma}G$, equipped with the s_σ-topology, such that $\mathrm{pr}_2 \circ \lambda_\sigma^E=\rho$.

(ii) If E' is any other second countable locally compact representation group for (G,N), with splitting projection ρ', there exist $f \in Z_b^2(G, s_\sigma(H_B^2)\hat{}_{\varphi_N})$ and a topological group isomorphism $\kappa_\sigma^{E'}$ of E' onto $s_\sigma(H_B^2)\hat{}_{\varphi_N}fG$ such that $\mathrm{pr}_2 \circ \kappa_\sigma^{E'}=\rho'$, when $s_\sigma(H_B^2)\hat{}_{\varphi_N}fG$ is equipped with the unique topology making it into a second countable locally compact representation group for (G,N) with splitting projection pr_2 and a Borel splitting section $g \mapsto (e_{s_\sigma(H_B^2)\hat{}}, g)$.

Proof: (i) We have already shown that $\mathrm{ev}^{s_\sigma}{}_{(g,g')}$ is continuous for all g,g' in G (Proposition 7) and that ev^{s_σ} is Borel (Proposition 5). We define the group isomorphism λ_σ^E by

$$\lambda_\sigma^E(a\sigma(g))=(\hat{\beta}_\sigma(\tau(a)),g) \quad (a \in \mathrm{Ker}\rho;\ g \in G), \qquad \text{(III. 4)}$$

where $\hat{}$ denotes the dual mapping and β_σ stands for $(s_\sigma \circ \gamma_B)^{-1}$. It is topological because it is Borel (Ref. 3, Remark 13).

(ii) Let σ' be a normalized Borel section associated with ρ', transport on $s_{\sigma'}(H_B^2)$ the compact open topology of $(\mathrm{Ker}\rho')\hat{}$ via $s_{\sigma'}\circ\gamma_B$, and endow $s_{\sigma'}(H_B^2)\hat{}_{\varphi_N}\mathrm{ev}^{s_{\sigma'}}G$ with the $s_{\sigma'}$-topology. The mapping

$$\imath_{\sigma\sigma'}: s_\sigma([\mu]) \mapsto s_{\sigma'}([\mu]) \quad (\mu\in Z_B^2(G,\mathbf{U}(1)_{\Phi_N}))$$

of $s_\sigma(H_B^2)$ onto $s_{\sigma'}(H_B^2)$ is a topological group isomorphism (by Ref. 10, Theorem 2.2). Notice that we need only a very special case of this theorem, so that the proof of Moore can be greatly simplified. Indeed, the two topologies transported from $s_\sigma(H_B^2)$, $s_{\sigma'}(H_B^2)$ onto $H_B^2(G,\mathbf{U}(1)_{\Phi_N})$ through s_σ^{-1}, $s_{\sigma'}^{-1}$, respectively, are Hausdorff and the corresponding transgression mappings are topological group isomorphisms. We define $f\in Z_0^2(G,s_\sigma(H_B^2)\hat{}_{\varphi_N})$ by $f=\hat{\imath}_{\sigma\sigma'}\circ \mathrm{ev}^{s_{\sigma'}}$; then $\lambda:(a,g)\mapsto(\hat{\imath}_{\sigma\sigma'}(a),g)$ is a topological group isomorphism of $s_{\sigma'}(H_B^2)\hat{}_{\varphi_N}\mathrm{ev}^{s_{\sigma'}}G$ onto $s_\sigma(H_B^2)\hat{}_{\varphi_N}fG$ and we have the result with $\kappa_\sigma^{E'}=\lambda\circ\lambda_{\sigma'}^{E'}$, where $\lambda_{\sigma'}^{E'}$ is defined by (III.4) with E' instead of E and σ' instead of σ. ■

Remark 8: Proposition 8 generalizes Proposition 3.1 of Ref. 10.

IV. APPLICATIONS

We shall now apply the results of Secs. II and III to three important (nondisjoint) types of Polish groups. In the following, G shall always denote a Polish group and N a closed normal subgroup of G of index 1 or 2.

(A) *G and N are such that $H_B^2(G,\mathbf{U}(1)_{\Phi_N})$ is countable.*

There exists a fibered Polish representation group for (G,N), namely the group $F_s(G,N)=s(H_B^2)\hat{}_{\varphi_N}\mathrm{ev}^sG$ constructed in Proposition 3 by means of a $(G,N)_B$-selector s. This is a second countable locally compact (resp. compact) group if so is G. The Poincaré group (or the Lorentz group), with its orthochronous subgroup as N, is an example.[20]

The following types (A_1) and (A_2) are subtypes of (A):

(A_1) *G admits a universal covering group $\widetilde{G}$ such that $\pi_1(G)$ (the fundamental group of G) is finite and $H_B^2(\widetilde{G},\mathbf{U}(1)_0)$ is trivial.*

Notice that $\widetilde{G}$ is Polish and that, since G is connected, $N=G$. We see that G is of type (A) because $\widetilde{G}$ is a splitting group for G (Proposition 6) and, if $\rho_G:\widetilde{G}\to G$ is the covering projection, then the discrete group $\mathrm{Ker}\rho_G$ is finite, so that

$$H_B^2(G,\mathbf{U}(1)_0)\approx(\mathrm{Ker}\rho_G)\hat{}/\mathrm{Ker}\,\mathrm{trg}_B^1$$

is finite.

A compact semisimple connected real Lie group G is of type (A_1). Indeed,

$$H_B^2(G,\mathbf{U}(1)_0)\approx\pi_1(G)\hat{}$$

(Ref. 21, Corollary 4.2) is finite and $F_s(G)=\widetilde{G}$ is the unique representation group for G up to topological group isomorphisms. Of type (A_1) are also the Bondi—Metzner—Sachs group B (cf. Ref. 22 and references therein) and the neutral components $\mathbf{P}_0$, respectively $\mathbf{L}_0$, of the Poincaré and Lorentz groups. Again, $F_s(B)$ [resp. $F_s(\mathbf{P}_0)$, resp. $F_s(\mathbf{L}_0)$] is the universal covering group of B (resp. of $\mathbf{P}_0$, resp. of $\mathbf{L}_0$) and is its unique representation group up to topological group isomorphisms.

(A_2) *G is a second countable compact group* (Ref. 23, Theorem 2.2 and its corollary).

Moreover, in this case $H_B^2(G,\mathbf{U}(1)_{\Phi_N})$ is a torsion group (and is finite if, in addition, G is a real Lie group).

(B) *G admits a topological extension (E,ρ) by a Polish G-module A_{φ_N} such that $H_c^1(E,\mathbf{U}(1)_{\Phi_N\circ\rho})$ and $H_B^2(E,\mathbf{U}(1)_{\Phi_N\circ\rho})$ are trivial.*

It follows from the exactness of the inflation-restriction sequence (B.4) that E is a Polish representation group for (G,N), i.e., that

$$H_B^2(G,\mathbf{U}(1)_{\Phi_N})\approx(\mathrm{Ker}\rho)\hat{}$$

via trg_B^1. A semisimple connected (finite-dimensional) real Lie group G is then of type (B) (cf. also Ref. 21, Theorem 4.4), with $E=\widetilde{G}$ and $\rho=\rho_G$ (the covering projection), and $\widetilde{G}$ is a representation group for G which is the unique second countable locally compact one up to topological group isomorphisms.

(C) *G is an almost connected second countable locally compact group.*

By "almost connected," we mean that the quotient group G/G_0 is compact, where G_0 is the neutral component of G. There exists always a second countable locally compact splitting group for (G,N) (Ref. 10, Proposition 2.2). Furthermore, we have the following result (Ref. 10, Propositions 2.6 and 2.7):

Let $\imath$ be the canonical injection of $\mathbf{Z}$ into $\mathbf{R}$. In order that (G,N) admit a second countable locally compact representation group, it is necessary and sufficient that $\mathrm{Im}(\tilde{\imath}_B)_*^2$ be closed in the finite-dimensional (Ref. 10, Theorem 1.1) real vector space $H_B^2(G,\mathbf{R}_{\varphi_N})$ endowed with the canonical topology.

Notice that the sufficiency of the condition just stated follows at once from the corollary to Proposition 5 (taking account of Ref. 10, Theorem 1.2), which shows, moreover, how a representation group for (G,N) can then be constructed. Examples are the Galilei group and its neutral component.[20,24]

If, in particular, G is a connected (finite-dimensional) real Lie group (and so $N=G$), it admits a (uniquely determined) connected and simply connected finite-dimensional real Lie splitting group (Ref. 21, Theorem 2.1; cf. also Ref. 24) which, however, is not in general a representation group. This can be seen in the case $G=\mathbf{T}$, where $\mathbf{T}$ itself is a representation group for $\mathbf{T}$ unique up to topological group isomorphisms. On the other hand, if G admits a connected real Lie representation group E, then the splitting group in question is $\widetilde{E}$. This is the case when G is, in addition, simply connected: $H_B^2(G,\mathbf{U}(1)_0)$ is a finite-dimensional real vector space because it is isomorphic to the Chevalley—Eilenberg cohomology space $H^2(\mathrm{Lie}(G),\mathbf{O}_0)$, where $\mathrm{Lie}(G)$ denotes the Lie algebra of G and the $\mathrm{Lie}(G)$-module $\mathbf{R}_0$ is trivial (Ref. 2, Theorems 4.1 and 5.1), and $(\tilde{\pi}_B)_*^2$ is a bijection (Ref. 23, Theorem A).

APPENDIX A: QUASIFIBERED EXTENSIONS

Let G be a Hausdorff topological group and let A_Ψ be a Hausdorff topological G-module. A topological extension (E,ρ) of G by A_Ψ is said to be *quasifibered* if there exists a normalized section continuous at e_G associated with ρ (Ref. 14, Déf. 3.2 and Prop. 3.5). If $\mathrm{Ext}_t^{QF}(G,A_\Psi)$ denotes the subset of $\mathrm{Ext}_t(G,A_\Psi)$ formed by all equivalence classes of quasifibered extensions (which becomes an Abelian group with the Baer multiplication as law of composition (Ref. 14, Sec. 9), then we have

$$\mathrm{Ext}_t(G,A_\Psi)=\mathrm{Ext}_t^{QF}(G,A_\Psi)$$

when G and A are metrizable (Ref. 14, Prop. 3.6).

The cochain complex suited to the study of quasifibered extensions of G by A_Ψ is the subcomplex $C_e^*(G,A_\Psi)$ of the Eilenberg—MacLane cochain complex $C^*(G,A_\Psi)$ such that, for $p>0$, $C_e^p(G,A_\Psi)$ is the subgroup of all elements of $C^p(G,A_\Psi)$ continuous at the neutral element of G^p and, for $p\leq 0$, $C_e^p(G,A_\Psi)=C^p(G,A_\Psi)$. We denote the relevant groups of $C_e^*(G,A_\Psi)$ by the corresponding Eilenberg—MacLane symbols with an additional subscript "e".

In analogy with the case of fibered extensions, a given $f\in Z_e^2(G,A_\Psi)$ is not always a factor set of a quasifibered extension of G by A_Ψ. So, we are led to consider the elements f of $Z_e^2(G,A_\Psi)$ *regular at* e_G, i.e., such that the mapping

$$g\mapsto f(g'^{-1},g')^{-1}f(g'^{-1},g)f(g'^{-1}g,g')$$

of G into A_Ψ is continuous at e_G for all $g'\in G$. Notice that, if f is an element of $Z_e^2(G,A_\Psi)$ regular at e_G, then every 2-cocycle cohomologous to f modulo $B_e^2(G,A_\Psi)$ is regular at e_G. The set

$$H_{er}^2(G,A_\Psi)=\{[f](\mathrm{mod}B_e^2(G,A_\Psi))\,|\,f\in Z_e^2(G,A_\Psi) \text{ and } f \text{ regular at } e_G\}$$

is a subgroup of $H_e^2(G,A_\Psi)$, and we have the following result of Calabi (Ref. 14, Cor. 3 to Prop. 9.2):

Proposition A.1: Let G be a Hausdorff topological group and let A_Ψ be a Hausdorff topological G-module. There exists a group isomorphism

$$\alpha_e:[(E,\rho)]\mapsto[f](\mathrm{mod}B_e^2(G,A_\Psi))$$

of $\mathrm{Ext}_t^{QF}(G,A_\Psi)$ onto $H_{er}^2(G,A_\Psi)$, where f is the factor set of (E,ρ) defined by a normalized section continuous at e_G associated with ρ.

It follows from Proposition A.1 that an element f of $Z_e^2(G,A_\Psi)$ is regular at e_G if and only if it is a factor set of a quasifibered extension of G by A_Ψ.

Proposition A.2: Let G be a Polish group and let A_Ψ be a Polish G-module fibering over G. Then there exists a group isomorphism γ_e of $H_{er}^2(G,A_\Psi)$ onto $H_B^2(G,A_\Psi)$.

Proof: We have $\gamma_e=\gamma_*\circ\alpha_b\circ\alpha_e^{-1}$, where α_e, α_b, γ_* are, respectively, as in Proposition A.1, Ref. 8 (Theorem 2), and Ref. 9 (Proposition 2). The mapping α_b is a group isomorphism by the theorem of Brown.[25] ∎

Corollary: Let G,A_Ψ be as in Proposition A.2 and let f be an element of $Z_e^2(G,A_\Psi)$ regular at e_G. There exists $h\in C_e^1(G,A_\Psi)$ such that $f\delta h\in Z_B^2(G,A_\Psi)$.

Proof: By reason of Proposition A.2, we can find $h'\in C^1(G,A_\Psi)$ such that $f\delta h'\in Z_B^2(G,A_\Psi)$. Moreover, f (resp. $f\delta h'$) is the factor set of a topological extension (E,ρ) of G by A_Ψ defined by a normalized section σ continuous at e_G (resp. by a normalized locally continuous Borel section σ') associated with ρ. If we define $h\in C_e^1(G,A_\Psi)$ by $h(g)=\sigma'(g)\sigma(g)^{-1}$, then $f\delta h=f\delta h'$. ∎

Remark: It follows from Proposition A.2 and its corollary that, for each element f of $Z_e^2(G,A_\Psi)$ regular at e_G, there exists $h\in C_e^1(G,A_\Psi)$ such that $f\delta h\in Z_B^2(G,A_\Psi)$ and

$$\gamma_e([f](\mathrm{mod}B_e^2(G,A_\Psi)))=[f\delta h](\mathrm{mod}B_B^2(G,A_\Psi)).$$

Furthermore, if f_1,f_2 are in $Z_B^2(G,A_\Psi)$ and $[f_1]=[f_2](\mathrm{mod}B_e^2(G,A_\Psi))$, then we have $[f_1]=[f_2](\mathrm{mod}B_B^2(G,A_\Psi))$.

APPENDIX B: ON THE INFLATION-RESTRICTION SEQUENCE

Let G be a Polish group, let B_Ψ be a Polish G-module, let (E,ρ) be a topological extension of G by a Polish G-module A_ψ, and let σ be a normalized Borel section associated with ρ. We denote by Ψ_ρ the topological operation $\Psi\circ\rho$ of E on B and notice that $\Psi_\rho|A$ is trivial. Remember that, for Polish groups, Borel 1-cocycles are continuous. The group $H_c^1(A,B_0)$ is identified, as usual, with $Z_c^1(A,B_0)$, and then $H_c^1(A,B_0)^G$ is the subgroup of all elements $h\in Z_c^1(A,B_0)$ satisfying the condition

$$\Psi(g)h(a)=h(\psi(g)a) \tag{B1}$$

for all $g\in G$ and all $a\in A$. The following group homomorphisms are particular cases of those (called, respectively, inflation, restriction, and transgression) of homological algebra.

(1) $\mathrm{inf}_b^n:H_b^n(G,B_\Psi)\to H_b^n(E,B_{\Psi_\rho})$ $(n=1,2)$, deduced from the group homomorphism $f\mapsto f\circ\rho^n$ of $Z_b^n(G,B_\Psi)$ into $Z_b^n(E,B_{\Psi_\rho})$ by passing to the quotients.

(2) $\mathrm{res}_b^1:H_c^1(E,B_{\Psi_\rho})\to H_c^1(A,B_0)^G$, deduced from the group homomorphism $h\mapsto h|A$ of $Z_c^1(E,B_{\Psi_\rho})$ into $Z_c^1(A,B_0)^G$ by passing to the quotients. Notice that $h|A$ satisfies (B1) on account of the cocycle identity for h.

(3) $\mathrm{trg}_b^1:H_c^1(A,B_0)^G\to H_b^2(G,B_\Psi)$, given by $\mathrm{trg}_b^1(h)=[h\circ f_\sigma]$, where f_σ is the factor set of (E,ρ) defined by σ. We have $h\circ f_\sigma\in Z_b^2(G,B_\Psi)$ because h satisfies (B1), and it is obvious that trg_b^1 does not depend on the choice of the Borel section σ.

Proposition B.1: Let G be a Polish group, let B_Ψ be a Polish G-module, and let (E,ρ) be a topological extension of G by a Polish G-module A_ψ. Then the sequence of groups

$$1\to H_c^1(G,B_\Psi)\xrightarrow{\mathrm{inf}_b^1}H_c^1(E,B_{\Psi_\rho})\xrightarrow{\mathrm{res}_b^1}H_c^1(A,B_0)^G \xrightarrow{\mathrm{trg}_b^1}H_b^2(G,B_\Psi)\xrightarrow{\mathrm{inf}_b^2}H_b^2(E,B_{\Psi_\rho}), \tag{B2}$$

[the inflation-restriction sequence for $(E,G,\rho;B_\Psi)$] is exact.

Proof: We shall not consider the sequence (B2) in its natural context, namely that of spectral sequences, but merely check exactness at each joint. Let σ be a normalized Borel section associated with ρ.

(a) *Exactness at* $H_c^1(G, B_\Psi)$: If $[h] \in \mathrm{Ker}\,\mathrm{inf}_b^1$, there exists $b \in B$ such that, for each $x \in E$, $h(\rho(x)) = b^{-1}(\Psi(\rho(x))b)$, i.e., $h \in B_c^1(G, B_\Psi)$.

(b) *Exactness at* $H_c^1(E, B_{\Psi_\rho})$: If $h \in Z_c^1(G, B_\Psi)$, then $(h \circ \rho)(a) = e_B$ for all $a \in A$, so that $\mathrm{Im}\,\mathrm{inf}_b^1 \subseteq \mathrm{Ker}\,\mathrm{res}_b^1$. On the other hand, if $[h] \in \mathrm{Ker}\,\mathrm{res}_b^1$, then $h(a) = e_B$ and $h(a\sigma(g)) = h(\sigma(g))$ for all $a \in A$ and all $g \in G$. It follows that $h \circ \sigma \in Z_c^1(G, B_\Psi)$ and $\mathrm{inf}_b^1([h \circ \sigma]) = [h]$, whence $\mathrm{Ker}\,\mathrm{res}_b^1 \subseteq \mathrm{Im}\,\mathrm{inf}_b^1$.

(c) *Exactness at* $H_c^1(A, B_0)^G$: Let $h \in Z_c^1(E, B_{\Psi_\rho})$ and put $h|A = l$. From the cocycle identity, we obtain $l \circ f_\sigma = \delta(h \circ \sigma)$; hence

$$\mathrm{trg}_b^1(\mathrm{res}_b^1([h])) = \mathrm{trg}_b^1(l) = [l \circ f_\sigma]$$

is the neutral element of $H_b^2(G, B_\Psi)$ and thus $\mathrm{Im}\,\mathrm{res}_b^1 \subseteq \mathrm{Ker}\,\mathrm{trg}_b^1$. If now l denotes an arbitrary element of $\mathrm{Ker}\,\mathrm{trg}_b^1$, there exists $h' \in C_b^1(G, B_\Psi)$ such that $l \circ f_\sigma = \delta h'$. We define a Borel mapping h of E into B by

$$h(a\sigma(g)) = l(a)h'(g) \quad (a \in A;\ g \in G)$$

(Ref. 19, Chap. I. Sec. 6, Proposition 4) and check that $h \in Z_c^1(E, B_{\Psi_\rho})$. As $\mathrm{res}_b^1([h]) = l$, we have $\mathrm{Ker}\,\mathrm{trg}_b^1 \subseteq \mathrm{Im}\,\mathrm{res}_b^1$.

(d) *Exactness at* $H_b^2(G, B_\Psi)$: If $h \in C_b^1(E, B_{\Psi_\rho})$ is such that $h|A \in H_c^1(A, B_0)^G$ and $h(a\sigma(g)) = h(a)h(\sigma(g))$ for all $a \in A$ and all $g \in G$, then

$$\delta h(a\sigma(g),\ a'\sigma(g')) = h(f_\sigma(g, g'))^{-1}\delta(h \circ \sigma)(g, g') \qquad \text{(B3)}$$

for all a, a' in A and all g, g' in G [the "δ" to the left- (resp. to the right-) hand side of (B3) denotes the coboundary operator of $C^*(E, B_{\Psi_\rho})$ (resp. of $C^*(G, B_\Psi)$)]. Now let l be an arbitrary element of $H_c^1(A, B_0)^G$ and define a Borel mapping h of E into B by $h(a\sigma(g)) = l(a)^{-1}$ $(a \in A;\ g \in G)$. It follows from (B3) that $\mathrm{inf}_b^2(\mathrm{trg}_b^1(l))$ is the neutral element of $H_b^2(E, B_{\Psi_\rho})$ and thus $\mathrm{Im}\,\mathrm{trg}_b^1 \subseteq \mathrm{Ker}\,\mathrm{inf}_b^2$. Conversely, if $[f] \in \mathrm{Ker}\,\mathrm{inf}_b^2$, there exists $h \in C_b^1(E, B_{\Psi_\rho})$ such that $f \circ \rho^2 = \delta h$. Since h satisfies the conditions stated above, we have $\mathrm{trg}_b^1((h|A)^{-1}) = [f]$ by virtue of (B3); hence $\mathrm{Ker}\,\mathrm{inf}_b^2 \subseteq \mathrm{Im}\,\mathrm{trg}_b^1$. ∎

If G and A are second countable locally compact groups, Proposition B.1 is a particular case of a result of Moore (Ref. 23, Chap. I, 5).

Taking account of Ref. 9, Proposition 2 and of (II.3), we have the following result, where the "B mappings" are defined in an obvious way and $\Phi_{N_\rho} = \Phi_N \circ \rho$.

Corollary: Let G be a Polish group, let N be a closed normal subgroup of G of index 1 or 2, and let (E, ρ) be a topological extension of G by a Polish G-module A_{φ_N}. Then the inflation-restriction sequence

$$1 \to H_c^1(G, \mathrm{U}(1)_{\Phi_N}) \xrightarrow{\mathrm{inf}_B^1} H_c^1(E, \mathrm{U}(1)_{\Phi_{N_\rho}}) \xrightarrow{\mathrm{res}_B^1} \hat{A}$$
$$\xrightarrow{\mathrm{trg}_B^1} H_B^2(G, \mathrm{U}(1)_{\Phi_N})$$
$$\xrightarrow{\mathrm{inf}_B^2} H_B^2(E, \mathrm{U}(1)_{\Phi_{N_\rho}}) \qquad \text{(B4)}$$

is exact.

Remark: The corollary is obviously still valid if, instead of $\mathrm{U}(1)_{\Phi_N}$, we have a Polish G-module B_Ψ fibering over G and, instead of A_{φ_N}, a Polish G-module A_ψ, provided $\hat{A}$ is replaced by $H_c^1(A, B_0)^G$.

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Inequalities and uncertainty principles

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Sobolev inequalities give lower bounds for quantum mechanical Hamiltonians. These inequalities are derived from commutator inequalities related to the Heisenberg uncertainty principle.

1. INTRODUCTION

This article is an attempt to bring together and survey various results that generalize and improve on the Heisenberg uncertainty principle. These results might be called local uncertainty principles, since they say that when the momentum uncertainty is small, not only is the position uncertainty large, but the probability of being localized at any particular point is very small. This information is useful for obtaining lower bounds to quantum mechanical Hamiltonians.

The more elementary local uncertainty principles are consequences of a general commutator inequality. This derivation, which uses only the most basic principles of quantum mechanics, shows why the inequalities depend so crucially on the dimension of space. The most powerful local uncertainty principle, the Sobolev inequality, is shown to be a consequence of one of these elementary principles. The utility of Sobolev inequalities in quantum mechanics is of course well known;[1,2] the purpose is to clarify their physical meaning.

The hydrogen atom provides an illustration of the point of view. The Hamiltonian is $H=p^2/(2m)-e^2/r$, where m and e are the mass and charge of the electron, and r is its distance from the nucleus. The expectation of H in a particular state is

$$\langle H\rangle=\langle p^2\rangle/2m-e^2\langle r^{-1}\rangle. \tag{1.1}$$

If this is bounded below independently of the state, we have a lower bound for H.

It is clear that

$$e^2\langle r^{-1}\rangle\leq\langle p^2\rangle/2m+(me^4/2)\langle r^{-1}\rangle^2/\langle p^2\rangle. \tag{1.2}$$

Thus it is sufficient to get a lower bound for $-(me^4/2)\times\langle r^{-1}\rangle^2/\langle p^2\rangle$, that is, a lower bound for $\langle p^2\rangle^{1/2}\langle r^{-1}\rangle^{-1}$. The ordinary Heisenberg uncertainty principle, in n-dimensional space, gives $\langle p^2\rangle^{1/2}\langle r^2\rangle^{1/2}\geq n\hbar/2$, where $\hbar$ is Planck's constant (rationalized). If it were true in general that $\langle r^2\rangle^{1/2}\leq\langle r^{-1}\rangle^{-1}$, this would give the lower bound $-(me^4/2\hbar^2)(2/n)^2$. However it is not difficult to imagine states where this fails; the probability is too concentrated near $r=0$, and $\langle r^{-1}\rangle$ can become exceedingly large. In fact the argument must be wrong, since the conclusion is so spectacularly false for $n=1$. The Hamiltonian in one dimension is not even bounded below.

There is a local uncertainty principle that does make the argument work, namely

$$\langle p^2\rangle^{1/2}\langle r^{-1}\rangle^{-1}\geq(n-1)\hbar/2. \tag{1.3}$$

This inequality says that the state cannot be too concentrated near $r=0$ without forcing a large kinetic energy. Since it is always true that $\langle r^{-1}\rangle^{-1}\leq\langle r\rangle\leq\langle r^2\rangle^{1/2}$, the inequality is qualitatively stronger than the Heisenberg uncertainty principle. In fact, it is enough to give the correct lower bound

$$\langle H\rangle\geq-(me^4/2\hbar^2)/[2/(n-1)^2]. \tag{1.4}$$

The rest of this article contains progressively stronger results. Section 2 has the abstract commutator inequalities and Secs. 3 and 4 apply these to one- and n-dimensional systems. The strong forms of the local uncertainty principle are valid only for $n\geq 3$; this is why weak short range potentials do not have negative energy bound states for $n\geq 3$. Section 5 has the derivation of Sobolev inequalities. Finally, Sec. 6 contains a brief survey of some recent developments related to these inequalities.

This article had its origin in lectures given to the participants at the Third International Conference on Group Theory in Physics, held at the CNRS in Marseille in June 1974. I am indebted to Richard Lavine for advice about commutator estimates.

2. COMMUTATOR INEQUALITIES

In general unbounded observables may not be added and multiplied freely, because of possible ambiguities, such as $\infty-\infty$. We shall mainly deal with situations where these problems do not arise. We also assume that the adjoint satisfies the usual relations $A^{**}=A$, $(A+B)^*=A^*+B^*$, $(AB)^*=B^*A^*$. The real observables are those satisfying $A=A^*$. The positive ones are of the form A^*A. The expectation of A in a fixed state is written $\langle A\rangle$. The expectation is linear in A. In addition it is positive: $A\geq 0$ implies $\langle A\rangle\geq 0$, and normalized so $\langle 1\rangle=1$.

If we use these properties to compute $0\leq\langle(A-iB)^*\times(A-iB)\rangle=\langle A^*A\rangle-i\langle A^*B-B^*A\rangle+\langle B^*B\rangle$, we obtain the basic commutator bound

$$i\langle A^*B-B^*A\rangle\leq\langle A^*A\rangle+\langle B^*B\rangle. \tag{2.1}$$

This inequality is called a commutator bound because when A and B are real it involves the commutator $AB-BA$.

If we apply the commutator bound to $A\langle A^*A\rangle^{-1/2}$ and $B\langle B^*B\rangle^{-1/2}$, we obtain $i\langle A^*B-B^*A\rangle\leq 2\langle A^*A\rangle^{1/2}\langle B^*B\rangle^{1/2}$. If we apply this in turn to a suitable complex multiple of A, we arrive at the Schwarz inequality

$$|\langle A^*B\rangle|\leq\langle A^*A\rangle^{1/2}\langle B^*B\rangle^{1/2}. \tag{2.2}$$

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One simple consequence of the Schwartz inequality is worth noting. If $Y > 0$, set $A = Y^{1/2}$ and $B = Y^{-1/2}$. Then we obtain the inequality of the harmonic mean,

$$\langle Y^{-1}\rangle^{-1} \leq \langle Y\rangle. \tag{2.3}$$

Our estimates will come from when the left-hand side of the commutator bound is positive. If B is real, and if $\exp(-iaA/\hbar)$ is defined for $a \geq 0$, we may define $B(a) = \exp(iaA^*/\hbar)B\exp(-iaA/\hbar)$. In general we do not wish to assume that $A = A^*$, so the evolution that sends B into $B(a)$ need not preserve the algebraic operations. In any case, the value of $dB(a)/da$ at $a = 0$ is $(i/\hbar)(A^*B - BA)$. Thus $(i/\hbar)\langle A^*B - BA\rangle$ is positive when $\langle B(a)\rangle$ is increasing with a. So we look for positive commutators by looking for some observable B that is increasing under some evolution.

3. ONE-DIMENSIONAL SYSTEMS

Quantum mechanics in one dimension is based on the commutation relation $pq - qp = -i\hbar$, where q and p are the position and momentum observables, and $\hbar > 0$ is constant. The Schwarz inequality gives $\hbar/2 \leq \langle p^2\rangle^{1/2} \times \langle q^2\rangle^{1/2}$ for any state. If we replace p by $p - \langle p\rangle$ and q by $q - \langle q\rangle$, the same argument works and gives the Heisenberg uncertainty principle

$$\Delta p \Delta q \geq \hbar/2. \tag{3.1}$$

The following local uncertainty principle is an attempt to improve on this.

Local uncertainty principle: Let a be any point and b any positive number. Then for every state

$$\text{Prob}\{|q - a| \leq b\} \leq 2b\Delta p/\hbar. \tag{3.2}$$

Proof: It is easy to see that $pq^k - q^kp = -i\hbar kq^{k-1}$ and hence that $p\phi(q) - \phi(q)p = -i\hbar\phi'(q)$. Thus by the Schwarz inequality

$$\hbar\langle\phi'(q)\rangle \leq 2\langle p^2\rangle^{1/2}\langle\phi(q)^2\rangle^{1/2}. \tag{3.3}$$

Apply this to $\phi(q) = (2/\pi)\arctan((q - x)/\epsilon)$. Since $|\phi(q)| \leq 1$, this gives

$$\hbar\langle\delta_\epsilon(q - x)\rangle \leq \langle p^2\rangle^{1/2}, \tag{3.4}$$

where $\delta_\epsilon(t) = (1/\pi)\epsilon/(t^2 + \epsilon^2)$ is an approximate delta function. Let $\chi(t) = 1$ where $|t| \leq b$ and $\chi(t) = 0$ elsewhere. Integrate the last inequality over $|x| \leq b$ and let $\epsilon \to 0$. We obtain

$$\hbar\,\text{Prob}\{|q| < b\} = \hbar\langle\chi(q)\rangle \leq 2b\langle p^2\rangle^{1/2}.$$

If we replace p and q by $p - \langle p\rangle$ and $q - a$ we arrive at the local uncertainty principle.

Let us see in what sense this is a more powerful result than the Heisenberg uncertainty principle. We know from the elementary Chebyshev inequality that

$$\text{Prob}\{|q - \langle q\rangle| \geq b\} \leq (\Delta q/b)^2. \tag{3.5}$$

Combine this with the local uncertainty principle. This gives

$$(1 - 2b\Delta p/\hbar) \leq (\Delta q/b)^2. \tag{3.6}$$

Choose $b = \hbar/(4\Delta p)$. We obtain

$$\tfrac{1}{2} \leq (4\Delta q\Delta p/\hbar)^2. \tag{3.7}$$

This is not quite the Heisenberg uncertainty principle, because the constant is wrong. But it is qualitatively an inequality of the same kind. The point is that it is difficult to imagine that one could reverse the argument and derive something like the local principle from the Heisenberg principle.

The local uncertainty principle is relevant to the question of lower bounds. For instance, consider a one-dimensional Hamiltonian $H = p^2/2m + v(q)$, where $\|v\|_1 = \int|v(x)|\,dx < \infty$. It follows from the proof of the local uncertainty principle that

$$\begin{aligned}|\langle v(q)\rangle| &\leq \|v\|_1\langle p^2\rangle^{1/2}/\hbar \\ &\leq \langle p^2\rangle/(2m) + m\|v\|_1^2/(2\hbar^2).\end{aligned} \tag{3.8}$$

Hence

$$\langle H\rangle = \langle p^2\rangle/2m + \langle v(q)\rangle \geq -m\|v\|_1^2/(2\hbar^2). \tag{3.9}$$

Even though v need not be bounded below, the total Hamiltonian is bounded below by a constant that depends on $\hbar$.

4. n-DIMENSIONAL SYSTEMS

We are mainly interested in inequalities for three-dimensional systems, but it is worthwhile to derive them for all dimensions n simultaneously. This helps to point out what is special about $n = 3$.

If $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ are vectors of observables, we write $A + B = (A_1 + B_1, \ldots, A_n + B_n)$ and $AB = A_1B_1 + \cdots + A_nB_n$. The expectation of A is the vector $\langle A\rangle = (\langle A_1\rangle, \ldots, \langle A_n\rangle)$ and we write $(\Delta A)^2 = \langle(A - \langle A\rangle)^2\rangle$.

Quantum mechanics in n dimensions concerns position and momentum observables $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ satisfying $p_jq_k - p_kq_j = -i\delta_{kj}\hbar$. Since $n\hbar/2 \leq \Delta p_1\Delta q_1 + \cdots + \Delta p_n\Delta q_n \leq \Delta p\Delta q$, the Heisenberg uncertainty principle in n dimensions is

$$\Delta p\Delta q \geq n\hbar/2. \tag{4.1}$$

We wish to decompose p^2 into radial and angular parts. The angular momentum in the jk plane is $J_{jk} = q_jp_k - q_kp_j$. The total angular momentum is $J^2 = \sum_{j<k}J_{jk}^2$. The angular momentum observables all commute with q^2. Thus it is easy to compute that

$$\begin{aligned}q^{-2}J^2 &= \tfrac{1}{2}\sum_{jk}J_{jk}q^{-2}J_{jk} \\ &= \sum_{jk}(p_kq_jq^{-2}q_jp_k - p_kq_jq^{-2}q_kp_j) \\ &= p^2 - pqq^{-2}qp.\end{aligned}$$

Thus we have the decomposition

$$p^2 = pqq^{-2}qp + q^{-2}J^2. \tag{4.2}$$

Note that $pq - qp = -in\hbar$, so pq and qp differ only by a constant.

Since the expression for the radial part of p^2 involves the term q^{-2}, which is singular at the origin, we must pause to examine when manipulations with such singular terms are justified. For this it is necessary to look more closely at the representation of the observables as operators.

When the observables are represented as operators acting in a Hilbert space, the states are given by unit vectors ψ. The expectation $\langle A\rangle = \langle\psi, A\psi\rangle$ is given by the inner product. In the Schrödinger representation the Hilbert space is a space L^2 of square integrable functions on $\mathbb{R}^n$. Then p_j and q_k are given by $-i\hbar\partial/\partial x_j$ and by multiplication by x_k. It is easy to check that p^2, q^2, and qp are given by $-\hbar^2\Delta$, multiplication by x^2, and $-i\hbar r\partial/\partial r$.

Many of our estimates will not be valid for $n=1$. The reason is that when $n=1$ a point has a nonzero capacity and cannot be ignored in the manipulations. However for $n \geq 2$ a point has zero capacity and is thus negligible.

Zero Capacity Principle: If $n \geq 2$, then all functions ψ in L^2 with finite kinetic energy $\langle\psi, p^2\psi\rangle$ may be approximated arbitrarily closely in kinetic energy by functions which vanish near some fixed point.

Proof: Consider the Hilbert space of all ψ with $\langle\psi, (p^2+1)\psi\rangle < \infty$. In order to show that the ψ which vanish near the origin are dense in this space, it is sufficient to show that the only vector ϕ orthogonal to these ψ is the zero vector.

Assume that $\langle\psi, (p^2+1)\phi\rangle = 0$ for all such ψ. Then $u = (p^2+1)\phi$ is a distribution that vanishes except at the origin. Furthermore, $\langle(p^2+1)^{-1}u, u\rangle = \langle\phi, (p^2+1)\phi\rangle < \infty$. But any distribution supported at a point must be a linear combination of Dirac delta measures and their derivatives. We can thus compute $\langle(p^2+1)^{-1}u, u\rangle$ by Fourier transformation and observe that this is finite only when $n=1$ and u is a multiple of a delta measure, or when $u=0$. Thus for $n \geq 2$, $u=0$ and so $\phi=0$ as well.

The above proof may be interpreted as saying that the capacity of a point to support a distribution u with $\langle(p^2+1)^{-1}u, u\rangle < \infty$ is zero when $n \geq 2$. The result shows that when $n \geq 2$ we may always calculate with wavefunctions that vanish near the point of singularity. The estimate then extends by continuity to all states with finite kinetic energy.

We now turn to calculations. It is important to note that the radial part of p^2 may be factored in two ways. In fact, we may commute q^{-2} past qp and $pq = qp - in\hbar$ to obtain

$$pqq^{-2}qp = q^{-2}(pq + 2i\hbar)qp$$
$$= q^{-2}qp(pq + 2i\hbar) = q^{-2}qpq^2pqq^{-2}.$$

If we write $r = (q^2)^{1/2}$, this becomes

$$(pqr^{-1})(r^{-1}qp) = (r^{-2}qpr)(rpqr^{-2}). \tag{4.3}$$

We now use the basic commutator bound. We insert the two factorizations in turn, first with $A = r^{-1}qp$ and second with $A = rpqr^{-2}$. In both cases A^*A is the radial part of p^2, in particular $\langle A^*A\rangle \leq \langle p^2\rangle$. We choose $B = \phi(q)$, where ϕ is a real function. The result of the calculation is

$$i(A^*B - BA) = \hbar[\partial\phi(q)/\partial r + \nu r^{-1}\phi(q)], \tag{4.4}$$

where $\nu = n-1$ in the first case and $3-n$ in the second case. The commutator bound and the zero capacity principle thus give a general inequality.

Commutator Inequality: Let $n \geq 2$ and $\nu = n-1$ or $3-n$. Let ϕ be a real function that is sufficiently differentiable and has no singularities except at the origin. Then in any state

$$\hbar\langle\partial\phi(q)/\partial r + \nu r^{-1}\phi(q)\rangle - \langle\phi(q)^2\rangle \leq \langle p^2\rangle. \tag{4.5}$$

One special case deserves more detail. Take $A = r^{-1}qp$ and $B = \beta r^{-1}$, where β is a constant. The commutator identity

$$A^*A - i(A^*B - B^*A) + B^*B = (A - iB)^*(A - iB)$$

gives

$$pqr^{-2}qp - (n-2)\hbar\beta r^{-2} + \beta^2 r^{-2} = (pq + i\beta)r^{-1}r^{-1}(qp - i\beta).$$

When $\beta = [(n-2)/2]\hbar$ this becomes

$$pqr^{-1}qp - [(n-2)/2]^2\hbar^2 r^{-2} = r^{-1}D^2r^{-1}, \tag{4.6}$$

where $D = (pq + qp)/2$ is the dilation generator. Since $\langle r^{-1}D^2r^{-1}\rangle \geq 0$, we have in particular the well-known $1/r^2$ bound

$$\langle p^2\rangle^{1/2}\langle r^{-2}\rangle^{-1/2} \geq [(n-2)/2]\hbar, \tag{4.7}$$

a local uncertainty principle valid for $n \geq 3$.

The behavior of the probabilities is thus different when $n \geq 3$. The probability of being in a ball of radius b depends quadratically on b.

Quadratic Local Uncertainty Principle: In dimensions $n \geq 3$, there is a constant c such that in any state

$$\text{Prob}\{|q - a| \leq b\} \leq c(b\Delta p/\hbar)^2. \tag{4.8}$$

Proof: Let $\chi(q) = 1$ when $|q| \leq b$ and 0 otherwise. Since $\chi(q)b^{-2} \leq q^{-2}$, the $1/r^2$ bound gives

$$\langle\chi(q)\rangle b^{-2} \leq 4/(n-2)^2\langle p^2\rangle/\hbar^2.$$

Since $p - \langle p\rangle$ and $q - a$ could have been used in place of p and q, this gives the result. The constant $c = 4/(n-2)^2$.

An alternative proof gives a sharper constant in dimension $n=3$. This argument is due to Lavine.

Proof: We use the second factorization corresponding to $\nu = 3-n$. When $n=3$ or when $n \geq 4$ and ϕ is negative, this gives

$$\langle\hbar\partial\phi(q)/\partial r - \phi(q)^2\rangle \leq \langle p^2\rangle. \tag{4.9}$$

Insert $\phi(x) = -\hbar[\pi/(2b)]\cot(\pi r/(2b))$ for $r \leq b$, 0 elsewhere. Since $\csc^2 - \cot^2 = 1$, the result is

$$\hbar^2(\pi/2b)^2\langle\chi(q)\rangle \leq \langle p^2\rangle.$$

This time the constant $c = 4/\pi^2$.

The quadratic local uncertainty principle is obviously false for $n=1$. It is less obvious that it is also false for $n=2$. To see this, take ψ to be a radial function that behaves like $\log\log(1/r)$ near the origin and is nice elsewhere. The expectation $\langle\psi, p^2\psi\rangle$ is proportional to $\langle\partial\psi/\partial r, \partial\psi/\partial r\rangle$. Furthermore, since $\partial\psi/\partial r = (r\log r)^{-1}$ near the origin, the integral behaves like $2\pi\int r^{-1}(\log r)^{-2}\,dr = 2\pi \times \int(\log r)^{-2}\,d(\log r)$, which is finite near $r=0$. On the other hand, $\text{Prob}\{r \leq b\} = 2\pi\int_0^b|\psi|^2\,r\,dr$ cannot be bounded by a multiple of b^2, since ψ is unbounded.

In this example the singularity of ψ is rather mild. This is the general situation, since there is a true inequality that is only slightly weaker.[3]

The exact bound for the hydrogen atom problem is also a consequence of the commutator estimates. Use the first factorization, so that $\nu = n-1$, and take $B=\beta$, a constant. This gives the $1/r$ bound

$$\hbar(n-1)\beta\langle r^{-1}\rangle - \beta^2 \leq \langle p^2\rangle. \tag{4.10}$$

If we take $\beta = \langle p^2\rangle^{1/2}$ we get the local uncertainty principle (1.3). Or we can take $\beta = 2me^2/[(n-1)\hbar]$ and obtain the lower bound (1.4) directly.

We can also obtain the ground state. The commutator bound is an equality precisely when $\|(A-iB)\psi\|^2=0$, that is $A\psi = iB\psi$. In our application this says that in the Schrödinger representation $-i\hbar\partial\psi/\partial r = i\beta\psi$. The solution of this is $\psi = C\exp(-\beta r/\hbar)$. Since the lower bound is assumed when $\beta = 2me^2/[(n-1)\hbar]$, the Bohr radius $\hbar/\beta$ in n dimensions is $(\hbar^2/me^2)(n-1)/2$.

5. SOBOLEV INEQUALITIES

Let p be a real number with $1 \leq p < \infty$. If ψ is a complex function such that $\int|\psi(x)|^p\,dx < \infty$, ψ will be said to belong to L^p. The L^p norm of ψ is defined by $\|\psi\|_p = (\int|\psi(x)|^p\,dx)^{1/p}$.

There is a useful generalization of this notion that is somewhat less known. We say that ψ is in weak L^p if there is a constant M such that for all $s>0$ the volume of the set where $|\psi| \geq s$ is bounded by $(M/s)^p$. The least such M is denoted $\|\psi\|_p^*$. It is easy to see that if ψ is in L^p, then it is in weak L^p and $\|\psi\|_p^* \leq \|\psi\|_p$. We need only observe that $\|\psi\|_p^p \geq \int_{|\psi|\geq s}|\psi(x)|^p\,dx \geq s^p\,\mathrm{vol}(|\psi|>s)$. On the other hand, in n dimensions $r^{-n/p}$ is in weak L^p but not in L^p.

In the following we write Ω_n for the volume of the ball of radius 1. The general formula in n dimensions is $\Omega_n = \pi^{n/2}/\Gamma((n/2)+1)$.

Strichartz Inequality[4,5]: Let $n \geq 3$. Then there is a constant C such that for every function v

$$|\langle v(q)\rangle| \leq C\|v\|_{n/2}^*\langle p^2\rangle/\hbar^2. \tag{5.1}$$

This inequality may be deduced from the $1/r^2$ bound by rearrangement. We save the proof for later. The proof will give the constant $C = [4/(n-2)^2]\Omega_n^{-2/n}$. (This is best possible.) The Sobolev inequality is obtained as a corollary by replacing weak L^p by L^p.

Sobolev Inequality[6,7]: Let $n \geq 3$. Then there is a constant C such that for every function v

$$|\langle v(q)\rangle| \leq C\|v\|_{n/2}\langle p^2\rangle/\hbar^2. \tag{5.2}$$

The constant C in the Sobolev inequality thus certainly does not exceed the constant C in the Strichartz inequality. However this is not the smallest possible constant, since the Sobolev inequality was derived here as a corollary of a rather different inequality. The sharpest constant is actually $C = 1/[\pi n(n-2)][\Gamma(n)/\Gamma(n/2)]^{2/n}$.[8] Notice that if we set $v = |\psi|^{r-2}$, where $1/r = \frac{1}{2} - 1/n$, the inequality becomes the classic Sobolev inequality

$$\|\psi\|_r^2 \leq C\langle\psi, p^2\psi\rangle/\hbar^2. \tag{5.3}$$

Thus it says that ψ is in L^r and hence cannot be too peaked.

The most interesting consequence of the Sobolev inequality is the no-binding theorem, which says that if $n \geq 3$, then a weak short-range potential v won't bind a quantum mechanical particle to make a state of strictly negative total energy. Weak and short range means here that the $L^{n/2}$ norm of v is small. More general criteria may be given,[9] but this norm has the appeal of being translation invariant. In addition, it often gives fairly good numerical results.[9]

No binding theorem: Let $n \geq 3$ and let v be a real function on $\mathbb{R}^n$. If $(2mC\|v\|_{n/2})^{n/2}/\hbar^n \leq 1$, then in any state

$$\langle H\rangle = \langle p^2/2m\rangle + \langle v(q)\rangle \geq 0.$$

Proof: Apply Sobolev's inequality. Thus if $2mC\|v\|_{n/2}/\hbar^2 \leq 1$, then in any state

$$-\langle v(q)\rangle \leq \langle p^2/2m\rangle.$$

Consider as an example the case $n=3$. The Yukawa potential $v(x) = -g\exp(-kr)/r$ is in $L^{3/2}$. Thus if g is small the Yukawa potential won't bind. The Coulomb potential $v(x) = -e^2/r$ is not in $L^{3/2}$, but it is the sum of a singular part in $L^{3/2}$ and a long range part which is bounded. For such potentials the inequality gives a lower bound.

Lower Bound Theorem: Let $n \geq 3$ and let v be a real function that is the sum of a function in $L^{n/2}$ and a function that is bounded below. Then the Hamiltonian $H = p^2/2m + v(q)$ is bounded below.

Proof: Write $v = v_k + v_k'$, where $v_k = v$ where $v \geq -k$ and $v_k = 0$ elsewhere. If k is sufficiently large, then v_k' is in $L^{n/2}$ with arbitrarily small norm. Thus if k is large enough so that v_k' won't bind, then

$$\langle H\rangle = \langle p^2/2m + v_k'(q)\rangle + \langle v_k(q)\rangle \geq -k.$$

On can also deduce lower bounds directly from the Sobolev inequality. It follows from the integral representation of fractional powers[1] that when $0<\alpha<1$ the inequality $0 \leq \langle B\rangle \leq \langle A\rangle$ for all states implies the inequality $0 \leq \langle B^\alpha\rangle \leq \langle A^\alpha\rangle$ for all states. Let $r > n/2$ and apply this with $\alpha = n/(2r)$ to the Sobolev inequality. This gives

$$\langle|v(q)|^\alpha\rangle \leq C^\alpha\|v\|_{n/2}^\alpha\langle p^{2\alpha}\rangle/\hbar^{2\alpha}.$$

Replace $|v|$ by $|v|^{1/\alpha}$ and use the estimate[10] $\langle p^{2\alpha}\rangle \leq \langle p^2\rangle^\alpha$. We arrive at the general result

$$|\langle v(q)\rangle| \leq C^\alpha\|v\|_r\langle p^2\rangle^\alpha/\hbar^{2\alpha}, \tag{5.4}$$

where $\alpha = n/(2r)$ and $r > n/2$, $n \geq 3$. This uncertainty principle gives a lower bound on $H = p^2/2m + v(q)$ in terms of $\|v\|_r$. Notice that it does not give a no-binding result, since the fractional power leads only to an inhomogeneous estimate for $\langle p^2\rangle$.

There is however another no-binding result that is a consequence of the quadratic local uncertainty principle and rearrangement. Assume $n \geq 3$. Let v be bounded with $|v| \leq M$ and assume that $v=0$ except on a set S of volume $\Omega_n b^n$. The result says that

$$|\langle v(q)\rangle| \leq (4/\pi^2)Mb^2\langle p^2\rangle/\hbar^2. \tag{5.5}$$

As a consequence, if $(4/\pi^2)(2mMb^2/\hbar^2) \leq 1$, then $\langle H\rangle = \langle p^2/2m\rangle + \langle v(q)\rangle \geq 0$ in every state.

The proofs that have been deferred until now are based on the notion of rearrangement.[10] We assume that χ is a positive function that approaches zero at infinity in the sense that for $s>0$, $\mathrm{vol}\{s<\chi\}<\infty$. Here we write $\{s<\chi\}$ for $\{x: s<\chi(x)\}$. The function χ may be decomposed in slices as

$$\chi=\int_0^\infty ds\, 1_{\{s<\chi\}}. \tag{5.6}$$

In this expression 1_M is the function that is 1 on the set M and 0 elsewhere, so $1_{\{s<\chi\}}(x)=1$ precisely when $s<\chi(x)$. The rearrangement $\chi^\sim$ of χ is defined as the positive function given by

$$\chi^\sim=\int_0^\infty ds\, 1_{\{s<\chi^\sim\}}, \tag{5.7}$$

where by definition $\{s<\chi^\sim\}$ is a ball centered at the origin with the same volume as $\{s<\chi\}$.

Lemma 1:

$$\int \chi\psi\, dx \le \int \chi^\sim\psi^\sim dx. \tag{5.8}$$

Proof: Use the decompositions $\chi=\int_0^\infty ds\, 1_{\{s<\chi\}}$ and $\psi=\int_0^\infty dt\, 1_{\{t<\psi\}}$. The volume where $s<\chi$ and $t<\psi$ is the volume of an intersection of two sets. On the other hand, the sets $s<\chi^\sim$ and $t<\psi^\sim$ are both balls centered at the origin, so the volume where $s<\chi^\sim$ and $t<\psi^\sim$ is the volume of one of the two sets. But the volume of an intersection is always smaller than the volume of either set.

Lemma 2: The function χ is in weak L^p if and only if its rearrangement $\chi^\sim$ satisfies

$$\chi^\sim \le \Omega_n^{-1/p}\|\chi\|_p^* r^{-n/p}. \tag{5.9}$$

Proof: Clearly χ is in weak L^p if and only if $\chi^\sim$ is. But $\chi^\sim$ is in weak L^p if and only if $\{s<\chi^\sim\}$ is contained in the ball of volume $(\|\chi\|_p^*/s)^p$ for all s. This, however, is precisely the ball $\{s<\Omega_n^{-1/p}\|\chi\|_p^* r^{-n/p}\}$. Thus by decomposing $\chi^\sim$ into slices we see that $\chi^\sim$ is in weak L^p if and only if $\chi^\sim\le\Omega_n^{-1/p}\|\chi\|_p^* r^{-n/p}$.

The following is the crucial lemma.

Lemma 3 [11]:

$$\int |\nabla\chi^\sim|^2 dx \le \int |\nabla\chi|^2 dx. \tag{5.10}$$

Proof: We show that this holds true on each shell where the functions have values between s and $s+ds$.

First, $\{s<\chi<s+ds\}$ is a shell of width $|\nabla\chi|^{-1}ds$. Its volume is $\int_{\chi=s}|\nabla\chi|^{-1}ds\, d\sigma$, where the integration is over surface area. On the other hand, $\{s<\chi^\sim<s+ds\}$ is a spherical shell of constant width $|\nabla\chi^\sim|^{-1}ds$. Its volume is $|\nabla\chi^\sim|^{-1}ds\,\tilde\sigma$, where $\tilde\sigma$ is the area of the sphere. Since by definition of rearrangement the two volumes are equal,

$$|\nabla\chi^\sim|^{-1}ds\,\tilde\sigma=\int_{\chi=s}|\nabla\chi|^{-1}ds\, d\sigma. \tag{5.11}$$

Second, since (again by the definition of rearrangement) the volumes of $\{s<\chi\}$ and $\{s<\chi^\sim\}$ are the same, the isoperimetric equality says that the area of the sphere is smaller than the area of the other surface. That is,

$$\tilde\sigma\le\sigma=\int_{\chi=s}d\sigma. \tag{5.12}$$

We can use these two facts to estimate

$$\begin{aligned}\int_{\{s<\chi^\sim<s+ds\}}|\nabla\chi^\sim|^2 dx &= |\nabla\chi^\sim|\tilde\sigma\, ds\\ &=\left(\int_{\chi=s}|\nabla\chi|^{-1}d\sigma\right)^{-1}\tilde\sigma^2 ds\\ &\le\left(\int_{\chi=s}|\nabla\chi|^{-1}\frac{d\sigma}{\sigma}\right)^{-1}\sigma\, ds.\end{aligned} \tag{5.13}$$

We can now complete the proof by using the inequality of the harmonic mean to show that this is bounded by

$$\left(\int_{\chi=s}|\nabla\chi|\frac{d\sigma}{\sigma}\right)\sigma\, ds=\int_{\{s<\chi<s+ds\}}|\nabla\chi|^2 dx. \tag{5.14}$$

Proof of the Strichartz inequality: Set $\chi=|\psi|$ and put these lemmas together. By Lemma 1,

$$\int v|\psi|^2 dx\le\int |v|\chi^2 dx\le\int |v|^\sim\chi^{\sim2}dx. \tag{5.15}$$

By Lemma 2,

$$\int |v|^\sim\chi^{\sim2}dx\le\Omega_n^{-2/n}\|v\|_{n/2}^*\int r^{-2}\chi^{\sim2}dx. \tag{5.16}$$

The $1/r^2$ bound gives

$$\int r^{-2}\chi^{\sim2}dx\le 4/(n-2)^2\int|\nabla\chi^\sim|^2 dx. \tag{5.17}$$

Finally, by Lemma 3

$$\int|\nabla\chi^\sim|^2 dx\le\int|\nabla\chi|^2 dx\le\int|\nabla\psi|^2 dx. \tag{5.18}$$

This completes the proof.

The other no-binding result may be proved the same way, using

$$\int|v|\chi^2 dx\le\|v\|_\infty\int_S\chi^2 dx\le\|v\|_\infty\int_B\chi^{\sim2}dx, \tag{5.19}$$

where B is a ball of radius b centered at the origin. Then by the quadratic local uncertainty principle

$$\int_B\chi^{\sim2}dx\le(4/\pi^2)b^2\int|\nabla\chi^\sim|^2 dx. \tag{5.20}$$

The proof is completed in the same way.

6. SURVEY OF RELATED RESULTS

(i) Bound states

One consequence of the Sobolev inequality is that if $n\ge 3$ and $(2m\|v\|_{n/2})^{n/2}\hbar^{-n}$ is sufficiently small, then $p^2/2m+v(q)$ has no strictly negative eigenvalues. In the classical limit $\hbar\to 0$ the number of such eigenvalues is asymptotically $\Omega_n(2m\|v\|_{n/2})^{n/2}(2\pi\hbar)^{-n}$.[12] Recently Lieb[13] and Cwikel[14] have shown that if $n\ge 3$ the number is bounded by a multiple of the classical expression.

(ii) Scattering

In the present treatment positive commutators have been used to determine behavior in space. However they may also be used to analyze time development. Lavine[15] has recently found estimates on the rate of barrier penetration by these methods.

(iii) Matter

Dyson and Lenard showed rigorously that the energy per particle in bulk matter is bounded below. Their arguments have three ingredients: positivity of the integral operator given by $1/|x-y|$, the exclusion principle for the fermions, and some sort of local uncertainty principle. Recently Lieb[16] and Thirring have made a substantial improvement on the lower bound. The technique uses Sobolev type inequalities[16,17] for many-fermion systems.

(iv) Entropy

Consider a quantum mechanical particle in some fixed state. Let ρ and σ be the position and momentum probability densities and $S(q) = -\int \rho \log\rho \, dx$ and $S(p) = -\int \sigma \log\sigma \, dp$ be the position and momentum entropies. Beckner[18] recently proved a theorem on Fourier transforms that extends results of Nelson and Gross on the quantum mechanical harmonic oscillator. This theorem implies[18,19] the inequality

$$\exp[S(q) + S(p)] \geq (e\pi\hbar)^n. \tag{6.1}$$

Entropy becomes negative when the probability distribution is peaked, so the inequality says that position and momentum cannot both be too peaked.

The entropy always gives a lower bound for certain expectations. If we integrate the elementary inequality $-\rho\log\rho + \rho \leq -\rho\log\psi + \psi$ and assume ψ is a positive function normalized so that $\int \psi\, dx = 1$, we obtain $S(q) \leq \langle -\log\psi(q)\rangle$. In the general case when ψ is not normalized one can apply this to $\psi/\int\psi\, dx$ to obtain $S(q) \leq \langle -\log\psi(q)\rangle + \log\int\psi(x)\, dx$.

The Beckner inequality may thus be rewritten in terms of expectations of arbitrary functions as

$$\exp(\langle -\log\psi(q)\rangle + \langle -\log\phi(p)\rangle)\left(\int \psi\, dx\right)\left(\int \phi\, dp\right) \geq (e\pi\hbar)^n. \tag{6.2}$$

If we take ψ and ϕ to be Gaussian probability densities with means $\langle q\rangle$ and $\langle p\rangle$ and standard deviations Δq and Δp, this becomes

$$(2\pi e/n)^n(\Delta p)^n(\Delta q)^n \geq (e\pi\hbar)^n, \tag{6.3}$$

the Heisenberg uncertainty principle. Thus the entropy inequality is an uncertainty principle in which each factor is smaller than the corresponding factor in the Heisenberg uncertainty principle, yet where the product has the same lower bound.

Consider the harmonic oscillator Hamiltonian $H = p^2/2m + m\omega^2q^2/2 - n\hbar\omega/2$ and let $\rho_0(x) = (\pi\hbar/m\omega)^{-n/2} \times \exp(-m\omega x^2/\hbar)$ and $\sigma_0(p) = (\pi\hbar m\omega)^{-n/2}\exp[-p^2/(m\omega\hbar)]$ be the Gaussian position and momentum probability densities in its ground state. Set $\rho = h\rho_0$ and $\sigma = j\sigma_0$ and define entropies $S_0(q) = -\int h \log h\rho_0\, dx$ and $S_0(p) = -\int j \times \log j\sigma_0\, dp$ with respect to the ground state. The entropy inequality becomes

$$S_0(q) + S_0(p) \geq -2\langle H\rangle/\hbar\omega. \tag{6.4}$$

It says that when the energy is low the probabilities resemble the ground state probabilities.

In this form the inequality is independent of dimension and extends to dimension $n=\infty$. The Gaussian measures $\rho_0\, dx$ and $\sigma_0\, dp$ are defined as infinite product measures. The harmonic oscillator Hamiltonian

$$H = \tfrac{1}{2}(m^{1/2}\omega q + im^{-1/2}p)^* \times (m^{1/2}\omega q + im^{-1/2}p)$$

also makes sense when $n=\infty$.

If we set $\psi = f\rho_0$ and $\phi = g\sigma_0$, the lower bound for entropy takes the form $S_0(q) \leq -\langle \log f(q)\rangle + \log\langle f(q)\rangle_0$, where the subscript 0 denotes expectation in the ground state. Note that the choice $f=1$ gives $S_0(q) \leq 0$. We also see that the inequality is equivalent to

$$\langle \log f(q)\rangle + \langle \log g(p)\rangle \leq 2\langle H\rangle/\hbar\omega + \log\langle f(q)\rangle_0 + \log\langle g(p)\rangle_0. \tag{6.5}$$

Nelson's inequality is obtained by setting $g=1$. If we then write $\log f = -2v/\hbar\omega$, we obtain

$$\langle H\rangle + \langle v(q)\rangle \geq -(\hbar\omega/2)\log\langle \exp(-2v(q)/\hbar\omega)\rangle_0. \tag{6.6}$$

This shows that even in infinite dimensions a function v that is unbounded below may still give a total Hamiltonian $H + v(q)$ that is bounded below, provided that $\langle \exp(-2v(q)/\hbar\omega)\rangle_0 < \infty$. This fact has been useful in quantum field theory.

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On the definition and properties of generalized 3-j symbols

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Whipple's work on the symmetries of ${}_3F_2$ functions with unit argument is applied to study the properties of 3-j symbols generalized to any arguments. It turns out that there are twelve sets of ten formulas (twelve sets of 120 generalized 3-j symbols) which are equivalent in the usual case. Whipple's parameters r provide a better description of the symmetries than the Regge symbol.

1. INTRODUCTION

The 3-j symbols, related to the Clebsh-Gordan coefficients of SU2, are so widely known by physicists that their notation is sometimes used in the literature to express quantities which are also generalized hypergeometric functions ${}_3F_2$ with unit argument.[1] For example, it is used for angular momenta which are multiples of $\frac{1}{4}$ in the "tree" theory of hyperspherical harmonics[2,3] or with negative values[4] for the discrete representations of SU(1, 1). Clearly, references to SU2 results are made by these authors to generalize a well-known formalism to their own problem.

For usual angular momenta, we know Regge's symmetries,[5] extended to the transformation $j \to -j-1$ by Yutsis.[6] The same coefficient can be obtained by different expressions which cannot be easily related: This point is well known as there are Racah's,[7] Wigner's[8] etc., formulas for usual 3-j symbols. But, in the usual case, all the coefficients of the ${}_3F_2$ functions are integers whereas some of them can be half-integer in the hyperspherical formalism.

A systematic study of all the possible formulas and the conditions of their validity has not been performed until now (at least, does not appear in the most complete monographies on this subject[9]).

Group theoretical studies lead to integerlike quantum numbers: integer multiples of $\frac{1}{2}$ or $\frac{1}{4}$. Strictly speaking, there is no way to perform an "analytical continuation" to the usual 3-j symbols. Sometimes, the same problem can be studied in a more pedestrian way, by relations between special functions: in this approach, some quantum numbers can take any value, even complex, and the analytical continuation makes sense. After the publication of an article[10] displaying seven different formulas for the same coefficient, the question arises how many of them there can be. The importance of this question is illustrated by the fact that the author could find no relation between his seven formulas and another one already published.[11]

To each ${}_3F_2$ with unit argument, 12 generalized 3-j symbols can be associated by permutation of the three numerator coefficients and the two denominator coefficients of the ${}_3F_2$. If there are n equivalent ${}_3F_2$, there are $12\times n$ equivalent generalized 3-j symbols. There is no need to consider only the ${}_3F_2$ which are finite sums, because, if there is only one finite ${}_3F_2$ among the n equivalent ones, any of the $12\times n$ equivalent 3-j symbols can be given by it. Furthermore a ${}_3F_2$ which is an infinite sum can be evaluated even, to some extent, when it does not converge mathematically. So the symmetry properties of generalized 3-j coefficients are related to those of ${}_3F_2$ of unit argument. The finite sums are only special cases for which the group of symmetry is wider.

The symmetry properties of the ${}_3F_2$ with unit argument were studied by Whipple[12–14] using a very convenient notation. Whipple's parameters provide a better representation of symmetry properties than the Regge symbol because this representation includes Yutsis' "mirror" symmetry[6] and indicates the breakdown of the usual rules when the usual relations between quantum numbers are not fulfilled. Whipple's work has been already applied[15] to study symmetries and relations between 3-j coefficients of SU(2) and SU(1, 1) where all the coefficients of the ${}_3F_2$ functions are integers.

In sec. 2 we shall summarize Whipple's work. Then we shall choose a definition for a generalized 3-j symbol; this definition is absolutely arbitrary but reduces to a usual formula when the arguments fulfill the usual relations. Whipple's theory leads to ten equivalent formulas. The existence of a negative integer leads to relations with a series of ten other formulas, but the existence of more than one negative integer can give rise to quite complex situations which are summarized in sec. 5. In sec. 6 we shall consider the inverse problem: which are the generalized 3-j symbols that a formula can define. To discuss this problem, which can be seen as symmetry properties or analytical continuation, we are obliged to discard Yutsis' phase rules and propose another convention which is not convenient. Unfortunately there is no simple solution to this problem.

In Appendix A we give a method to check the properties described here, even if the series diverge. For completeness, Appendix B gives all the recurrence relations between a generalized 3-j symbol and two of its 30 neighbors in terms of Whipple's parameters.

2. RELATIONS BETWEEN ${}_3F_2$

Whipple introduced six parameters $r_0 - r_5$ such that their sum is zero and

$$\alpha_{lmn} = \tfrac{1}{2} + r_l + r_m + r_n, \quad \beta_{mn} = 1 + r_m - r_n. \tag{1}$$

With them he defined the functions

$$F_p(0;45) = \frac{1}{\Gamma(\alpha_{123})\Gamma(\beta_{40})\Gamma(\beta_{50})} \times {}_3F_2[\alpha_{145}, \alpha_{245}, \alpha_{345}; \beta_{40}, \beta_{50}; 1], \tag{2}$$

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$$F_n(0;45)=\frac{1}{\Gamma(\alpha_{045})\Gamma(\beta_{04})\Gamma(\beta_{05})}\times {}_3F_2\,[\alpha_{023},\alpha_{013},\alpha_{012};\ \beta_{04},\beta_{05};1].$$

The function $F_n(0;45)$ is derived from $F_p(0;45)$ by changing the sign of all the r's. By permutation of the suffixes 60 F_p and 60 F_n are found. If there is no negative integer in the numerator parameters, these series converge only if the real part of α_{123} (or α_{045}) is positive.

The transformation[16]

$${}_3F_2[a,b,c;e,f;1]=\frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\times{}_3F_2[e-a,f-a,s;s+b,s+c;1] \tag{3}$$

with $s=e+f-a-b-c$ can be written $F_p(0;45)=F_p(0;23)$.

By interchange of a, b, c and use of the transformation on the results we obtain the ten $F_p(0;ij)$. Consequently, among the 120 functions F_p and F_n, there are only 12 different ones which can be denoted by $F_p(i)$ and $F_n(i)$ for $i=0-5$. This transformation has been used by Racah[7] and other authors.[4,10]

There is a relation between three of these functions,

$$\frac{\sin\pi\beta_{ij}}{\pi\Gamma(\alpha_{ijk})}F_p(k)=\frac{F_n(i)}{\Gamma(\alpha_{jlm})\Gamma(\alpha_{jln})\Gamma(\alpha_{jmn})}-\frac{F_n(j)}{\Gamma(\alpha_{ilm})\Gamma(\alpha_{iln})\Gamma(\alpha_{imn})} \tag{4}$$

where i, j, k, l, m, n are all different. Changing the signs of the r, a relation between $F_n(k)$, $F_p(i)$, and $F_p(j)$ is obtained. From these relations, a relation between three F_p or F_n can be obtained. This set of relations makes it possible to express any F_n or F_p as a linear combination of two of them. They have been used[17] in the study of Wigner coefficients for continuous representations of SU(1,1).

If one of the α, α_{lmn}, is a negative integer, the series which include it terminate; there are 18 of them. These series can be written in reversed order leading to

$$\Gamma(\alpha_{jkl})\Gamma(\alpha_{jkm})\Gamma(\alpha_{jkn})F_p(i)=(-)^{\alpha_{lmn}}\Gamma(\alpha_{jkl})\Gamma(\alpha_{ikl})\,\Gamma(\alpha_{jkl})F_n(l), \tag{5}$$

where l, m, n are any of the indices of the integer α and i, j, k are any of the other indices. The relation (5) which involves 60 functions can be obtained as a particular case of the relation (4).

3. GENERALIZED 3-j SYMBOLS

The definition of a generalized 3-j symbol is quite arbitrary. We can choose

$$\begin{pmatrix} a & b & c\\ \alpha & \beta & \gamma\end{pmatrix}=\exp[i\pi(a-b-\gamma)]\frac{\delta(\alpha+\beta+\gamma)}{\Gamma(-a+c-\beta+1)\Gamma(-b+c+\alpha+1)}\times\left[\frac{\Gamma(a+\alpha+1)\Gamma(b-\beta+1)\Gamma(c+\gamma+1)\Gamma(c-\gamma+1)}{\Gamma(a-\alpha+1)\Gamma(b+\beta+1)\Gamma(a+b-c+1)}\times\frac{\Gamma(a-b+c+1)\Gamma(b+c-a+1)}{\Gamma(a+b+c+2)}\right]^{1/2}$$
$$\times{}_3F_2[-b-\beta,\ -a+\alpha,\ -a-b+c;\ -a+c-\beta+1,\ -b+c+\alpha+1;1] \tag{6}$$

for any complex value of a, b, c, α, and β. However, the real part of the argument of the Γ function in the square root must be positive in order to define this square root as the analytical continuation to the positive value when the imaginary part of a, b, c, α, and β vanishes.

By identifying the ${}_3F_2$ to $F_p(0;45)$ we obtain for the Whipple's parameters:

$$\begin{aligned} r_1&=\tfrac{1}{6}(3+6a+2\gamma-2\beta), & r_4&=\tfrac{1}{6}(-3-6a+2\gamma-2\beta),\\ r_2&=\tfrac{1}{6}(3+6b+2\alpha-2\gamma), & r_5&=\tfrac{1}{6}(-3-6b+2\alpha-2\gamma),\\ r_3&=\tfrac{1}{6}(3+6c+2\beta-2\alpha), & r_0&=\tfrac{1}{6}(-3-6c+2\beta-2\alpha). \end{aligned} \tag{7}$$

The related Regge symbol is

$$\begin{vmatrix} -a+b+c & a-b+c & a+b-c\\ a+\alpha & b+\beta & c+\gamma\\ a-\alpha & b-\beta & c-\gamma \end{vmatrix}=\begin{vmatrix} -\alpha_{015} & -\alpha_{024} & -\alpha_{345}\\ -\alpha_{034} & -\alpha_{145} & -\alpha_{025}\\ -\alpha_{245} & -\alpha_{035} & -\alpha_{014} \end{vmatrix} \tag{8}$$

with $a+b+c=1-\alpha_{045}$. All of the α and β can be obtained from (8), using $\alpha_{ijk}=1-\alpha_{lmn}$ and $\beta_{ij}=1+\alpha_{kli}-\alpha_{klj}$ for any value of k and l. We shall consider α_{045} and the α of the Regge symbol as negativelike and avoid them as arguments of Γ functions. In the following we shall use i,j,k for any one of the indices 0, 4, 5 and l,m,n for any one of 1, 2, 3. So, the positivelike α are, α_{lmn} and α_{ilm}.

With this notation

$$\begin{pmatrix} a & b & c\\ \alpha & \beta & \gamma\end{pmatrix}=\exp[i\pi(r_5-r_4)]R_p(0)F_p(0;45), \tag{9}$$

where

$$R_p^2(0)=\frac{\Gamma(\alpha_{123})\Gamma(\alpha_{124})\Gamma(\alpha_{125})\Gamma(\alpha_{134})\Gamma(\alpha_{135})\Gamma(\alpha_{234})\Gamma(\alpha_{235})}{\Gamma(\alpha_{012})\Gamma(\alpha_{013})\Gamma(\alpha_{023})} \tag{10}$$

is the product of all the $\Gamma(\alpha_{\lambda\mu\nu})$ with $\lambda,\mu,\nu\neq 0$ where the $\Gamma(\alpha_{\lambda\mu\nu})$ of negativelike α have been replaced by $\Gamma(1-\alpha_{\lambda\mu\nu})^{-1}$. The definition (9) describes only the 12 3-j symbols which correspond to the even transformations of the Regge symbol which do not change the second diagonal. The other even transformations (even permutations of a, b, c) lead to $F_p(4;05)$ and $F_p(5;04)$. The odd transformations give $F_n(1;23)$, $F_n(2;13)$, and $F_n(3;12)$. We shall use the notation

$$\begin{aligned} \bar F_p(\lambda)&=R_p(\lambda)F_p(\lambda),\\ \bar F_n(\lambda)&=R_n(\lambda)F_n(\lambda),\\ R_p(\lambda)R_n(\lambda)&=1, \end{aligned} \tag{11}$$

where $R_p(\lambda)$ is defined as $R_p(0)$ for $\lambda=1,2,3,4,5$. Note that the numerator of $R_p(i)$ includes seven Γ functions and its denominator only three, and that these figures are inverted for $R_p(l)$.

The Regge symbol (8) corresponds to six general 3-j symbols which can be expressed with two of them by the following relations:

$$\sin\pi(r_4-r_5)\bar{F}_p(0)+\sin\pi(r_5-r_0)\bar{F}_p(4)$$
$$+\sin\pi(r_0-r_4)\bar{F}_p(5)=0, \qquad (12)$$
$$\sin\pi(r_0-r_4)\bar{F}_n(l)=\sin\pi\alpha_{4mn}\bar{F}_p(0)-\sin\pi\alpha_{0mn}\bar{F}_p(4),$$

where the $\bar{F}_p$ and $\bar{F}_n$ are the 3-j symbols up to a phase. But $F_p(0;45)$ can be replaced in (9) by any of the nine other $F_p(0;\lambda\mu)$, leading to ten different formulas. In other words, among the 120 permutations of r_1, r_2, r_3, r_4, and r_5 which lead to other values of $a, b, c, \alpha, \beta, \gamma$, permutations of r_1, r_2, r_3 introduce no change in (9); the permutation of r_4 and r_5 changes the phase if r_4-r_5 is not an integer, whereas any permutation limited to F_p does not change the value. Extension of the permutation to R_p is related to the symmetries of 3-j symbols and will be studied later.

Among these nine other formulas, three are infinite sums for usual arguments. There are:

$$F_p(0;12)=\frac{1}{\Gamma(-a-b-c)\Gamma(a+c-\beta+2)\Gamma(b+c+\alpha+1)} \quad (13)$$
$$\times {}_3F_2[a+b+c+2,\ b-\beta+1,\ a+\alpha+1;\ a+c-\beta+2,\ b+c+\alpha+2;1],$$

$$F_p(0;13)=\frac{1}{\Gamma(-a+\alpha)\Gamma(a+c-\beta+2)\Gamma(2c+2)}$$
$$\times {}_3F_2[a+b+c+2,\ c+\gamma+1,\ a-b+c+1;\ a+c-\beta+2,\ 2c+2;1], \qquad (14)$$

and $F_p(0;23)$ which differs from (14) by the exchange of a and b and the change of sign for α, β, γ.

The six other formulas are:

$$F_p(0;14)=\frac{1}{\Gamma(c-\gamma+1)\Gamma(a+c-\beta+2)\Gamma(-a+c-\beta+1)}$$
$$\times {}_3F_2[b-\beta+1,\ c+\gamma+1,\ -b-\beta;\ a+c-\beta+2,\ -a+c-\beta+1;\ 1], \qquad (15)$$

$$F_p(0;15)=\frac{1}{\Gamma(-a+b+c+1)\Gamma(a+c-\beta+2)\Gamma(-b+c+\alpha+1)}$$
$$\times {}_3F_2[a+\alpha+1,\ a-b+c+1,\ -b-\beta;\ a+c-\beta+2,\ -b+c+\alpha+1;\ 1], \qquad (16)$$

$$F_p(0;34)=\frac{1}{\Gamma(a+\alpha+1)\Gamma(2c+2)\Gamma(-a+c-\beta+1)}$$
$$\times {}_3F_2[c+\gamma+1,\ -a+b+c+1,\ -a-b+c;\ 2c+2,\ -a+c-\beta+1;1] \qquad (17)$$

and $F_p(0;25)$, $F_p(0;24)$, and $F_p(0;35)$ which differ respectively from (15), (16), and (17) by the same exchange of a and b and change of sign for α, β, γ.

These formulas present no interest except for negative angular momenta. To each of them is related a Regge symbol including some negative numbers of which the transformation leads to six new generalized 3-j symbols corresponding to $\bar{F}_p(l)$ and $\bar{F}_n(i)$. They are not equal to (6) in the general case. The five other formulas quoted in Ref. 9 are respectively $\bar{F}_p(3;25)$, $\bar{F}_p(2;01)$, $\bar{F}_p(3;01)$, $\bar{F}_n(0;23)$, and $\bar{F}_n(5;13)$ with the notation (7); they describe different generalized 3-j symbols except for the two $\bar{F}_p(3)$.

Dixon's theorem[13,14] sums the expression (6) when $\alpha=\beta=0$ and gives

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix}=\exp[i\pi(a-b)]\cos\left[\pi\frac{a+b-c}{2}\right]$$
$$\times\left[\frac{\Gamma(-a+b+c+1)\Gamma(a-b+c+1)\Gamma(a+b-c+1)}{\Gamma(a+b+c+2)}\right]^{1/2}$$
$$\times\frac{\Gamma[(a+b+c)/2+1]}{\Gamma[(-a+b+c)/2+1]\Gamma[(a-b+c)/2+1]\Gamma[(a+b-c)/2+1]} \qquad (18)$$

which generalizes the well known formula to any value of a, b, and c. They are different results for the other F_p and F_n; for example $F_p(4)$ and $F_p(5)$ differ from (18) by the argument of the cosine.

4. EXISTENCE OF ONE INTEGER

There are relations between some $\bar{F}_p(\lambda)$ and $\bar{F}_n(\mu)$ when $\alpha_{\lambda\mu\nu}$ is a negative integer (or zero) but also when $\beta_{\lambda\mu}$ is an integer. Let us consider this case.

If β_{ij} is an integer, $r_i - r_j$ is also integer and the relations (4) reduce to

$$\bar{F}_p(j)=(-)^{r_i-r_j}\bar{F}_p(i), \quad \bar{F}_n(j)=\bar{F}_n(i). \qquad (19)$$

If β_{lm} is an integer,

$$\bar{F}_p(m)=\bar{F}_p(l), \quad \bar{F}_n(m)=(-)^{r_m-r_l}\bar{F}_n(l). \qquad (20)$$

If β_{il} is an integer,

$$\pi^2\bar{F}_n(i)=\sin\pi\alpha_{klm}\sin\pi\alpha_{kln}\bar{F}_n(l), \qquad (21)$$
$$\pi^2\bar{F}_p(l)=(-)^{r_i-r_l}\sin\pi\alpha_{klm}\sin\pi\alpha_{kln}\bar{F}_p(i).$$

In the last case

$$\sin\pi\alpha_{klm}\sin\pi\alpha_{kln}=\sin\pi\alpha_{jlm}\sin\pi\alpha_{jln}$$
$$=(-)^{r_i-r_l}\sin\pi\alpha_{jlm}\sin\pi\alpha_{klm}$$
$$=(-)^{r_i-r_l}\sin\pi\alpha_{jln}\sin\pi\alpha_{kln}. \qquad (22)$$

For the usual 3-j symbols, r's are integers plus $\frac{1}{6}$, $\frac{1}{2}$, or $\frac{5}{6}$. When, on the contrary all of them are integers, the α are half-integers and

$$(-)^{r_j-r_k}\bar{F}_p(i)=\pi^2\bar{F}_p(l), \quad \pi^2\bar{F}_n(i)=(-)^{r_m-r_n}\bar{F}_n(l) \qquad (23)$$

but there is no relation between the F_n and the F_p.

There are only three negativelike α among the arguments of $F_p(0)$ which are α_{145}, α_{245}, and α_{345} $(-b-\beta, -a+\alpha$, and $-a-b+c)$. If α_{145} is zero or a negative integer, relation (5) reads

$$\bar{F}_p(0)=\frac{\pi^2}{\sin\pi\alpha_{124}\sin\pi\alpha_{125}}\bar{F}_p(2)=\frac{\pi^2}{\sin\pi\alpha_{134}\sin\pi\alpha_{135}}\bar{F}_p(3)$$
$$=(-)^{b+\beta}\bar{F}_n(1)=\frac{\pi^2}{\sin\pi\alpha_{125}\sin\pi(r_3-r_4)}\bar{F}_n(4)$$
$$=\frac{\pi^2}{\sin\pi\alpha_{124}\sin\pi(r_3-r_5)}\bar{F}_n(5). \qquad (24)$$

If any other α_{ijk} is a negative integer, a similar relation can be obtained by a permutation of indices. However, if α_{045} is a negative integer the relation is

$$\bar{F}_p(1)=\bar{F}_p(2)=\bar{F}_p(3)=(-)^{\alpha_{045}}\bar{F}_n(0)=(-)^{\alpha_{045}}\bar{F}_n(4)$$
$$=(-)^{\alpha_{045}}\bar{F}_n(5). \qquad (25)$$

Relation (24) holds as long as the sines do not vanish. If there are other integers among the α, we need a more careful study. Relation (24) provides 60 equivalent formulas to compute expression (6), but there are still six other groups of ten expressions which are independent. The three term relations (4) are such that these six other $\overline{F}_p$ or $\overline{F}_n$ cannot be expressed only with those bound by relation (24).

When relations (24) are used, some unwanted Γ functions appear in the denominator for $\overline{F}_p(l)$ and $\overline{F}_n(i)$; two of them can be inverted, introducing two other sines which cancel those of relation (24) in some cases. In particular, when $b+\beta$ or $a-\alpha$ is an integer, from $F_p(3;45)$, we get

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(a-b-\gamma)]\frac{\Gamma(a+c+\beta+1)\Gamma(b+c-\alpha+1)}{\Gamma(a+b-c+1)}R_p(3) \times {}_3F_2[-a-b-c-1,-b-\beta,-a+\alpha;-a-c-\beta,-b-c+\alpha;1]. \quad (26)$$

When $b+\beta$ or $a-b+c$ is an integer, from $F_p(2;45)$ we get

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(a-b-\gamma)]\frac{\Gamma(a+b-\gamma+1)\Gamma(2b+1)}{\Gamma(a-\alpha+1)}R_p(2) \times {}_3F_2[-a-b-c-1,-b-\beta,-a-b+c;-a-b+\gamma,-2b;1]. \quad (27)$$

When $a-\alpha$ is an integer from $F_p(3;24)$ and $F_p(3;25)$ we get

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(2a-b+\beta)]\frac{\Gamma(b+c-a+1)\Gamma(c-\gamma+1)}{\Gamma(b-c+\alpha+1)}R_p(3) \times {}_3F_2[-a+b-c,b-\beta+1,-a+\alpha;b-c+\alpha+1,-a-c-\beta;1], \quad (28)$$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(2a-b+\beta)]\frac{\Gamma(b+c-\alpha+1)\Gamma(1+c-\gamma)}{\Gamma(b-c+\alpha+1)}R_p(3) \times {}_3F_2[-c-\gamma,a+\alpha+1,-a+\alpha;b-c+\alpha+1,-b-c+\alpha;1]. \quad (29)$$

When $b+\beta$ is an integer from $F_p(2,15)$, we get

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(a-2b+\alpha)]\frac{\Gamma(a+\alpha+1)\Gamma(2b+1)}{\Gamma(a-b+\gamma+1)}R_p(2) \times {}_3F_2[a-b-c,a-b+c+1,-b-\beta;a-b+\gamma+1,-2b;1]. \quad (30)$$

A last kind of argument in the ${}_3F_2$ is obtained from $F_p(3;12)$,

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(a-b-\gamma)] \times \frac{R_p(3)}{\sin\pi\alpha_{134}\sin\pi\alpha_{135}\Gamma(-a-b-c-1)\Gamma(a-c-\beta+1)} \frac{1}{\Gamma(b-c+\alpha+1)}\,{}_3F_2[a+b-c+1,b-\beta+1,a+\alpha+1;a-c-\beta+1,b-c+\alpha+1;1], \quad (31)$$

where $l=1$ when $b+\beta$ is an integer, $l=2$ when $a-\alpha$ is an integer, or l has any of these values when $2c$ is an integer.

Formulas (6), (13)–(17), and (26)–(31) display the twelve possible types of arguments for the ${}_3F_2$; the 120 ${}_3F_2$ can be obtained from them by the permutation of a, b, c and change of sign for α, β, γ. Formulas (6), (13), (26), and (31) lead to six ${}_3F_2$ and all the others to twelve. For usual arguments (6), (26), and (27) include three negative integers as numerator coefficients of the ${}_3F_2$; not taking symmetries into account there are 24 such formulas. There are known[9] results of Van der Waerden for (6) and Bandzaitis and Yutsis for the others. Also, in the usual case, there are two negative integers in formulas (28)–(30) which represent 36 ${}_3F_2$. They are Wigner's, Racah's and Majumdar's formulas respectively. There is only one negative integer in formulas (15)–17 and none in formulas (13), (14), and (31); nevertheless, these formulas could be used for negative quantum numbers.

As there are Γ functions in definition (2) of F_p and F_n we must look more carefully to those with a negative integer argument. When one of the β is a negative integer, $-n$, the Γ cancels the n first terms of the series, relations (19)–(21) could be obtained by this prescription. When one α, α_{145} for example, is a negative integer, the series $F_p(0;23)$, $F_p(2;03)$, $F_p(3;02)$, $F_n(1;45)$, $F_n(4;15)$, and $F_n(5;14)$ in the definition of which $\Gamma(\alpha_{145})$ appears are divergent because α_{145} is their convergence indicator. The result is mathematically undefined. However, we can sum, to some extent, these series as described in Appendix A and we can check that $F_p(0;23)=F_p(0;45)$ for any complex value of α_{145}, if the real part of $-\alpha_{145}$ is not too large. There is no such problem of convergence for the 18 finite series.

5. EXISTENCE OF MORE THAN ONE INTEGER

As the sum of two α's without common indices is unity, two negative α have one or two common indices. When they have two common indices, they can appear as coefficients of the same ${}_3F_2$; when they have only one common index, one and the positive value of the other can appear together, leading to quite complicated situations where some of the previous results break down.

When the α's which are negative integers have two common indices, relations (24) and (25) written for each of them are compatible. The second one adds a $\overline{F}_p$ and $\overline{F}_n$ to relation (24), but a third or a fourth negative integer adds only one $\overline{F}_n$. The conditions that α_{045}, α_{145}, α_{245}, and α_{345} are negative integers give no relation between $\overline{F}_p(0)$, $\overline{F}_p(4)$, and $\overline{F}_p(5)$ and these last two remain independent.

If $\alpha_{145} = -n$ and $\alpha_{045} = -m$ with $m > n$, the 18 finite series related by (24) are limited by n but $\bar{F}_p(0;14)$, $\bar{F}_p(0;15)$, $\bar{F}_n(1;02)$, and $\bar{F}_n(1;03)$ in which there is $\beta_{01} = 1 + m - n$, start from $n - m$ instead of zero. Among the 18 finite series related by (25), four includes $-n$ and the others are limited by $-m$.

To illustrate the conflicting situation, let us take $b \pm \beta$ integer. As $\alpha_{124} = 1 + b - \beta$, the coefficients of $\bar{F}_p(2)$ and $\bar{F}_n(5)$ become infinite in relation (24); conversely, the coefficients of $\bar{F}_p(2)$ and $\bar{F}_n(5)$ become infinite because $b + \beta$ is integer in the relation between these functions and $\bar{F}_p(4)$ obtained from $\alpha_{035} = -b + \beta$. The relations obtained from α_{035} and α_{145} are not compatible because the ratio of the coefficients of $\bar{F}_p(2)$ and $\bar{F}_n(5)$ is not the same in the two relations. However, formulas transformed as (26)—(30) no longer include an infinite coefficient. A closer look at the ten $\bar{F}_p(2)$ shows that, in the absence of other integers, $\bar{F}_p(2;01)$, $\bar{F}_p(2;04)$, $\bar{F}_p(2;13)$, and $\bar{F}_p(2;34)$ are infinite sums which must vanish by continuity with respect to $-b + \beta$; in relation (24) they are indefinite ($\infty \times 0$). There is $-2b$ among the denominator parameters of $\bar{F}_p(2;05)$, $\bar{F}_p(2;15)$, $\bar{F}_p(2;35)$, and $\bar{F}_p(2;45)$; for them $[\sin\pi\alpha_{124}\Gamma(-2b)]^{-1}$ can be replaced by $(-)^{b+\beta}\pi^{-1}\Gamma(2b+1)$ if $b - \beta$ is not an integer. $\bar{F}_p(2;15)$ and $\bar{F}_p(2;45)$ are finite sums because they include $-b - \beta$. $\bar{F}_p(2;05)$ and $\bar{F}_p(2;15)$ include $-b + \beta$ and are infinite sums, but if $-b + \beta$ becomes an integer, the terms from $b - \beta + 1$ to $2b$ vanish and the finite sum limited by $-b + \beta$ plus the infinite sum beyond $2b$ verifies relation (24). For $\bar{F}_p(2;14)$, $s = -b + \beta$ and $[\sin\pi\alpha_{124}\Gamma(s)]^{-1}$ can be replaced by $\pi^{-1}\Gamma(1 + b - \beta)$. The last one, $\bar{F}_p(2;03)$, of which the convergence parameters is $-b - \beta$ becomes a finite series related to $\bar{F}_p(4)$.

In conclusion, we can replace $[\sin\pi\alpha_{124}]^{-1}$ by $(-)^{-b+\beta}\pi^{-1}\Gamma(0)$ and $\Gamma(0)$ can be used to change $\Gamma(0)\Gamma(-n)^{-1}$ into $(-)^n\Gamma(n+1)$ for any positive integer n appearing in a meaningful case (five among the ten). Such a limit must be taken carefully because a too simple correspondence $\sin\pi n^{-1} \to (-)^n\Gamma(0)$ is misleading [$\sin\pi n \equiv \sin\pi(1-n)$ for any value of n]. The set of relations can be written:

$$\bar{F}_p(0) = \frac{\pi^2}{\sin\pi\alpha_{135}\sin\pi\alpha_{134}}\bar{F}_p(3) = (-)^{\alpha_{145}}\bar{F}_n(1)$$
$$= \frac{\pi^2}{\sin\pi\alpha_{125}\sin\pi\beta_{43}}\bar{F}_n(4) \approx (\alpha_{145})\frac{\pi\Gamma(0)}{\sin\pi\beta_{54}}\bar{F}_p(2)$$
$$\approx (\alpha_{145})(-)^{\alpha_{145}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{32}}\bar{F}_n(5), \tag{32}$$

$$\bar{F}_p(4) = \frac{\pi^2}{\sin\pi\alpha_{013}\sin\pi\alpha_{125}}\bar{F}_p(1) = (-)^{\alpha_{035}}\bar{F}_n(3)$$
$$= \frac{\pi^2}{\sin\pi\alpha_{235}\sin\pi\beta_{01}}\bar{F}_n(0) \approx (\alpha_{035})\frac{\pi\Gamma(0)}{\sin\pi\beta_{50}}\bar{F}_p(2)$$
$$\approx (\alpha_{035})\frac{\pi\Gamma(0)}{\sin\pi\beta_{12}}\bar{F}_n(5), \tag{33}$$

$$\bar{F}_p(5) = \frac{\sin\pi\alpha_{125}}{\sin\pi\alpha_{012}}\bar{F}_p(0) + \frac{\sin\pi\alpha_{235}}{\sin\pi\alpha_{234}}\bar{F}_p(4), \tag{34}$$

$$\bar{F}_n(2) = \frac{\sin\pi\alpha_{134}}{\sin\pi\beta_{40}}\bar{F}_p(0) - \frac{\sin\pi\alpha_{013}}{\sin\pi\beta_{40}}\bar{F}_p(4), \tag{35}$$

where $\approx(\alpha_{\lambda\mu\nu})$ means, when the series is defined, one should take the series limited by $\alpha_{\lambda\mu\nu}$ or the finite series plus an infinite one (the finite series not limited by $\alpha_{\lambda\mu\nu}$, without an infinite one does not verify the relation). In this case $\beta_{52} = -2b$ is a negative integer; relation (21) fails, but gives the value of the infinite series (beyond $2b+1$) of $\bar{F}_p(2)$ in terms of $\bar{F}_n(5)$ and $\bar{F}_p(5)$ and $\bar{F}_n(2)$ to a sign.

When $\alpha_{245} = -a + \alpha$ is also an integer, there is compatibility between α_{145} and α_{245} but not between α_{245} and α_{035}. Relation (32) is completed by

$$\bar{F}_p(0) = (-)^{\alpha_{245}}\bar{F}_n(2) \approx (\alpha_{245})\frac{\pi\Gamma(0)}{\sin\pi\beta_{54}}\bar{F}_p(1)$$
$$= (-)^{\alpha_{145}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{32}}\bar{F}_n(5). \tag{36}$$

A detailed verification of $\bar{F}_n(5)$ shows no effect of α_{035}: one needs only the first series, finite of infinite; when there are two parts of the series they have opposite signs; the only undefined expressions are $\bar{F}_n(5;04)$ and $\bar{F}_n(5;34)$. Relation (33) reduces to

$$\bar{F}_p(4) = (-)^{\alpha_{035}}\bar{F}_n(3) = \frac{\pi^2}{\sin\pi\alpha_{235}\sin\pi\beta_{01}}\bar{F}_n(0)$$
$$\approx (\alpha_{035})\frac{\pi\Gamma(0)}{\sin\pi\beta_{50}}[\bar{F}_p(1), \bar{F}_p(2)] \approx (\alpha_{035})\frac{\pi\Gamma(0)}{\sin\pi\beta_{12}}\bar{F}_n(5), \tag{37}$$

but the relation with $\bar{F}_n(5)$ holds only when α_{035} is present, otherwise, it is $\infty \times 0$ because β_{12} vanishes. Relation (34) stays and relation (35) disappears.

When $\alpha_{034} = -a - \alpha$ is an integer, the modifications of relations(32)—(35) are similar but interchanged with respect to $\bar{F}_p(0)$ and $\bar{F}_p(4)$.

With four negative integers, there are two conflicting situations. In the first one, the negative integers are on the same row or column of the Regge symbol. With the last row, from $\alpha_{145} = -b - \beta$ and $\alpha_{045} = -a - b - c - 1$, we get

$$\bar{F}_p(0) = (-)^{\alpha_{145}}\bar{F}_n(1) \approx (\alpha_{145})\frac{\pi\Gamma(0)}{\sin\pi\beta_{45}}\bar{F}_p(2)$$
$$\approx (\alpha_{145})\frac{\pi\Gamma(0)}{\sin\pi\beta_{54}}\bar{F}_p(3) \approx (\alpha_{145})(-)^{\alpha_{145}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{32}}\bar{F}_n(4)$$
$$\approx (\alpha_{145})(-)^{\alpha_{145}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{23}}\bar{F}_n(5), \tag{38}$$

and from α_{045} and $\alpha_{034} = -a - \alpha$, or $\alpha_{025} = -c - \gamma$, a similar relation for $\bar{F}_p(4)$ and $\bar{F}_p(5)$. These three sets of formulas are related by

$$\bar{F}_p(0) = \frac{\sin\pi\alpha_{012}}{\sin\pi\alpha_{124}}\bar{F}_p(4) + \frac{\sin\pi\alpha_{013}}{\sin\pi\alpha_{135}}\bar{F}_p(5). \tag{39}$$

Note the difference of sign between $\bar{F}_p(2)$ and $\bar{F}_p(3)$ or $\bar{F}_n(4)$ and $\bar{F}_n(5)$ in expression (38), whereas there is no difference of sign in relation (25) which holds when α_{045} only is a negative integer.

The second conflicting arrangement with four integers is $a \pm \alpha$ and $b \pm \beta$ integer. In this case

$$\bar{F}_p(0) = \frac{\pi^2}{\sin\pi\alpha_{135}\sin\pi\alpha_{134}}\bar{F}_p(3) = (-)^{\alpha_{145}}\bar{F}_n(1)$$

$$= (-)^{\alpha_{245}}\bar{F}_n(2) = (-)^{\alpha_{145}} \frac{\pi\Gamma(0)}{\sin\pi\beta_{32}} [\bar{F}_n(4), \bar{F}_n(5)]$$

$$\approx (-)^{r_4 - r_5}\Gamma(0)\Gamma(0)[\bar{F}_p(1), \bar{F}_p(2)] \quad (40)$$

but the relation with $\bar{F}_p(1)$ and $\bar{F}_p(2)$ holds only when α_{245} or α_{145} respectively are present. Also

$$\bar{F}_p(4) = (-)^{r_4 - r_5}\bar{F}_p(5) = (-)^{\alpha_{035}}\bar{F}_n(3) = \frac{\pi^2}{\sin\pi\alpha_{235}\sin\pi\beta_{01}}\bar{F}_n(0)$$

$$= \frac{\pi\Gamma(0)}{\sin\pi\beta_{50}}[\bar{F}_p(1), \bar{F}_p(2)] \approx (-)^{\alpha_{035} + r_1 - r_2}\Gamma(0)\Gamma(0) \times [\bar{F}_n(4), \bar{F}_n(5)]. \quad (41)$$

The transformation to an hyperspherical basis of a two-body harmonic oscillator wavefunction leads to a generalized 3-j symbol[10] for which $a \pm \alpha$ are the radial quantum numbers, $a \pm (b - c)$ are the hyperspherical and the hyperradial quantum numbers; in the simplest cases $b \pm \beta$ and $c \pm \gamma$ are half-integer. Seven different formulas were given, which lead to the same result for any value of two parameters ($b \pm \beta$ for example). We found

$$\bar{F}_p(0) = \frac{\pi^2}{\sin\pi\alpha_{234}\sin\pi\alpha_{235}}[\bar{F}_p(2), \bar{F}_p(3)] = (-)^{r_0 - r_5}\bar{F}_p(5)$$

$$= \frac{\pi^2}{\sin\pi\alpha_{234}\sin\pi\beta_{51}}[\bar{F}_n(0), \bar{F}_n(5)] = (-)^{\alpha_{245}}\bar{F}_n(2)$$

$$= (-)^{\alpha_{345}}\bar{F}_n(3) = \frac{\pi\Gamma(0)}{\sin\pi\beta_{45}}\bar{F}_p(1) = (-)^{\alpha_{245}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{31}}\bar{F}_n(4)$$

$$= \frac{\sin\pi\alpha_{023}}{\sin\pi\alpha_{234}}\bar{F}_p(4) + \frac{\sin\pi\beta_{40}}{\sin\pi\alpha_{234}}\bar{F}_n(1). \quad (42)$$

A trivial symmetry of this problem was the change of sign for the magnetic quantum numbers which is the change of sign for the r and permutation of (045) with (123), as can be seen in (7). The seven formulas of Ref. 10 are $F_n(4)$ [by symmetry $F_p(1)$]. A $\bar{F}_p(5;24)$ was previously published.[11] In fact, there are eight formulas with two negative integers among the numerator parameters of the ${}_3F_2$ [four $\bar{F}_p(1)$ and one $\bar{F}_p(0), \bar{F}_p(2), \bar{F}_p(3)$, and $\bar{F}_p(5)$] and twenty formulas with only one negative integer [four $\bar{F}_p(0), \bar{F}_p(1), \bar{F}_p(2), \bar{F}_p(3)$, and $\bar{F}_p(5)$]. $\bar{F}_p(4)$ and $\bar{F}_n(1)$ cannot be used except by their linear combination shown in (42); anyway, they include no negative integer. Consequently there is no conflict in this situation with four integers.

When $\alpha_{015} = a - b - c$ is also negative integer we get

$$\bar{F}_p(0) = (-)^{r_0 - r_5}\bar{F}_p(5) = (-)^{\alpha_{245}}\bar{F}_n(2) = (-)^{\alpha_{345}}\bar{F}_n(3)$$

$$= \frac{\pi\Gamma(0)}{\sin\pi\beta_{54}}[\bar{F}_p(2), \bar{F}_p(3), -\bar{F}_p(1)]$$

$$= (-)^{\alpha_{345}}\frac{\pi\Gamma(0)}{\sin\pi\beta_{12}}[\bar{F}_n(0), \bar{F}_n(5), -\bar{F}_n(4)], \quad (43)$$

$$\bar{F}_p(4) = (-)^{\alpha_{015}}\bar{F}_n(1) \approx (-)^{r_5 - r_0}\Gamma(0)\Gamma(0)[\bar{F}_p(2), \bar{F}_p(3)]$$

$$\approx (-)^{\alpha_{015} + r_3 - r_2}\Gamma(0)\Gamma(0)[\bar{F}_n(0), \bar{F}_n(5)], \quad (44)$$

but the relations denoted by $\approx$ hold only three times, the other values being $\infty \times 0$. When $\alpha_{145} = -b - \beta$ is a negative integer

$$\bar{F}_p(0) = (-)^{\alpha_{145}}\bar{F}_n(l) = (-)^{r_0 - r_5}\bar{F}_p(5)$$

$$= \frac{\pi^2}{\sin\pi\alpha_{124}\sin\pi\beta_{53}}[\bar{F}_n(0), \bar{F}_n(5)] = \frac{\pi\Gamma(0)}{\sin\pi\beta_{45}}\bar{F}_p(l)$$

$$= (-)^{\alpha_{145} + r_2 - r_3}\Gamma(0)\Gamma(0)\bar{F}_n(4) \quad (45)$$

and $\bar{F}_p(4)$ is independent. There is a similar relation when $\alpha_{025} = -c - \gamma$ is a negative integer.

The most important case is when α_{ijk} and the nine α_{lij} are negative integer because it is the usual one.[15] Then

$$\bar{F}_p(0) = (-)^{\alpha_{145}}\bar{F}_n(l) = (-)^{r_0 - r_4}\bar{F}_p(4) = (-)^{r_0 - r_5}\bar{F}_p(5)$$

$$= (-)^{r_4 - r_5}\Gamma(0)\Gamma(0)\bar{F}_p(l) = (-)^{\alpha_{145} + r_2 - r_3}\Gamma(0)\Gamma(0)\bar{F}_n(i) \quad (46)$$

leading to 120 formulas for the same 3-j symbol. In (45) and (46) l stands for 1, 2, and 3 and i stands for 0, 4 or 5. Among them 96 are finite sums.

6. SYMMETRIES

Up till now we considered the generalized 3-j symbol (6) and we investigated how many formulas can be used to obtain it. Conversely, each formula can be interpreted as twelve generalized 3-j symbols, which means that there can be symmetry properties between 1440 generalized 3-j symbols. To study them, we have only to consider the six r parameters (7).

The six r parameters divide into two subsets: r_0, r_4, r_5 on one side and r_1, r_2, r_3 on the other side. Their permutations and the change of their sign generate the 1440 3-j symbols. We can establish the following properties:

(1) r_1, r_2, r_3 can be permuted. This symmetry is trivial and introduces no sign.

(2) r_0, r_4, r_5 can be permuted two by two if their differences is an integer. Permutation of r_4 and r_5 is trivial. Permutation of r_0 and r_4 leads to

$$\exp[i\pi(r_5 - r_4)]\bar{F}_p(0) \to \exp[i\pi(r_5 - r_0)]\bar{F}_p(4)$$

$$= -\exp[i\pi(r_5 - r_0)]\left(\frac{\sin\pi(r_4 - r_5)}{\sin\pi(r_5 - r_0)}\bar{F}_p(0) + \frac{\sin\pi(r_0 - r_4)}{\sin\pi(r_5 - r_0)}\bar{F}_p(5)\right) \quad (47)$$

which means invariance if $r_0 - r_4$ is an integer.

(3) The transformations above are the even transformations of the Regge symbol. The simplest odd transformation is the change of sign for $\alpha, \beta, \gamma, r_4 \to -r_1$, $r_5 \to -r_2$, $r_0 \to -r_3$, which leads to

$$\exp[i\pi(r_5 - r_4)]\bar{F}_p(0) \to \exp[i\pi(r_1 - r_2)]\bar{F}_n(3)$$

$$= \frac{\exp[i\pi(r_1 - r_2)]}{\sin\pi(\alpha_{125} + \alpha_{345})}[\sin\pi\alpha_{125}\bar{F}_p(0) - \sin\pi\alpha_{345}\bar{F}_p(5)]. \quad (48)$$

If $a + b + c = -\alpha_{345}$ is an integer,

$$\begin{pmatrix} a & b & c \\ -\alpha & -\beta & -\gamma \end{pmatrix} = \exp[i\pi(a + b - c + 2\gamma)]\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (49)$$

which reduces to the usual relation when $\alpha_{014} = -c + \gamma$ is also an integer. In such a relation r_1, r_2, r_3 can be permuted. If α_{145} or α_{245} is an integer we obtain other relations related to even permutations of a, b, c in the right-hand side of (49).

These symmetries are the usual Regge's symmetries. Permutations of a r_i with a r_l lead to Yutsis's mirror symmetry; they do not keep the structure of the square roots $R_p(\lambda)$ or $R_n(\lambda)$ and introduce negativelike arguments in the Γ functions. In that, we cannot follow Yutsis' rule because it is not an analytical continuation. Let us consider how Yutsis introduces his rule: The change $l \to -l-1$ into $(l+m)!/(l-m)!$ introduces a "phase"

$$\frac{\Gamma(-l+m)\Gamma(l-m+1)}{\Gamma(-l-m)\Gamma(l+m+1)} = \frac{\sin\pi(l+m)}{\sin\pi(l-m)}. \quad (50)$$

For any complex value of m, this reduces to $(-)^{2l+1}$ when l is an integer or half-integer; for any complex value of l, this phase is $(-)^{2m}$ for any integer or half-integer value of m; when l and m are both integer or half integer, this phase is mathematically undefined but is taken as being $(-)^{2m}$ by Yutsis, in contradiction with the analytical continuation from l to $-l-1$, as discussed above. The square root introduces an ambiguity of sign which depends on the path of analytical continuation.

By permutation of r_1 and r_4 in (9) we get

$$\begin{pmatrix} -a-1 & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp(i\phi) \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (51)$$

with

$$\exp(i\phi) = \exp[i\pi(-2a-1)] \left(\frac{\Gamma(\alpha_{012})\Gamma(\alpha_{013})\Gamma(\alpha_{245})\Gamma(\alpha_{345})}{\Gamma(\alpha_{024})\Gamma(\alpha_{034})\Gamma(\alpha_{125})\Gamma(\alpha_{135})} \right)^{1/2}$$
$$= \exp[i\pi(-2a-1)] \left(\frac{\sin\pi(c-b+a)\sin\pi(\alpha+a)}{\sin\pi(c-b-a)\sin\pi(\alpha-a)} \right)^{1/2}. \quad (52)$$

This phase vanishes for $a = -\frac{1}{2}$. It is the phase for a path which is symmetric for $a \to -a-1$ in the complex plane; its value is ± 1 for all integer or half-integer a. If the path is chosen along the real axis the phase of $\sin\pi(\alpha+a)/\sin\pi(\alpha-a)$ increases or decreases by π with respect to the sign of the imaginary part of α when a increases for a half-unit. Consequently, when the signs of the imaginary parts of $(c-b)$ and α are identical, the square root is $\exp[i\pi(2a+1)]$ for any integer or half-integer value of a; when these signs are opposite, the square root is 1 for these values. We obtain

$$\begin{pmatrix} -a-1 & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-)^{(a+1/2)[\mathrm{Sign}\{\mathrm{Im}(c-b)\} - \mathrm{Sign}\{\mathrm{Im}(\alpha)\}]} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (53)$$

for any value of b, c, α, and β when a is an integer or half-integer. The path can be deformed; then $\mathrm{Sign}\{\mathrm{Im}(c-b)\} = 1$ means that the zeros $b-c+n\pi$ are above the path and the poles $c-b+n\pi$ below it.

Permutation of r_5 and r_2 leads to a phase deduced from (52) by cyclic permutation of a, b, c. Permutation of r_0 and r_3 leads to

$$\begin{pmatrix} a & b & -c-1 \\ \alpha & \beta & \gamma \end{pmatrix} = \left(\frac{\sin\pi(a-b+c)\sin\pi(\gamma+c)}{\sin\pi(a-b-c)\sin\pi(\gamma-c)} \right)^{1/2} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (54)$$
$$= (-)^{(c+1/2)[\mathrm{Sign}\{\mathrm{Im}(b-a)\} - \mathrm{Sign}\{\mathrm{Im}(\gamma)\}]} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$$

when $r_3 - r_0 = 2c+1$ is an integer, using (21). If $b+\beta$ or $a-\alpha$ is an integer, (24) can be used, but it is more restrictive than (21). We can note

(1) the phase for the change of sign of a, b, c is cyclic.

(2) there is no phase for $a \to -a-1$ when a is half-integer.

(3) if there is a phase $(-)^{2a+1}$, this phase disappears when the sign of magnetic quantum numbers is changed: Relation (49) holds with the actual values of a, b, c in contrast with Yutsis' notation.

(4) the phase for $a \to -a-1$, $b \to -b-1$ is the product of the phases for $a \to -a-1$ by the phase for $b \to -b-1$ with a changed into $-a-1$. As a and b must be half integers, we can define the path with respect to the imaginary parts of α, β, and c; roles of a and b can be interchanged.

(5) there is a difficulty for $a \to -a-1$, $b \to -b-1$, and $c \to -c-1$ when anyone of a, b, and c is half integer because no parameter is left to define the path. However, very different relations can be found when there are more than one restriction on the r.

For example, let us consider the relation (51) when $(c-b)$ and α are integers: When all the poles are on the real axis, the square root of (52) reduces to a sign $(-)^{2c-2b+2\alpha}$ and

$$\begin{pmatrix} -a-1 & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \exp[i\pi(-2a-1)] \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (55)$$

for any value of a. If $c-b$ or α is half-integer the symmetric path cannot be used because it goes through the singularity; a detailed study of the behavior of $R_p^2(0)$ in the vicinity of $a = -\frac{1}{2}$ when going along a small circle shows a change of sign between $a = -\frac{1}{2} + \epsilon$ and $a = -\frac{1}{2} - \epsilon$ which compensates the phase $2c - 2b + 2\alpha$ and the relation (55) holds. A similar relation can be obtained for any complex value of c only if $b \pm \beta$ and $a \pm \alpha$ are negative integers, using (40).

In conclusion, the analytic continuation $a \to -a-1$ for integer or half-integer values of a, when the other quantum numbers are complex, introduces a phase ± 1, depending on the path, but never i as in Yutsis' notation. Furthermore there is a phase $\pm \exp[i\pi(2a+1)]$ for any complex value of a if some other quantum numbers are integer or half-integer.

7. SUMMARY AND CONCLUSION

A generalized 3-j symbol is defined by two sets of three parameters (r_1, r_2, r_3) and (r_0, r_4, r_5) where r_0 plays a special role. To compute each of them, there are ten formulas related to permutations of r_1, r_2, r_3, r_4, r_5 and there are twelve independent definitions with respect to what is selected to be r_0 and an overall sign for the r. There are three-term relations between any of these independent definitions. However, if the differences between two r or some combinations of them are integers, the twelve definitions become identical, leading to 120 formulas among which there are 12 different patterns for the usual conditions on the angular momenta.

Each formula can be interpreted in six independent ways as generalized 3-j symbols (permutation of r_1, r_2, r_3). Permutation of r_0, r_4, and r_5 gives new coefficients if their difference is an integer. Exchange of (r_1, r_2, r_3) with (r_4, r_5, r_0) and change of sign gives relation to another set of coefficients if r_l and r_m can be found among r_1, r_2, r_3 such that $\frac{1}{2}+r_0+r_l+r_m$ is a positive integer, but this relation includes a coefficient which reduces to the usual phase for angular momenta. If all the differences $r_i - r_j$ are integer we get the 72 coefficients of Regge's symmetry. In all these permuations, the arguments of Γ functions remain positivelike and there is no problem of phase.

Keeping r_0 fixed, permutations of the other r's give relations to other generalized 3-j symbols which are usually considered as analytic continuation of the usual ones. These relations reduce to a phase when the permuted r differs by an integer. This phase is always + or − and depends on the path of analytic continuation (a phase i can be obtained only if the path goes through a zero and a pole). So, there are 12 sets of 120 generalized 3-j coefficients for any value of the r. Each of them can be obtained using ten different series, finite or infinite, which can be summed up when all the r are small enough even if they diverge, using the method described in Appendix A. When the differences between the r are all integers there are 1440 related 3-j symbols and, in principle, 120 formulas for each of them.

The following patterns:

$$\begin{vmatrix} + & + & + \\ + & + & + \\ + & + & + \end{vmatrix}_+ \quad \begin{vmatrix} - & - & + \\ + & + & - \\ + & + & - \end{vmatrix}_+ \quad \begin{vmatrix} + & + & - \\ - & - & + \\ - & - & + \end{vmatrix}_- \quad \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}_- \qquad (56)$$

are respectively those of the Regge's symbol for the usual 3-j symbols and after the change $a \to -a-1$ for 1, 2, or 3 angular momenta (the sign of the sum of rows is given in subscripts). Instead of the analytic continuation, the sign of which cannot be easily defined, we suggest that any generalized 3-j symbol be defined by such or such expression with respect to the first pattern.

This work did not take into account the pattern

$$\begin{matrix} \begin{vmatrix} + & + & - \\ + & + & + \\ + & + & + \end{vmatrix}_+ & \begin{vmatrix} - & + & - \\ + & - & + \\ + & - & + \end{vmatrix}_+ & \begin{vmatrix} - & - & + \\ + & + & - \\ + & + & - \end{vmatrix}_- \\ \begin{vmatrix} + & + & - \\ - & - & + \\ - & - & + \end{vmatrix}_+ & \begin{vmatrix} - & - & + \\ - & + & - \\ - & + & - \end{vmatrix}_- & \begin{vmatrix} - & - & + \\ - & - & - \\ - & - & - \end{vmatrix}_- \end{matrix} \qquad (57)$$

which occurs if the usual relations between arguments are kept except that $c > a+b$, because we know no example of these coefficients. Their study is more difficult, due to the definition of positivelike arguments of Γ functions and the relative position of negative integers. Note that (56) and (57) do not present all the possible patterns.

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APPENDIX A: EVALUATION OF DIVERGENT SERIES

Let us consider the infinite series

$$ {}_3F_2[a,b,c;e,f;1] = 1 + \frac{abc}{ef\times 1} + \frac{abc}{ef\times 1}\,\frac{(a+1)(b+1)(c+1)}{(e+1)(f+1)\times 2} + \cdots \qquad \text{(A1)}$$

and its convergence parameter

$$s = e+f-a-b-c. \qquad \text{(A2)}$$

The series (A1) converges only if $\mathrm{Re}(s)>0$. In fact, it converges quite slowly if $\mathrm{Re}(s)$ is not at least 2 or 3. Let us introduce a hypergeometric series and a parameter D,

$$D\times {}_2F_1[A,B;C;1] = D\times\left[1+\frac{A\times B}{C\times 1}+\frac{A\times B}{C\times 1}\,\frac{(A+1)(B+1)}{(C+1)\times 2}+\cdots\right] \qquad \text{(A3)}$$

such that the nth term of (A1) and the nth term of (A2) coincide to a relative error n^{-4}. We get

$$\begin{aligned} &A+B-C=-s=a+b+c-e-f,\\ &A^2+B^2-C^2=a^2+b^2+c^2-e^2-f^2,\\ &A^3+B^3-C^3=a^3+b^3+c^3-e^3-f^3,\\ &D=\frac{\Gamma(e)\Gamma(f)\Gamma(A)\Gamma(B)}{\Gamma(C)\Gamma(a)\Gamma(b)\Gamma(c)}. \end{aligned} \qquad \text{(A4)}$$

The sum S of ${}_3F_2[a,b,c;e,f;1]-D_2F_1[A,B;C;1]$ converges if $\mathrm{Re}(s)>-3$ leading to the result

$$\begin{aligned} {}_3F_2[a,b,c;e,f;1] &= S + D_2F_1[A,B;C;1]\\ &= S+\frac{\Gamma(e)\Gamma(f)\Gamma(s)\Gamma(A)\Gamma(B)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(s+A)\Gamma(s+B)}. \end{aligned} \qquad \text{(A5)}$$

In particular, the relation

$$\frac{1}{\Gamma(e)\Gamma(f)\Gamma(s)}\,{}_3F_2[a,b,c;e;f;1] = \frac{S}{\Gamma(e)\Gamma(f)\Gamma(s)}+\frac{\Gamma(A)\Gamma(B)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(s+A)\Gamma(s+B)} \qquad \text{(A6)}$$

holds for $s=0,-1,-2$ when the term S disappears, but is invalid for the other negative integer values.

This method of evaluation for the ${}_3F_2$ is consistent with Thomae's transformation as, for sufficiently small coefficients, the ten $F_p(i)$ or $F_n(i)$ give the same result.

TABLE I. Shift of Whipple's parameters for contiguous functions.

	F	$F(A+1)$	$F(B+1)$	$F(C+1)$	$F(D-1)$	$F(E-1)$	$F(+)$
$r_0=\frac{1}{6}[5+2A+2B+2C-4D-4F]$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
$r_1=\frac{1}{6}[-1+2A-4B-4C+2D+2E]$	0	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$r_2=\frac{1}{6}[-1-4A+2B-4C+2D+2E]$	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$r_3=\frac{1}{6}[-1-4A-4B+2C+2D+2E]$	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$r_4=\frac{1}{6}[-1+2A+2B+2C+2D-4E]$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$r_5=\frac{1}{6}[-1+2A+2B+2C-4D+2E]$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
Notation	$F_p(0)$	$F_p^{-(2,3)}(0)$	$F_p^{-(1,3)}(0)$	$F_p^{-(1,2)}(0)$	$F_p^{(0,5)}(0)$	$F_p^{(0,4)}(0)$	$F_p^{(4,5)}(0)$

APPENDIX B: RECURRENCE RELATIONS

Recurrence relations between contiguous ${}_3F_2[A,B,C;D;E;z]$ have been studied by Rainville[18] but we know of no study of the recurrence relation for ${}_3F_2[A,B,C;D,E;1]$ based on their symmetries.

The simplest recurrence relations are

$$A\{{}_3F_2[A,B,C;D,E;z]-{}_3F_2[A+1,B,C;D,E;z]\}+\frac{ABCz}{DE}\,{}_3F_2[A+1,B+1,C+1;D+1,E+1;z]=0 \tag{B1}$$

and

$$(D-1)\{{}_3F_2[A,B,C;D,E;z]-{}_3F_2[A,B,C;D-1,E;z]\}+\frac{ABCz}{DE}\,{}_3F_2[A+1,B+1,C+1;D+1,E+1;z]=0 \tag{B2}$$

Eliminating $F(+)={}_3F_2[A+1,B+1,C+1,D+1,E+1;z]$, we get four independent relations between ${}_3F_2[A,B,C;D,E;z]$ and five of the ten contiguous functions. Table I gives the Whipple's parameters r in terms of A,B,C,D,E and their shift for contiguous functions.

We can note the occurrence of four shifts by $\frac{1}{3}$ and two by $-\frac{2}{3}$ or four shifts by $-\frac{1}{3}$ and two by $\frac{2}{3}$. Consequently we shall note by $F_p^{(i,j)}(0)$ the function defined by (2) for which r_i and r_j are increased by $\frac{2}{3}$ with respect to those of $F_p(0)$ and the other r decreased by $\frac{1}{3}$; similarly for $F_p^{-(i,j)}(0)$, r_i and r_j are decreased by $\frac{2}{3}$. With respect to Whipple's parameters, $F(+)$ is a contiguous function whereas in the usual sense it is not.

Therefore, there are 30 contiguous $F_p^{\pm(i,j)}(0)$ of $F_p(0)$; the ones with positive shift are given in Table II in terms of A,B,C,D,E and of 3-j symbols. Replacing the ${}_3F_2$ by $F_p(0)$ in (B.1) or (B.2) and using the symmetry property and also relation (4), one gets a recurrence relation between $F_p(0)$ and any two of its contiguous functions. All these recurrences can be collected into three types, of which the most important is

$$h(ij;kl)F_p(0)-g'(i,j)F^{-(i,j)}(0)-g(k,l)F_p^{(k,l)}(0)=0, \tag{B3}$$

where

$$h(ij,kl)=\tfrac{1}{4}+\tfrac{1}{2}(r_k+r_l-r_i-r_j)+r_ir_j+r_kr_l+r_i^2+r_j^2+r_k^2+r_l^2-\tfrac{1}{2}\sum_\lambda r_\lambda^2, \tag{B4}$$

$$g(i,j)=\alpha_{ijk}\alpha_{ijl}\alpha_{ijm},\quad i,j,k,l,m\neq 0$$
$$=1,\qquad i \text{ or } j=0 \tag{B5}$$

$$g'(i,j)=-\alpha_{klm},\qquad i,j,k,l,m\neq 0$$
$$=\alpha_{klm}\alpha_{kln}\alpha_{kmn}\alpha_{lmn},\quad i \text{ or } j=0,\ k,l,m,n\neq i,j. \tag{B6}$$

From (B3) we obtain

$$f(ij;kl)F_p(0)+g(i,j)F_p^{(i,j)}(0)-g(k,l)F_p^{(k,l)}(0)=0, \tag{B7}$$

$$f'(ij;kl)F_p(0)+g'(i,j)F_p^{-(i,j)}(0)-g'(k,l)F_p^{-(k,l)}(0)=0, \tag{B8}$$

where

$$f(ij;kl)=h(ij;kl)-h(ij;ij)$$
$$=\tfrac{1}{2}(r_k+r_l-r_i-r_j)+r_kr_l+r_k^2+r_l^2-r_ir_j-r_i^2-r_j^2, \tag{B9}$$

$$f'(ij;kl)=h(kl;kl)-h(ij;kl)$$
$$=\tfrac{1}{2}(r_i+r_j-r_k-r_l)+r_kr_l+r_k^2+r_l^2-r_ir_j-r_i^2-r_j^2. \tag{B10}$$

When written for the $\overline{F}_p(0)$, relation (B3) becomes

$$h(ij;kl)\overline{F}_p(0)-\overline{g}'(i,j)\overline{F}_p^{-(i,j)}(0)-\overline{g}(kl)\overline{F}_p^{(k,l)}(0)=0, \tag{B11}$$

TABLE II. Shift of ${}_3F_2$ and 3-j parameters for contiguous Whipple's parameters.

i,j	A	B	C	D	E	a	b	c	α	β	γ
0,1	0	−1	−1	−1	−1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
0,2	−1	0	−1	−1	−1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
0,3	−1	−1	0	−1	−1	0	0	0	−1	1	0
0,4	0	0	0	0	−1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
0,5	0	0	0	−1	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
1,2	0	0	−1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
1,3	0	−1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
1,4	1	0	0	1	0	0	0	0	0	−1	1
1,5	1	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
2,3	−1	0	0	0	0	−0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
2,4	0	1	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
2,5	0	1	0	0	1	−0	0	0	1	0	−1
3,4	0	0	1	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
3,5	0	0	1	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
4,5	1	1	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0

where

$$\bar{g}(i,j)=\epsilon(i,j)(\alpha_{ijk}\alpha_{ijl}\alpha_{ijm}\alpha_{ijn})^{1/2}, \tag{B12}$$

$$\bar{g}'(i,j)=\epsilon(i,j)(\alpha_{klm}\alpha_{kln}\alpha_{kmn}\alpha_{lmn})^{1/2}, \tag{B13}$$

with

$$\epsilon(i,j)=\begin{cases} 1 \text{ if } i \text{ or } j=0 \text{ and } i,j\neq 4 \text{ or } 5, \\ -1 \text{ if } i \text{ and } j\neq 0 \text{ and } i \text{ or } j=4 \text{ or } 5. \end{cases} \tag{B14}$$

In (B12) are all the $\alpha_{\lambda\mu\nu}$ with i and j; in (B13) are all the $\alpha_{\lambda\mu\nu}$ without i and j.

When written on the 3-j symbols, the phase (B14) is simple because $\epsilon(i,j)=1$ except for $\epsilon(4,5)=-1$. With the coefficients given by (B4), (B7), (B8), (B12), and (B13) we can write a recurrence relation similar to (B11) between a 3-j symbol and any two of the 30 contiguous ones.

One of the simplest cases is $i=1,\ j=4,\ k=2,\ l=5$ for which we get

$$-(1+c-\gamma)(c+\gamma)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} - [(-c-\gamma)(-b+\beta)(1+b+\beta)\ (1+c-\gamma)]^{1/2} \times \begin{pmatrix} a & b & c \\ \alpha & \beta+1 & \gamma+1 \end{pmatrix} - [(-c-\gamma)(1+a+\alpha)(1+c-\gamma) (-a+\alpha)]^{1/2} \begin{pmatrix} a & b & c \\ \alpha+1 & \beta & \gamma-1 \end{pmatrix} = 0 \tag{B15}$$

for any value of a, b, c, α, β, and γ.

As can be seen in Table II, these recurrences do not include the ones on an angular quantum number because the change of a into $a+1$ does not lead to a contiguous coefficient. A recurrence between $a, a-1, a+1$ can be derived from three elementary recurrence relations. Our definition of a contiguous ${}_3F_2$ differs from Bailey's definition.[19]

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Exact statistical mechanics of some classical 1D systems

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The classical partition function Q_N is calculated in *closed form* for the following 1D N-body "hard-core" potentials, $V=\Sigma_{i=1}^N (gx_i+b/|x_i-x_{i-1}|)$, i.e., a Coulomb nearest neighbor "chain" in a uniform field, and $V=(1/2)\Sigma_{i=0}^N \Sigma_{j=0}^N \exp(|x_i-x_j|)$, a "fluid" with exponential interactions. The Q_N for both systems is separated into a product of N, *similar*, tractable integrals each depending on a *different* value of the index i. All thermodynamic variables are obtained in closed form. In the limit, as $N\to\infty$, most of them do *not linearly* increase with the size of the system, i.e., they are not "extensive." This is also discussed in terms of the "stability" and "temperedness" properties of the potentials. Nevertheless, both systems do have a heat capacity which is "extensive."

1. INTRODUCTION

There are few N-body problems whose classical partition function is known exactly. Even for one-dimensional systems[1] with interparticle potentials other than linear or quadratic in the space coordinates, formidable calculational difficulties arise. Additional external fields can, in many cases,[2] complicate matters even further.

We will evaluate *exactly* and in *closed form* configurational partition functions

$$Q_N \equiv \int d\mathbf{x}_N \exp[-\beta V(\mathbf{x}_N)], \tag{1.1}$$

with $\beta\equiv 1/kT$, $\mathbf{x}_N\equiv(x_1,\ldots,x_i,\ldots,x_N)$ and T the temperature, for certain classical one-dimensional (1D) systems with *anharmonic* N-body potentials, $V(\mathbf{x}_N)$. In some cases, an external, uniform field will be included in $V(\mathbf{x}_N)$.

The multiple integral Q_N, (1.1), is, for general anharmonic potentials, extremely difficult to compute. We have been able to find, however, for two specific examples, nontrivial coordinate transformations, which *separate* Q_N into a product of N, *similar*, tractable integrals each depending on a *different* value of the index i ($i=1,2,\ldots,N$). Such systems are to be distinguished from the simpler ones discussed in Appendix C, in which an obvious transformation separates the Q_N into a product of N *identical* integrals, *independent* of the numbering of the degrees of freedom.

Section 2 deals with the nearest neighbor Coulomb potential in a uniform field

$$V(\mathbf{x}_N)=\sum_{i=1}^{N}\{gx_i+b/|x_i-x_{i-1}|\}, \tag{1.2}$$

with $g,b>0$, as an example of a wider class of potentials which are amenable to our treatment. Takahashi,[3] Gürsey,[4] *et al.* introduced a similar separation method for linear assemblies with general nearest neighbor potentials. Their results, however, are not applicable when an external field is present in the problem, as in (1.2). This more general case was also treated by Montroll,[5] who obtained an expression for the equation of state of a 1D system of particles with *arbitrary* nearest neighbor potential.

In Sec. 3 we treat the exponential potential

$$V(\mathbf{x}_N)=\tfrac{1}{2}\sum_{i=0}^{N}\sum_{i=0}^{N}\exp(|x_i-x_j|), \tag{1.3}$$

in which we allow interactions between *all* particles. Kac, in 1959, calculated exactly the partition function for a 1D "gas" with pair interaction $-\exp(-|x_i-x_j|)$ plus a finite size hard core.[6] His approach, however, and his results for the fundamental thermodynamic variables are quite different from our own.

In each of the above two cases, (1.2) and (1.3), we display in Secs. 2B and 3B, *explicit* formulas for the free energy F, the internal energy U, the heat capacity C, etc. In Sec. 2A we also obtain the average length L and the thermal expansion coefficient α of the Coulomb gravitating (CG) "chain," (1.2).

In 2B and 3B, taking the $N\to\infty$ limit, we find for *both* systems that the "bulk"-limit of F/N and/or F/L is *not* finite. Hence *not* all appropriate thermodynamic variables will be *extensive*, i.e., proportional to the size of the system. The divergence of U/N appears as a constant, dependent on N but *not* dependent on T. Therefore, the C/N for both systems turns out to have a *finite* value in the infinite N limit. The divergent term in U/N could, of course, be removed by subtracting the appropriate function of N from the Hamiltonian. This, however, does not affect the calculation of L which still remains a non extensive quantity.

2. COULOMB INTERACTION IN A UNIFORM FIELD

Consider a one-dimensional "chain" of N particles in a uniform field of strength g, which are coupled to each other by some nearest neighbor interaction $v(r)$, where $r\equiv|x_i-x_{i-1}|$ and $i=1,2,\ldots,N$. The potential energy for such a system is

$$V(\mathbf{x}_N)=\sum_{i=1}^{N}\{gx_i+v(|x_i-x_{i-1}|)\}. \tag{2.1}$$

One end of the chain is *fixed* at the origin of the coordinates, i.e.,

$$x_0(t)\equiv 0 \quad \text{for all } t, \tag{2.2}$$

while the other end is *free* to move between zero and infinity. In addition, the particles have 0-diam hard core, i.e., they do not go "through" each other,

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$$0 = x_0 \le x_1 \le x_2 \le \cdots \le x_N < \infty. \tag{2.3}$$

As an example we evaluate Q_N, (1.1), for the "chain" with Coulomb interaction $v(r) = 1/r$, and compute its thermodynamic properties. The potential energy is

$$V(\mathbf{x}_N) = \sum_{i=1}^{N} \{x_i + b/(x_i - x_{i-1})\}, \tag{2.4}$$

for any $b > 0$ and $g = 1$. Equation (2.4) may be thought of as the potential for an array of N identically charged particles subject to a uniform gravitational field ("CG" system).

We introduce the simple but important identity

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} (N+1-i)(x_i - x_{i-1}), \tag{2.5}$$

and perform a *linear* transformation

$$r_i \equiv x_i - x_{i-1}, \quad i = 1, 2, \ldots, N, \tag{2.6}$$

whose Jacobian is easily shown to be unity.[5] Then upon substitution from (2.5) and (2.6) in (2.4) we find that (1.1) yields

$$Q_N = \int_0^\infty dr_1 \cdots \int_0^\infty dr_N \exp\left[-\beta \sum_{i=1}^{N} \{(N+1-i)r_i + b/r_i\}\right]. \tag{2.7}$$

We have thus *separated* the N-fold integral into a product of N single integrals each depending on a *different* value of the integer variable

$$n \equiv N + 1 - i \tag{2.8}$$

and can rewrite (2.7) as

$$Q_N = \prod_{n=1}^{N} \int_0^\infty dr \exp[-\beta(nr + b/r)]. \tag{2.9}$$

All expressions will become dimensionless, if one inserts the appropriate dimensional constants.

The one-dimensional integral on the rhs of (2.9) is known[7] in closed form:

$$q_n \equiv \int_0^\infty dr \exp[-\beta(nr + b/r)] = 2(b/n)^{1/2} K_1(2\beta\sqrt{nb}), \tag{2.10}$$

where K_1 is a modified Bessel function of the third kind.[8] Hence the Q_N for the CG system finally becomes

$$Q_N = 2^N (b^N/N!)^{1/2} \prod_{n=1}^{N} K_1(2\beta\sqrt{nb}). \tag{2.11}$$

It clearly follows from the above discussion that the Q_N for a general potential (2.1) can be similarly evaluated if the integral

$$q_n = \int_{r_0}^\infty dr \exp[-\beta(nr + v(r))], \tag{2.12}$$

where r_0 is the hard core diameter,[9] can be found in closed form as an explicit function of n.

A. Thermodynamics of the Coulomb-gravitating system

The full canonical partition function[1] is

$$Z_N = \lambda^{N/2} Q_N, \quad \lambda \equiv 2\pi m/h^2\beta, \tag{2.13}$$

with m the mass of each particle, h Planck's constant, and Q_N given by (2.11). Note that a factor $1/N!$ is not present in (2.13) because we are concerned with *one* specific particle ordering,[1] (2.3). We now compute the free energy

$$F \equiv -kT \log Z_N$$
$$= -kT\left\{\frac{N}{2}(\log 4b\lambda - \log N!) + \sum_{n=1}^{N} \log K_1(2\beta\sqrt{nb})\right\}, \tag{2.14}$$

the internal energy per particle

$$\frac{U}{NkT} \equiv -\frac{1}{NkT}\frac{\partial}{\partial\beta}\log Z_N$$
$$= \frac{3}{2} + \frac{1}{N}\sum_{n=1}^{N} 2\beta\sqrt{nb}\,\frac{K_0(2\beta\sqrt{nb})}{K_1(2\beta\sqrt{nb})}, \tag{2.15}$$

in units of kT, and the specific heat (at constant pressure, $P = 0$)

$$\frac{C}{Nk} \equiv \frac{1}{Nk}\frac{\partial U}{\partial T}$$
$$= \frac{7}{4} + 2b\beta^2(N+1) - \frac{1}{N}\sum_{n=1}^{N}\left\{\frac{1}{2} + 2\beta\sqrt{nb}\,\frac{K_0(2\beta\sqrt{nb})}{K_1(2\beta\sqrt{nb})}\right\}^2, \tag{2.16}$$

which varies between the values 1 at $T = 0$ and $\frac{3}{2}$ at $T = \infty$, in units of Nk. The average length of the "chain" is

$$L \equiv \langle x_N \rangle = Q_N^{-1}\int_0^\infty dx_N \cdots \int_0^{x_3} dx_2 \int_0^{x_2} dx_1\, x_N \exp[-\beta V]. \tag{2.17}$$

We indicate in (A6, A7) how to evaluate this N-fold integral:

$$L = \sqrt{b}\sum_{n=1}^{N}\{K_2(2\beta\sqrt{nb})/\sqrt{n}K_1(2\beta\sqrt{nb})\}. \tag{2.18}$$

The coefficient of thermal expansion, α, is obtained as

$$\alpha \equiv \frac{1}{L}\frac{\partial L}{\partial T}$$
$$= 3k\beta + \frac{2k\beta^2 b}{L}\sum_{n=1}^{N}\left\{1 - \left[\frac{K_2(2\beta\sqrt{nb})}{K_1(2\beta\sqrt{nb})}\right]^2\right\}. \tag{2.19}$$

In deriving the above formulas we make use of the derivative and recursion relations for the K_ν Bessel functions; cf. Ref. 8 Sec. 9.6.26, 27.

B. The $N \to \infty$ limit

In Appendix A we prove that the free energy, *per particle*, of the CG system *diverges* as the number of degrees of freedom, N, tends to infinity! Moreover, we calculate there explicitly that

$$L/N \underset{N\to\infty}{\sim} 2\sqrt{b}/\sqrt{N} \tag{2.20}$$

and also that the coefficient of thermal expansion, α, satisfies

$$\alpha \underset{N\to\infty}{\sim} \log N/N^{1/2}; \tag{2.21}$$

hence L and (αL) are *not* extensive parameters of the system. Equation (2.20) may be explained by noticing, from Appendix A, that our limiting procedure for $N \to \infty$ is equivalent to taking the $\beta \to \infty$, i.e., $T \to 0$, limit of the finite N case. Therefore, as N increases, the distance between the ith and the $(i-1)$th particle, d_i, tends to the (mechanical-) equilibrium value of the separated potential in (2.7),

$$V_i(r) \equiv (N+1-i)r + b/r, \tag{2.22}$$

i.e., $d_i \to (b/N+1-i)^{1/2}$. Summing up the d_i's we also recover (2.20) by this method:

$$L = \sum_{i=1}^{N} d_i \underset{N\to\infty}{\to} \sqrt{b}\sum_{n=1}^{N} n^{-1/2} \underset{N\to\infty}{\sim} 2\sqrt{bN},$$

where we have used definition (2.8) and (A1a).

It is clear from the above considerations that the CG system, in the "bulk"-limit, behaves like a "column of atmosphere" rather than a "chain" model for a one-dimensional solid. Already from the analogy with the well-known problem of the ideal gas "atmosphere," it would appear that *not all* thermodynamic variables of the CG system will be extensive.

Another reason why we might expect such a result in the thermodynamic limit stems from the following argument: It is known[10] that a *sufficient* condition for an N-body system to possess extensive free energy is that its potential energy be both "*stable*" and "*tempered.*" Stability requires that $V(\mathbf{x}_N)/N$ have a finite lower bound which will prevent "implosion," i.e., a "collapse" of the Q_N in phase space as $N\to\infty$. The potential (2.4) is indeed stable in that sense, since it is clearly bounded from below by zero. Temperedness on the other hand (meaning no "explosions" as $N\to\infty$) holds if and only if the *pair* potential, between *any two* particles, satisfies

$$v(r)\le Ar^{-1-\epsilon}, \tag{2.23}$$

for some $A,\epsilon, r_0>0$ and all $r\ge r_0$. The pair potential (2.22) does *violate* inequality (2.23) but acts between nearest neighbors only. This serves as one more motivation for investigating the extensivity properties of the CG system.

There still is an important variable of this system which *is extensive* in the "bulk"-limit, namely, the heat capacity. Since the large N behavior of U/N turns out to be dominated by an additive term which is independent of T,

$$U/N \underset{N\to\infty}{\sim} kT+4\sqrt{bN}/3, \tag{2.24}$$

(cf. Appendix A). it follows that

$$\lim_{N\to\infty}(C/Nk)=1. \tag{2.25}$$

This last result, (2.25), is derived in Appendix A directly from the exact finite N formula for C, (2.16).

3. A "FLUID" WITH EXPONENTIAL INTERACTIONS

The second one dimensional model is a "fluid" with an attractive interaction between *any* two particles ("EAP"), described by the potential

$$V(\mathbf{x}_N)=\tfrac{1}{2}\sum_{\substack{i=0\\(i\ne j)}}^{N}\sum_{j=0}^{N}\exp(|x_i-x_j|)$$
$$=\sum_{i=1}^{N}\sum_{j=0}^{i-1}\exp(x_i-x_j). \tag{3.1}$$

We impose again fixed-free boundary conditions and the ordering of the particles (2.3), as in the previous example. This time the N-fold integral Q_N can be separated into a product of N *similar* integrals by a *nonlinear* transformation

$$z_i\equiv\sum_{j=0}^{i-1}\exp(x_i-x_j),\quad i=1,2,\ldots,N. \tag{3.2}$$

As before, each integral will be a function of a different value of the numbering parameter n. The separation is carried out in detail in Appendix B and yields

$$Q_N=\prod_{n=1}^{N}E_1(n\beta), \tag{3.3}$$

where $E_1(z)$ is the exponential integral

$$E_1(z)\equiv\int_z^{\infty}\frac{e^{-z}}{z}\,dz. \tag{3.4}$$

The special properties, asymptotic formulas, etc. of $E_1(z)$ can be found, for example, in Chap. 5 of Ref. 8.

A. Thermodynamics of the exponential system

The free energy for this system is immediately obtained from Z_N [cf. (2.13) and (2.14)], with Q_N given by (3.3):

$$F=-kT\left\{\frac{N}{2}\log\lambda+\sum_{n=1}^{N}\log E_1(n\beta)\right\}. \tag{3.5}$$

We derive the internal energy and the specific heat for the EAP model

$$U=\frac{NkT}{2}+kT\sum_{n=1}^{N}\frac{\exp(-n\beta)}{E_1(n\beta)} \tag{3.6}$$

and

$$\left[C=\frac{Nk}{2}+k\sum_{n=1}^{N}\left((1+n\beta)\left(\frac{\exp(-n\beta)}{E_1(n\beta)}\right)-\left(\frac{\exp(-2n\beta)}{E_1^2(n\beta)}\right)\right)\right]. \tag{3.7}$$

The low and high temperature limits of C/Nk are $\frac{3}{2}$ and $\frac{1}{2}$, respectively.

B. The $N\to\infty$ limit

Following the discussion in Sec. 2B, we observe that the potential energy of the EAP system, (3.1), although "stable" (bounded below by zero), is certainly *not* "tempered", since $v(r)=e^r$ does not satisfy (2.23). In Appendix B we prove that F/N indeed *diverges* when $N\to\infty$, as suspected from the nontemperedness of the EAP pair interaction. Although there are obvious differences between the two 1D models treated here, it turns out that the large N behavior of their internal energy and specific heat is quite similar! In the EAP system, the divergence of U/N appears again in the form of an additive constant

$$U/N\underset{N\to\infty}{\sim}\tfrac{3}{2}kT+N/2. \tag{3.8}$$

Hence, C/Nk has a finite, nonzero limit

$$\lim_{N\to\infty}(C/Nk)=\tfrac{3}{2}, \tag{3.9}$$

which we derive in Appendix B, starting again from the exact, finite N equation for C, (3.7).

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APPENDIX A: $N\to\infty$ LIMIT AND LENGTH OF CG SYSTEM

The large N divergence of F/N, (2.14), for the Coulomb gravitating system is easy to prove. The crucial step is to realize that for *all* $\beta, b>0$, there exists an integer $n_0\ge 1$, such that $K_1(2\beta\sqrt{nb})<1$, for all $n\ge n_0$; cf. Ref. 8, Fig. 9.7. Hence the last term in

(2.14) adds a positive contribution to F/N, since the first n_0 terms of this sum disappear as $N\to\infty$. We, therefore, conclude by inspection

$$F/N \underset{N\to\infty}{\sim} (kT/2)\log N!$$

In taking the $N\to\infty$ limit of U/N, C_v/Nk, L/N, etc., we will make use of

$$S_{1/2}\equiv N^{-1}\sum_{n=n_0}^{N} n^{-1/2} \underset{N\to\infty}{\sim} 2N^{-1/2}, \quad \therefore S_{1/2}\underset{N\to\infty}{\to} 0, \tag{A1a}$$

$$S_1\equiv N^{-1}\sum_{n=n_0}^{N} n^{-1} \underset{N\to\infty}{\sim} N^{-1}\log N, \quad \therefore S_1\underset{N\to\infty}{\to} 0, \tag{A1b}$$

$$S_p\equiv N^{-1}\sum_{n=n_0}^{N} n^{-p} \underset{N\to\infty}{\sim} N^{-1}, \quad \therefore S_p\underset{N\to\infty}{\to} 0, \tag{A1c}$$

for all $p>1$ and any integer $n_0\geq 1$, and

$$\frac{K_0(z)}{K_1(z)} \underset{z\to\infty}{\sim} 1-\frac{1}{2z}+\frac{3}{8z^2}-\frac{3}{8z^3}, \tag{A1d}$$

$$\frac{K_2(z)}{K_1(z)} \underset{z\to\infty}{\sim} 1+\frac{3}{2z}+\frac{3}{8z^2}-\frac{3}{8z^3}. \tag{A1e}$$

The first three are derived approximating the sums by integrals, while the last two follow directly from the asymptotic expansion of $K_\nu(z)$ (see Ref. 8, 9.7.2).

We then argue as follows: For *all* values of β and b (except zero), there exists some integer $n_0\geq 1$ such that, for all $n\geq n_0$, $z\equiv 2\beta\sqrt{nb}$ is large enough that the Bessel function ratios present in (2.14)–(2.19) can be replaced by their asymptotic expansions (A1d)–(A1e). Thus we obtain for the CG 1D "atmosphere"

$$U/N \underset{N\to\infty}{\sim} kT+(2\sqrt{b}/N)\sum_{n=n_0}^{N}\sqrt{n}+(3/16\beta^2\sqrt{b})S_{1/2} - (3/16\beta^3 b)S_1 \tag{A2}$$

[dimensional constants are left out everywhere, cf. (2.9)] and

$$C/Nk \underset{N\to\infty}{\sim} 1+(3/8\beta\sqrt{b})S_{1/2}-(9/16\beta^2 b)S_1, \tag{A3}$$

where the first n_0 terms of each sum in Eqs. (2.15) and (2.16) have dropped out since they vanish as $N\to\infty$. Noting that

$$N^{-1}\sum_{n=n_0}^{N}\sqrt{n} \underset{N\to\infty}{\sim} 2\sqrt{N}/3, \tag{A4}$$

and taking (A1a)–(A1c) into account, we immediately arrive at the results (2.24) and (2.25).

The length L of the system, defined in (2.17), with V given by (2.4), is calculated as follows:

$$x_N=\sum_{i=1}^{N} r_i, \tag{A5}$$

cf. (2.6). Substituting (A5) in (2.17) one finds that the N-fold integral in the numerator yields a sum of N terms, each term being the product of N one-dimensional integrals. Dividing through by Q_N and after some cancellation one gets

$$L=\sum_{n=1}^{N}(w_n/q_n), \tag{A6}$$

with

$$w_n\equiv\int_0^\infty dr\, r\exp[-\beta(nr+b/r)],$$

q_n given in (2.10), and n defined by (2.8). The integral w_n is also found in the literature

$$w_n=2(b/n)K_2(2\beta\sqrt{nb}); \tag{A7}$$

cf. Ref. 7, 3.471-9, whence Eq. (2.18) readily follows.

In obtaining the large N limit of L/N and α [(2.18), (2.19)] we follow the limiting procedure outlined at the beginning of this appendix. We thus arrive at

$$L/N \underset{N\to\infty}{\sim} \sqrt{b}\{S_{1/2}+(3/4\beta\sqrt{b})S_1\}, \tag{A8}$$

and

$$\alpha \underset{N\to\infty}{\sim} (3k/4\sqrt{b})S_1/[S_{1/2}+(3/4\beta\sqrt{b})S_1]. \tag{A9}$$

Dividing the numerator and denominator in (A9) by $S_{1/2}$ and using (A1a)–(A1c) to find

$$S_1/S_{1/2} \underset{N\to\infty}{\sim} \log N/N^{1/2}, \tag{A10}$$

we finally deduce from (A8)–(A10):

$$L/N\sim 2(b/N)^{1/2} \quad\text{and}\quad \alpha \underset{N\to\infty}{\sim} \log N/N^{1/2}.$$

APPENDIX B: EAP SYSTEM $N\to\infty$ SEPARATION AND LIMIT

The Q_N for the EAP example can be separated into a product of N single integrals by means of the nonlinear transformation (3.2) to z_i variables.

The Jacobian matrix for the *inverse* transformation, with components

$$(\mathbf{J}^{-1})_{ij}\equiv\frac{\partial z_i}{\partial x_j}=\begin{cases}-\exp(x_i-x_j), & j<i,\\ 0, & j>i,\\ z_i, & j=i,\end{cases} \tag{B1}$$

has zeros in *all* entries above the main diagonal. The determinant of such a matrix is equal to the product of its diagonal elements, i.e.,

$$\det(\mathbf{J}^{-1})=\prod_{i=1}^{N} z_i, \tag{B2}$$

whence

$$\det(\mathbf{J})=1/\det(\mathbf{J}^{-1})=1\Big/\prod_{i=1}^{N} z_i. \tag{B3}$$

This is actually the Jacobian determinant we need. Thus, in terms of the new coordinates, the Q_N for the EAP "fluid" becomes

$$Q_N=\int_1^\infty dz_1\cdots\int_i^\infty dz_i\cdots\int_N^\infty dz_N\exp\left(-\beta\sum_{i=1}^{N} z_i\right)\Big/\prod_{i=1}^{N} z_i, \tag{B4}$$

where the lower limit i enters as a result of (3.2) and the fixed-free boundary conditions.[11] Equation (B4) finally yields

$$Q_N=\prod_{n=1}^{N} E_1(n\beta), \tag{B5}$$

where we have replaced the symbol i by n for reasons of consistency and $E_1(z)$ is the exponential integral defined in (3.4).

The behavior of F/N for large N is determined with the aid of the asymptotic formula 5.1.51 of Ref. 8 for $E_1(z)$. Dropping the first n_0 terms of the sum in (3.5) —just as we did for the CG system—and using the Taylor expansion of $\log(1+x)$, we find

$$F/N \underset{N\to\infty}{\sim} N^{-1}\sum_{n=n_0}^{N} n + (kT)^2 S_1 - (kT)^3 S_2, \tag{B6}$$

where S_1 and S_2 have been defined in Appendix A. They are seen to vanish as $N\to\infty$, while the sum in (B6) increases proportionally to N^2 rendering F/N divergent as the particle number approaches infinity.

In order to obtain U/N and C/Nk as $N\to\infty$, we invert the asymptotic expansion mentioned above and obtain

$$\frac{e^{-z}}{E_1(z)} \underset{z\to\infty}{\sim} z + 1 - \frac{1}{z} + \frac{3}{z^2}. \tag{B7}$$

Applying once more the limiting technique of the previous appendix to Eq. (3.6) and (3.7) leads to

$$U/N \underset{N\to\infty}{\sim} \tfrac{3}{2}kT + N/2 - (kT)^2 S_1 + 3(kT)^3 S_2 \tag{B8}$$

and

$$C/Nk \underset{N\to\infty}{\sim} \tfrac{3}{2} - 2kTS_1 + 9(kT)^2 S_2, \tag{B9}$$

where we made use of (B7). Finally, using (A1a)–(A1c) again, results (3.8) and (3.9) follow immediately from (B8) and (B9).

APPENDIX C: PURELY NEAREST NEIGHBOR CHAINS

The Q_N for the clearly separable *fixed-free* systems with potential energy

$$V(\mathbf{x}_N) = \sum_{i=1}^{N} v(|x_i - x_{i-1}|), \tag{C1}$$

is simply

$$Q_N = q^N, \tag{C2}$$

where

$$q \equiv \int_0^\infty \exp[-\beta v(r)]\,dr. \tag{C3}$$

In contrast with the models discussed in Secs. 2 and 3, the thermodynamic variables *per particle* do *not* depend on N and coincide with those of the corresponding one-dimensional subsystem.

Here, we evaluate the Q_N in closed form for two representative examples with anharmonic potentials. The first one is

$$V(\mathbf{x}_N) = \sum_{i=1}^{N} \{|x_i - x_{i-1}|^s + b^s/|x_i - x_{i-1}|^s\}, \tag{C4}$$

for any constants $b, s > 0$. The q integral (C3) for this system is

$$q = \int_0^\infty dr\,\exp[-\beta(r^s + b^s/r^s)] = (2b^{1/2}/s)K_{1/s}(2\beta b^{s/2}), \tag{C5}$$

cf., e.g., Ref. 7, 3.478-4, where $K_{1/s}$ is the Bessel function K_ν, of order $\nu = 1/s$ [for $s=1$, see (2.10) with $n=1$]. The case $s=2$ is the "nearest neighbor" Calogero potential.[12]

From the knowledge of the Q_N, (C2), the C/Nk for arbitrary s is found to vary *monotonically* from 1, at $T=0$, to $\frac{1}{2}+1/s$, at $T=\infty$. Thus, in the case $s=2$

$$C/Nk = 1, \quad \text{for } all\ T. \tag{C6}$$

This is reminiscent of a similar result obtained for the *original* Calogero model[13] and adds to the peculiarities of the $r^2 + b/r^2$ potential.

The second example is

$$V(\mathbf{x}_N) = \sum_{i=1}^{N} \{2\gamma(x_i - x_{i-1})^2 + \tfrac{1}{2}(x_i - x_{i-1})^4\}, \tag{C7}$$

for any real γ. The q integral (C3) is

$$q = \begin{cases} \tfrac{1}{2}\gamma^{1/2}\exp(\gamma^2\beta)K_{1/4}(\gamma^2\beta), & \gamma > 0, \\ (\pi/2\sqrt{2})\,|\gamma|^{1/2}\exp(\gamma^2\beta)[I_{1/4}(\gamma^2\beta) + I_{-1/4}(\gamma^2\beta)], & \gamma < 0, \end{cases} \tag{C8}$$

where $I_{\pm 1/4}$ are modified Bessel functions of the first and second kind. The q integral for $\gamma > 0$ was known before, cf. Ref. 7, 3.323-3. When $\gamma > 0$, (C7) becomes the potential energy of a "hard-core" Fermi–Pasta–Ulam "chain" (FPU),[14] under fixed free boundary conditions. *In the $\gamma < 0$ case the C/Nk, obtained analytically from (C8) and (C2), is not a monotonic function of the temperature.* From the value 1, at $T=0$, it passes through a maximum and a minimum and asymptotically approaches $\frac{3}{4}$ as $T\to\infty$.

[1] E. H. Lieb and D. C. Mattis, *Mathematical Physics in One Dimension* (Academic, New York, 1966). See the Introduction in Chap. 1 for an overview of the subject.

[2] Notably, the 2D Ising model with nonzero magnetic field.

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[6] M. Kac, Phys. Fluids 2, 8 (1959).

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[8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Sec. 9.6.

[9] For $r_0 > 0$, the integral (2.12) can be calculated exactly in some simple cases, e.g., the harmonic $v(r) \propto r^2$ and the logarithmic interaction $v(r) \propto \log r$.

[10] D. Ruelle, *Statistical Mechanics*: *Rigorous Results* (Benjamin, New York, 1969). See the discussion in Chap. 3 leading to Theorem 3.3.12.

[11] The hard core diameter r_0 need not be zero here. For $r_0 > 0$, the lower limit of the ith integral in (B4) becomes $c_i \equiv \sum_{j=1}^{i}\exp(jr_0) > i$. And while the form of the form of the subsequent finite N results will differ slightly from the zero-hard-core case, it can be shown that no significant changes occur in the large N behavior of the system.

[12] F. Calogero first solved exactly the quantum mechanical N-body problem with harmonic plus inverse square pair interactions between *all* particles, cf. F. Calogero, J. Math. Phys. 12, 419 (1971).

[13] See last paragraph in Sec. 2 and Ref. 15, of Calogero's paper, mentioned above.

[14] E. Fermi, J. Pasta, and S. Ulam, p. 978 in *Collected Papers of Enrico Fermi*, *Vol. II* (Univ. of Chicago Press, Chicago, 1965), reprinted in *Nonlinear Wave Motion*, edited by A. C. Newell, Lectures in Appl. Math., Vol. 15 (Am. Math. Soc., Providence, R.I., 1974).

Conformal extensions of the Galilei group and their relation to the Schrödinger group[a)]

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Various authors have considered a conformal extension C_{G0} of the Galilei group which in some sense is the nonrelativistic limit of the conformal extension of the Poincaré group, and have also established an invariance group for the free-particle Schrödinger equation, the "Schrödinger group." Here we establish the most general conformal extension C_G of the Galilei group, which is found to be identical to the group of the most general coordinate transformations that permit the use of noninertial frames of reference and of curvilinear coordinates in Galilei-invariant theories, which was considered by one of us some time ago, and is a gauge group containing a number of arbitrary functions. Both C_{G_0} and the Schrödinger group are subgroups of C_G containing the Galilei group, but otherwise they do not overlap. The Hamilton–Jacobi and Schrödinger equations for particles which are free or interact via inverse-square potentials are shown to be invariant under the Schrödinger group, and a further invariance of the Hamilton–Jacobi equation is established.

I. INTRODUCTION

In 1909, Cunningham and Bateman[1] realized that Maxwell's equations are invariant not only under the 10-parameter Poincaré (= inhomogeneous Lorentz) group, but under the wider 15-parameter conformal group C_P. Since then, conformal invariance has been considered in many areas of physics,[2] and in recent years has found renewed interest in high energy physics.[3]

For our present purposes, the general conformal group C is most concisely defined as the group of all transformations which in any Lorentz space with metric tensor $g_{\mu\nu}$ locally leave the light cone invariant. However, in the following we shall mainly be interested in Minkowski space and its metric tensor $\eta_{\mu\nu}$. The corresponding conformal group[4] C_P is more appropriately called the conformal extension of the Poincaré group; it is briefly discussed in Sec. III.

In connection with the renewed interest in conformal invariance in particle physics, a conformal extension of the Galilei group was considered in a study of Galilei-invariant field theories by Hagen[5] and this group was studied in detail by Roman *et al.*[6] Simultaneously, it was realized by Niederer[7] that the Schrödinger equation for a free particle is invariant under a wider group of transformations (the "Schrödinger group") than the Galilei group, identical with the group considered by Hagen. The relation of this group to the conformal group was studied by Barut[8] and Niederer,[9] both of whom compared the Schrödinger group to the nonrelativistic limit C_{G_0} of the conformal extension of the Poincaré group. A similar study was undertaken for the Hamilton–Jacobi equation by Boyer and Peñafiel.[10]

Our own interest in Galilean analogs to the conformal group C_P arose from a continuing investigation of possible dynamics of interacting particles.[11] In Sec. III we show that if such Galilean analogs are based on the nonrelativistic analog of Eq. (1), a group C_G very much wider than that considered in Refs. 5–10 results, which is identical with a group considered by one of us some time ago in a different context,[12] and is a gauge group containing a number of arbitrary functions. Even if we restrict it further than required by this analogy, we obtain a gauge group which is wider than the Schrödinger group. To obtain these results, it is convenient to use a formalism for the Galilei group introduced earlier,[13,14] which is outlined in Sec. II. Both the latter and C_{G_0} are subgroups of C_G containing the Galilei group, but otherwise they do not overlap. In Sec. IV, we present a simple proof of the invariance of the Hamilton–Jacobi and Schrödinger equations for free particles or particles interacting via inverse-square potentials under the Schrödinger group as well as a further invariance of the Hamilton–Jacobi equation. The relation of our results to previous work is discussed in Sec. V.

II. UNIFIED TREATMENT OF THE POINCARE AND GALILEI GROUPS

We consider the linear group of transformations of the Cartesian space coordinates x^1, x^2, x^3, and the time $t = x^0$

$$x'^{\mu} = \alpha^{\mu}{}_{\rho}\, x^{\rho} + \xi^{\mu}, \tag{1}$$

where the $\alpha^{\mu}{}_{\rho}$'s and ξ^{μ}'s are constant parameters. Here and in the following, summation over repeated indices is understood, Greek indices always range from 0 to 3, and Roman ones from 1 to 3.

The Poincaré group is the group of transformation (1), restricted by the condition[15]

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$$\eta_{\mu\nu}\alpha^{\mu}{}_{\rho}\alpha^{\nu}{}_{\sigma}=\eta_{\rho\sigma}, \tag{2P}$$

where $\eta_{\mu\nu}$ is a nonsingular symmetric tensor with signature -2. We can define its inverse by

$$\eta_{\mu\rho}\eta^{\rho\nu}=\delta_{\mu}^{\nu}. \tag{3P}$$

Equations (2P) and (3P) imply

$$\eta^{\mu\nu}\alpha^{\rho}{}_{\mu}\alpha^{\sigma}{}_{\nu}=\eta^{\rho\sigma}. \tag{4P}$$

We shall take the nonvanishing components of these tensors to be

$$\eta_{00}=1,\quad \eta_{11}=\eta_{22}=\eta_{33}=-c^{-2}, \tag{5P}$$

$$\eta^{00}=1,\quad \eta^{11}=\eta^{22}=\eta^{33}=-c^{2}. \tag{6P}$$

The full inhomogeneous Galilei group is the group of transformations (1), restricted by the conditions[13]

$$g_{\mu\nu}\alpha^{\mu}{}_{\rho}\alpha^{\nu}{}_{\sigma}=g_{\rho\sigma}, \tag{2G}$$

$$h^{\mu\nu}\alpha^{\rho}{}_{\mu}\alpha^{\sigma}{}_{\tau}=h^{\rho\sigma}, \tag{4G}$$

where the tensors $g_{\mu\nu}$ and $h^{\mu\nu}$ are singular; we can choose as their nonvanishing components

$$g_{00}=1, \tag{5G}$$

$$h^{11}=h^{22}=h^{33}=-1, \tag{6G}$$

and thus

$$g_{\mu\rho}h^{\rho\nu}=0. \tag{3G}$$

Clearly $g_{\mu\nu}$ and $h^{\mu\nu}$ are the limits $c\to\infty$ of $\eta_{\mu\nu}$ and $c^{-2}\eta^{\mu\nu}$, respectively; since they are independent, so are the relations (2G) and (4G).[16]

Equations (2), (4)—(6) imply that the Jacobian J of transformation (1) equals ± 1 in both cases. Thus both the Poincaré and the Galilei group consist of four parts, corresponding to the four combinations of the signs of J and of $\alpha^{0}{}_{0}$. The part with $J=\operatorname{sgn}\alpha^{0}{}_{0}=1$ forms a subgroup, the proper orthochronous Poincaré and Galilei group, respectively.

The space of the Poincaré group is metric, with a metric tensor $\eta_{\mu\nu}$, and a four-dimensional infinitesimal distance defined by

$$ds^{2}\equiv\eta_{\mu\nu}\,dx^{\mu}\,dx^{\nu}. \tag{7P}$$

For the Galilei group, we could also introduce such a distance through

$$ds^{2}\equiv g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}. \tag{7G}$$

However, the "metric" $g_{\mu\nu}$ is singular, and thus the space is not Riemannian; the separation (7G) is a pure time interval, and assigns a separation zero to any two simultaneous events.

Unlike $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, $g_{\mu\nu}$, and $h^{\mu\nu}$ cannot be used to lower and raise indices reversibly, and in general co- and contravariant vectors are distinct quantities. Since the Christoffel symbols and the curvature tensor defined from $\eta_{\mu\nu}$ vanish, the metric space characterized by $\eta_{\mu\nu}$ is flat. No analogous statements can be made for the space characterized by $g_{\mu\nu}$; however, if we introduce vanishing affine connections $\Gamma^{\rho}_{\mu\nu}$ by definition, the corresponding curvature tensor also vanishes, and thus this affinely connected space also is flat.

III. CONFORMAL EXTENSIONS OF THE POINCARÉ AND GALILEI GROUPS

The conformal extension C_P of the Poincaré group is the group of all coordinate transformation

$$x'^{\mu}=x'^{\mu}(x^{\rho}) \tag{8}$$

that connect line elements of the form

$$ds^{2}=\phi^{-2}(x^{\rho})\eta_{\mu\nu}\,dx^{\mu}\,dx^{\nu}\equiv\bar{\eta}_{\mu\nu}\,dx^{\mu}\,dx^{\nu} \tag{9P}$$

with each other,[4,17] and thus preserve the light cones $ds^{2}=0$. Clearly, the Poincaré transformations form a subgroup; another subgroup is that of the scale transformations (dilatations)

$$x'^{\mu}=C^{-1}x^{\mu}. \tag{10P}$$

It can be shown that the most general conformal transformation is the product of a Poincaré transformation and a "Haantjes transformation" (product of dilatations and acceleration transformations)

$$x'^{\mu}=\frac{x^{\mu}-C^{-1}l^{\mu}\eta_{\rho\sigma}x^{\rho}x^{\sigma}}{C-2\eta_{\alpha\beta}l^{\alpha}x^{\beta}+C^{-1}\eta_{\gamma\delta}l^{\gamma}l^{\delta}\eta_{\kappa\lambda}x^{\kappa}x^{\lambda}}, \tag{11P}$$

where C and l^{μ} are five arbitrary constant parameters, and thus the conformal extension of the Poincaré group is a 15-parameter group. The Galilean limit of this transformation (the "Galilean Haantjes transformation") is

$$x'^{\mu}=\frac{x^{\mu}-C^{-1}l^{\mu}g_{\rho\sigma}x^{\rho}x^{\sigma}}{C-2g_{\alpha\beta}l^{\alpha}x^{\beta}+C^{-1}g_{\gamma\delta}l^{\gamma}l^{\delta}g_{\kappa\lambda}x^{\kappa}x^{\lambda}}, \tag{11Ga}$$

which from Eq. (5G) is equivalent to

$$t'=\frac{t}{C-l^{0}t},\quad \mathbf{r}'=\frac{C\mathbf{r}-\mathbf{l}t^{2}}{(C-l^{0}t)^{2}}. \tag{11Gb}$$

This set of transformations together with the Galilei transformations forms a 15-parameter group C_{G_0} which has a structure very similar to that of the conformal extension of the Poincaré group, and has therefore been considered occasionally as the appropriate definition of the Galilean conformal group.[8] It is, however, by no means the most general conformal extension of the Galilei group.

Before proceeding with a study of this extension, we note that C_P in the interpretation adopted here[2] is to be understood as a group of transformations on the coordinates, but not on the metric tensor. Therefore, ds^{2} is not an invariant and $\bar{\eta}_{\mu\nu}$ does *not* equal $\eta'_{\mu\nu}$, but instead is given by

$$\bar{\eta}_{\mu\nu}=\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\eta_{\rho\sigma}\equiv\phi^{-2}\eta_{\mu\nu}, \tag{12P}$$

where the factor of $\eta_{\rho\sigma}$ arises from the transformation of $dx^{\mu}\,dx^{\nu}$, i.e., the expression (9P) arises from $\eta_{\mu\nu}\,dx'^{\mu}\,dx'^{\nu}$ rather than $\eta'_{\mu\nu}\,dx'^{\mu}\,dx'^{\nu}$. A similar interpretation must be adopted for the transformations of the conformal extension of the Galilei group.

We can define a "contravariant" $\bar{\eta}^{\mu\nu}$ as the inverse of $\bar{\eta}_{\mu\nu}$ from a relation corresponding to (3P) to obtain

$$\bar{\eta}^{\mu\nu}=\frac{\partial x^{\mu}}{\partial x'^{\rho}}\frac{\partial x^{\nu}}{\partial x'^{\sigma}}\eta^{\rho\sigma}=\phi^{2}\eta^{\mu\nu}, \tag{13P}$$

which does not equal $\eta'^{\mu\nu}$.

To obtain the most general extension of the Galilei group, we proceed directly from the Galilean analog of preservation of light cones. In the limit $c \to \infty$ the light cones $ds^2 = 0$ with ds^2 given by (7P) degenerate to planes of constant time (absolute simultaneity), for which $ds^2 = 0$ with the Galilean ds^2 (7G). The most general transformations C_G maintaining this condition are

$$x'^0 = x'^0(x^0), \quad dx'^0/dx^0 > 0 \text{ for all } x^0, \qquad \text{or } < 0 \text{ for all } x^0, \tag{14Ga}$$

$$x'^m = x'^m(x^\rho). \tag{14Gb}$$

Clearly, these transformations contain both the Galilei group and the group (11G) as special cases, but are much more general.

With the interpretation adopted above, we now have

$$ds^2 = \phi^{-2}(x^\rho) g_{\mu\nu}\, dx^\mu\, dx^\nu \equiv \bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{9G}$$

where

$$\bar{g}_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma} = \phi^{-2} g_{\mu\nu}. \tag{12G}$$

However, if we wish to define a "contravariant" $\bar{h}^{\mu\nu}$ from a relation corresponding to (3G) in analogy to the procedure used above to obtain (13P), we only get

$$\bar{h}^{\mu\nu} = \omega^2(x^\lambda) \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma} h^{\rho\sigma}, \tag{13G}$$

which is not necessarily proportional to $h^{\mu\nu}$ and contains an arbitrary factor $\omega^2(x^\lambda)$ because of the degenerate form of (3G). However, because of the form (14G) of the coordinate transformations we have at least

$$\bar{h}^{\mu 0} = \bar{h}^{0\mu} = 0. \tag{15G}$$

The relations (14G) are precisely those obtained in Ref. 12 as the most general coordinate transformations allowed that permit the use of noninertial frames of reference and curvilinear coordinates without changing the physical content of Galilei-invariant theories. The only restriction on (analytical) coordinate transformations imposed there was the exclusion of coordinate systems for which signals emitted at a time t_0 could arrive at some points of the systems at $t > t_0$ and at others at $t < t_0$.

It should be noted that imposition of the corresponding restriction on coordinate transformations for Poincaré-invariant theories does *not* lead to the conformal extension of the Poincaré group C_P. The condition on the description of signals stated above implies (in addition to preservation of light cones) that the space- or timelike character of separations [i.e., the sign of ds^2 in (9P)] is maintained, a condition not satisfied by the acceleration transformations. This condition leads to a set of restrictions on the transformed metric tensor $\eta'_{\mu\nu}$.[4,18,19] In the Galilei case, no such additional condition is implied, due to the collapse of the cone to a plane.

Some time ago, Zeeman[20] showed that the requirement of preservation of light cones in Minkowski space and of orientation of timelike vectors implies the "causality group," defined as the product of the orthochronous Poincaré group and the dilatation group. This result actually is an immediate consequence of the long-known fact that C_P is the widest group of transformations in Minkowski space which preserves the light cones, but that the subgroups of acceleration transformations and of antichronous Poincaré transformations do not preserve time orientation. A requirement of "causality" for Newtonian space—time analogous to Zeeman's for Minkowski space would demand preservation of absolute simultaneity and of time orientation, and thus the subgroup of orthochronous transformations of C_G defined by (14).

Thus Zeeman's statement "causality implies the Lorentz group" is valid only in Minkowski space; furthermore, as already discussed in Ref. 12 (Footnote (49) in connection with the transformations (14Ga), it is too strong a requirement to demand preservation of time orientation, "since this would assign physical meaning to the obviously conventional orientation of the time axis.... Allowing both signs does not contradict the 'causality condition' that a signal should not arrive earlier than it was emitted, which can be looked upon as a definition either of 'signal' or of 'earlier'." Therefore antichronous transformations need not be excluded, and the physically required causality conditions do not impose any restrictions on C_G, and in the case of C_P only exclude the acceleration transformations and impose the restrictions on $\eta'_{\mu\nu}$ mentioned above.

Because of the difference between the relation (13P) and (13G) there is a clear qualitative difference between the group of transformations C_G allowed by a conformal extension of the Galilei group and the group C_{G_0} obtained as the Galilean limit of G_P. On the other hand, we can subject the transformations of C_G to arbitrary restrictions to achieve a closer similarity to, or even identity with, the group C_{G_0}.

The weakest restriction on the transformations (14G) that reduces Eq. (13G) to a form reminiscent of (13P) is the requirement

$$\bar{h}^{\mu\nu} = \Omega^2(x^\rho) h^{\mu\nu}. \tag{16G}$$

This only restricts the transformations (14Gb), but not (14Ga). From (13G) and (14G) we obtain

$$\frac{\partial x^m}{\partial x'^r} \frac{\partial x^n}{\partial x'^r} = \left(\frac{\Omega}{\omega}\right)^2 \delta^{mn}, \tag{17a}$$

which implies

$$\frac{\partial x'^m}{\partial x^r} \frac{\partial x'^n}{\partial x^r} = \left(\frac{\Omega}{\omega}\right)^{-2} \delta^{mn}; \tag{17Gb}$$

no restrictions are imposed on $\partial x'^m / \partial x^0$.

The most general transformation satisfying the condition (17Gb) is

$$x'^m = F(x^0) \frac{\alpha^m{}_r(x^0) x^r + \alpha^m{}_r(x^0) \epsilon^r(x^0) x^n x^n}{1 + 2\epsilon^l x^l + \epsilon^k \epsilon^k x^l x^l} + \xi^m(x^0), \tag{18Ga}$$

where

$$\alpha^m{}_r \alpha^n{}_r = \delta^{mn}, \tag{18Gb}$$

which together with Eq. (14Ga) defines a group C_{G_1}, and for which

$$\frac{\Omega^2(x^\rho)}{\omega^2(x^\rho)}=\frac{F^2(x^0)}{(1+2\epsilon^l x^l+\epsilon^k\epsilon^k x^l x^l)^2}. \tag{19G}$$

The following distinct subgroups of the transformations (14Ga) and (18G), corresponding to simple forms of the functions F and ξ^m, are easily recognized [where we always first state the transformation (14Ga), written in terms of the time variables, and then the values of some of the functions and parameters appearing in (18Ga); those not specified explicitly are unrestricted constants, and all quantities not given explicitly as functions of t are understood to be constants]:

I. The Galilei group:

$$t'=t+\xi^0;$$

$$F=1,\quad \epsilon^r=0,\quad \xi^m=\alpha^m{}_0 t+\xi_0{}^m.$$

II. The Galilean Haantjes transformation (11G):

$$t'=\frac{t}{C-l^0 t};$$

$$F=\frac{C}{(C-l^0t)^2},\quad \alpha^m{}_r=\delta^m_r,$$

$$\epsilon^r=0,\quad \xi^m=\frac{-l^m t^2}{(C-l^0t)^2}.$$

III. The three-dimensional conformal transformation:

$$t'=t;$$

$$F=1,\quad \alpha^m{}_r=\delta^m_r,\quad \xi^m=0.$$

IV. The "Schrödinger dilatation"[7]:

$$t'=C^{-2}t;$$

$$F=C^{-1},\quad \alpha^m{}_r=\delta^m_r,\quad \epsilon^r=0,\quad \xi^m=0.$$

V. The "Schrödinger expansion"[7]:

$$t'=Ft;$$

$$F=(1-l^0t)^{-1},\quad \alpha^m{}_r=\delta^m_r,\quad \epsilon^r=0,\quad \xi^m=0.$$

Clearly there are many more subgroups. In particular, it should be noted that since any dilatations of x^0 and of the x^m are independent, their ratio is arbitrary, and thus the dilatation subgroup of the Haantjes transformations and the Schrödinger dilatations are only two particular cases of another subgroup of C_G (overlapping the subgroup II and containing IV):

VI. The general dilatations:

$$t'=B^{-1}t;$$

$$F=D^{-1},\quad \alpha^m{}_r=\delta^m_r,\quad \epsilon^r=0,\quad \xi^m=0.$$

We can further restrict our transformations by requiring in Eq. (16G)

$$\Omega^{-1}=\phi(x^0). \tag{20Ga}$$

However, this still leaves an arbitrariness beyond that of C_p because of the presence of the arbitrary function $\omega(x^\rho)$, and indeed imposes no restriction whatever on the transformation (18G). To obtain the full Galilean analogue to Eq. (13P), we must require in addition to (20Ga) that

$$\omega(x^\rho)=1, \tag{20Gb}$$

since these relations imply

$$\bar{h}^{\mu\nu}=\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}h^{\rho\sigma}=\phi^2 h^{\mu\nu}; \tag{21G}$$

the difference between (13P) and (21G) then (apart from the fact that $h^{\mu\nu}$ is degenerate) is that in the Galilean case ϕ can only be a function of x^0 alone.

Therefore Eqs. (20G) and (19G) require that ϵ^r vanishes, and thus the transformations (18G) reduces to

$$x'^m=F(x^0)\alpha^m{}_r(x^0)x^r+\xi^m(x^0). \tag{22G}$$

This is the group of orthogonal coordinate systems undergoing arbitrary accelerations as well as time-dependent dilatations. It, together with the time transformations (14Ga), forms a group C_L, the "Leibniz group" recently discussed by Barbour and Bertotti in a different context.[21] C_L includes the subgroups I, IV, and V listed above, but both the Galilean acceleration transformation and the three-dimensional conformal transformation are excluded. The product of these three subgroups is the Schrödinger group C_s,

$$t'=\frac{t+\bar{\xi}^0}{C^2[1-l^0(t+\bar{\xi}^0)]}$$

$$x'^m=\frac{\alpha^m{}_r x^r+\alpha^m{}_0 t+\bar{\xi}^m_0}{C[1-l^0(t+\bar{\xi}^0)]}, \tag{23G}$$

[where all parameters are constants, and the $\alpha^m{}_i$'s are subject to conditions (18Gb)], which thus is a subgroup both of the conformal extension C_G of the Galilei group and of its subgroup C_L restricted by Eq. (20G). However, it is not a subgroup of the group C_{G_0} discussed above (the product of the Galilei transformation and the Galilean Haantjes transformations). To obtain this group, we can not require condition (20G), but must instead only demand (16G) and restrict the transformation group (18G) to the product of the subgroups I and II listed above.

As noted before, both C_P and C_{G_0} are 15-parameter groups. Since from Eq. (18Gb) only three of the $\alpha^m{}_r$ are independent, the Schrödinger group C_s is a 12-parameter group. On the other hand, the conformal extension C_G of the Galilei group defined by (14G) contains four arbitrary functions $x'^0(x^0)$ and $x'^m(x^\rho)$ and its restricted forms defined by (14Ga) and (16G) or (20G) contain 11 or 8 arbitrary functions of x^0 alone, respectively [x'^0, F, ξ^m, and $\alpha^m{}_r$ in both cases, plus ϵ^r in the case (18G)]; thus they all are gauge groups.

IV. THE INVARIANCE GROUPS OF THE SCHRÖDINGER AND THE HAMILTON-JACOBI EQUATION

As noted in the Introduction, a number of authors have recently investigated the invariance groups of the free-particle Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi=0 \tag{24}$$

and Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\frac{\partial S}{\partial x^r}\frac{\partial S}{\partial x^r}=0. \tag{25}$$

All of these investigations of the Schrödinger equation worked with Eq. (24) and with finite transformations. However, it is much more convenient to work with the variational principle $\delta I = 0$ for Eq. (24), where

$$I = \int \left[\frac{\hbar}{2i} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} \frac{\partial \psi^*}{\partial x^r} \frac{\partial \psi}{\partial x^r} \right] d^4x, \tag{26}$$

and with infinitesimal transformations.

Using the Galilean tensors discussed in Sec. II, Eqs. (25) and (26) can be written

$$\frac{\partial S}{\partial t} - \frac{1}{2m} h^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0 \tag{27}$$

and

$$I = \int \left[\frac{\hbar}{2i} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} h^{\mu\nu} \frac{\partial \psi^*}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} \right] d^4x. \tag{28}$$

Before considering the various conformal extensions of the Galilei group introduced in Sec. III, it will be instructive to consider arbitrary coordinate transformations. Then in Eq. (28) we must also take into account that the integrand must transform as a scalar density rather than a scalar (a distinction which is not relevant for the Galilei group).[22] In a metric space, this is achieved by introducing the square root of the absolute value of the determinant of the metric in the integrand. While the four-dimensional space considered here does not have a nonsingular metric, the general coordinate transformations (14G) still have a nonvanishing Jacobian

$$J \equiv \left| \frac{\partial x'^\mu}{\partial x^\rho} \right| = \frac{dx'^0}{dx^0} \left| \frac{\partial x'^m}{\partial x^r} \right|, \tag{29}$$

and we can use a factor J^{-1} in the integrand to obtain the desired transformation property. This factor equals $[g_{00} |h|^{-1}]^{1/2}$, where

$$h \equiv \det h^{mn}. \tag{30}$$

This (apart from notation) is identical to the standard procedure adopted for obtaining the Schrödinger equation in curvilinear coordinates (where $g_{00} = 1$, and $-h^{mn}$ is the inverse of the metric tensor of 3-space). Thus Eq. (28) is replaced by

$$I = \int \left[\frac{\hbar}{2i} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} h^{\mu\nu} \frac{\partial \psi^*}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} \right] [g_{00} |h|^{-1}]^{1/2} d^4x. \tag{31}$$

As discussed in Sec. III, we consider conformal transformations as transformations on the coordinates alone, but not on the tensors $g_{\mu\nu}$ and $h^{\mu\nu}$, in conformity with the usual interpretation of transformations under the group C_P. Thus we have to investigate whether it is possible to maintain the form of the Hamilton–Jacobi and of the Schrödinger equation under these conditions. Clearly, for the Hamilton–Jacobi equation this will be the case if in Eq. (27) a transformation of the coordinates and of S will yield an equation of the same form, possibly multiplied by an over-all factor. For invariance of the Schrödinger equation following from the variational principle (31), on the other hand, it is necessary that the transformation of the coordinates (including the volume element) and of ψ will leave the integrand of (31) invariant up to a divergence. It should be noted that with the interpretation adopted here the factor $[g_{00} |h|^{-1}]^{1/2}$ does *not* change under conformal transformations, but that d^4x changes to $J\, d^4x$.

This interpretation has no effect on the Galilei invariance of the equations. However, it is clear that the equations are not invariant under the full group C_G which involves the general transformations (14G), or even under the transformations restricted only by the condition (16G) leading to (18Ga). We shall therefore investigate instead the possible invariance under the various subgroups.

We first consider the well-known case of subgroup I, i.e., the behavior of Eqs. (25) or (27) and (28) or (31) under Galilei transformations. Clearly, space and time translations as well as rotations leave them unchanged, with S and ψ transforming as scalars. However, for the Galilei "boosts"

$$t' = t, \quad x'^m = x^m + \epsilon^m t, \tag{32}$$

Eq. (25) becomes

$$\frac{\partial S}{\partial t'} + \epsilon^r \frac{\partial S}{\partial x'^r} + \frac{1}{2m} \frac{\partial S}{\partial x'^r} \frac{\partial S}{\partial x'^r} = 0. \tag{33}$$

This can easily be seen to be of the form (25) in the transformed quantities if we take

$$S' = S + m\epsilon^r x'^r. \tag{34}$$

Applying the transformations (32) to Eq. (28), we obtain

$$I = \int \left[\frac{\hbar}{2i} \left(\psi \frac{\partial \psi^*}{\partial t'} - \psi^* \frac{\partial \psi}{\partial t'} + \psi \epsilon^r \frac{\partial \psi^*}{\partial x'^r} - \psi^* \epsilon^r \frac{\partial \psi}{\partial x'^r} \right) + \frac{\hbar^2}{2m} \frac{\partial \psi^*}{\partial x'^r} \frac{\partial \psi}{\partial x'^r} \right] d^4x'. \tag{35}$$

To establish the invariance of the Schrödinger equation, it is sufficient to establish the invariance of the variational principle under infinitesimal transformations. It can easily be verified that, for infinitesimal ϵ^r, Eq. (35) is of the form (28) in the transformed quantities if we take

$$\psi' = \psi(1 + i\hbar^{-1} m \epsilon^r x'^r), \tag{36}$$

which is the infinitesimal form of multiplication of ψ by a phase factor.

Now we consider the general dilatations VI. Then Eq. (25) becomes

$$\frac{1}{B} \frac{\partial S}{\partial t'} + \frac{1}{2mD^2} \frac{\partial S}{\partial x'^r} \frac{\partial S}{\partial x'^r} = 0. \tag{37}$$

This is of the same form as Eq. (25) provided that we choose

$$S' = BD^{-2} S, \tag{38}$$

and thus the free-particle Hamilton–Jacobi equation is invariant under VI (up to a factor $D^2 B^{-2}$) as well as under its subgroup IV.

For Eq. (31), the transformations VI yield

$$I=\int\left[\frac{\hbar}{2i}\frac{1}{B}\left(\psi\frac{\partial\psi^*}{\partial t'}-\psi^*\frac{\partial\psi}{\partial t'}\right)-\frac{\hbar^2}{2m}\frac{1}{D^2}\frac{\partial\psi^*}{\partial x'^r}\frac{\partial\psi}{\partial x'^r}\right]BD^3d^4x'. \tag{39}$$

This is of the form (26) for the transformed quantities only if

$$B=D^2,\quad \psi'=D^{3/2}\psi, \tag{40}$$

i.e., only for the subgroup IV (with $D\equiv C$) of the transformations VI.[23]

Now we consider the subgroup V. Then Eq. (25) becomes

$$\frac{1}{(1-l^0t)^2}\left(\frac{\partial S}{\partial t'}+l^0x'^r\frac{\partial S}{\partial x'^r}+\frac{1}{2m}\frac{\partial S}{\partial x'^r}\frac{\partial S}{\partial x'^r}\right)=0. \tag{41}$$

It can easily be verified that this reduces to the form (25) apart from an irrelevant over-all factor $(1-l^0t)^{-2}$ provided that we choose

$$S'=S+\frac{ml^0x'^rx'^r}{2(1+l^0t')}. \tag{42}$$

To investigate the invariance of Eq. (31) under the subgroup V, it is simpler to consider only infinitesimal transformations, with $l^0\to\lambda$. Then Eq. (31) is transformed to

$$I=\int\left\{\frac{\hbar}{2i}\left[\left(\psi\frac{\partial\psi^*}{\partial t'}-\psi^*\frac{\partial\psi}{\partial t'}\right)(1+2\lambda t)+\left(\psi^*\frac{\partial\psi}{\partial x'^r}x^r-\psi\frac{\partial\psi^*}{\partial x'^r}\right)x^r\lambda\right]-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x'^r}\frac{\partial\psi}{\partial x'^r}(1+2\lambda t)\right\}(1-5\lambda t)\,d^4x', \tag{43}$$

which is of the form (26) for the transformed quantities if we choose

$$\psi'=\psi\left(1-\tfrac{3}{2}\lambda t-\frac{i\lambda m}{2\hbar}x^rx^r\right)=\psi\left(1-\tfrac{3}{2}\lambda t'-\frac{i\lambda m}{2\hbar}x'^rx'^r\right). \tag{44}$$

Thus the Schrödinger equation is invariant under the Schrödinger group, and the Hamilton–Jacobi equation is invariant under a 13-parameter group, the product of the subgroups I, V, and VI. Neither equation is invariant under subgroups II or III.

Obviously, these statements remain correct if we consider N noninteracting particles instead of just one free particle. In this case, of course, there exist additional transformations that leave the equations invariant which, however, are of no interest for our discussion.

In the presence of interactions Eq. (25) is replaced by

$$\frac{\partial S}{\partial t}+\sum_{k=1}^{N}\frac{1}{2m_k}\frac{\partial S}{\partial x_k{}^r}\frac{\partial S}{\partial x_k{}^r}+V=0, \tag{45}$$

and Eq. (26) by

$$I=\int\left[\frac{\hbar}{2i}\left(\psi\frac{\partial\psi^*}{\partial t}-\psi^*\frac{\partial\psi}{\partial t}\right)-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\frac{\partial\psi^*}{\partial x_k^r}\frac{\partial\psi}{\partial x_k^r}-\psi^*\psi V\right]d^4x, \tag{46}$$

with corresponding changes in the subsequent equations. For interactions of the form

$$V=\tfrac{1}{2}\sum_{k,l=1}^{N}V_{kl}(r_{kl}),\quad r_{kl}\equiv[(x_k^r-x_l^r)(x_k^r-x_l^r)]^{1/2}, \tag{47}$$

these equations are, of course, invariant under the Galilei transformations I regardless of the form of V_{kl}. It can readily be verified that the Hamilton–Jacobi equation remains invariant under the transformations V and VI, and the Schrödinger equation under V and IV, however, only if

$$V_{kl}=C_{kl}r_{kl}^{-2}. \tag{48}$$

The invariance of this particular potential was not recognized in Ref. 7 in which the name "Schrödinger group" was suggested (but was noted later by Burdet and Perrin[24]). On the other hand, it was known to Jacobi[25] that the equations of motion of a Newtonian N-body system with interactions of the form (48) are invariant under the transformations IV and V in addition to those of the Galilei group, and therefore the Schrödinger group should more appropriately be called the Jacobi–Schrödinger group.

V. DISCUSSION

In Sec. III we briefly discussed the conformal extension C_P of the Poincaré group. It can be characterized by a tensor $\bar{\eta}_{\mu\nu}$ related to the Minkowski metric $\eta_{\mu\nu}$ by Eq. (12P); its inverse $\bar{\eta}^{\mu\nu}$ is given by (13P). The transformations of C_P are explicitly given by Eq. (11P), which has the simple Galilean limit C_{G_0} given by Eqs. (11G). Both C_P and C_{G_0} are 15-parameter groups.

However, we can instead define conformal extensions of the Galilei group directly. The most general conformal extension C_G is given by the transformations (14G), for which the tensor $\bar{g}_{\mu\nu}$ is related to the Galilean "metric" $g_{\mu\nu}$ by Eq. (12G), which is analogous to Eq. (13P) and indeed is its Galilean limit. However, since neither $g_{\mu\nu}$ nor $\bar{g}_{\mu\nu}$ possess an inverse, the analog $\bar{h}^{\mu\nu}$ of $h^{\mu\nu}$ requires an independent definition, which is only restricted by the conformal analog of Eq. (3G). The most general $\bar{h}^{\mu\nu}$ allowed by this satisfies Eq. (13G), which is a much less restrictive relation between $\bar{h}^{\mu\nu}$ and $h^{\mu\nu}$ than the corresponding relation (13P) between $\bar{\eta}^{\mu\nu}$ and $\eta^{\mu\nu}$. A relation more closely analogous to (13P) is Eq. (16G), which together with (12G) defines a group C_{G_1} of transformations given by Eqs. (14Ga) and (18G). An even closer analogy with (13P) is obtained by imposing Eq. (21G), which together with (12G) defines a group of transformations C_L given by Eqs. (14Ga) and (22G). From their definitions, C_L is a subgroup of C_{G_1}, which is a subgroup of C_G. All three groups are gauge groups.

If the arbitrary functions in these groups are restricted in various ways, a number of subgroups can be obtained. The most important ones are the 15-parameter group C_{G_0}, which is a subgroup of C_{G_1}, but not of C_L, and the 12-parameter Jacobi–Schrödinger group C_S, which is a subgroup of C_L. Both C_{G_0} and C_S contain the Galilei group as a subgroup, but otherwise they do not overlap.

In Sec. IV we established the invariance of the free-particle Schrödinger equation under C_S by investigating the behavior of the variational principle (26) for this equation under the infinitesimal transformations of C_S.

Unlike other authors,[5-8] we did not have to consider the mass m as a quantity subject to transformations, nor did we have to define new transformations from those of C_{G_0} to absorb a change of mass into the coordinate transformations (as was done in Ref. 8). We also established the invariance of the free-particle Hamilton–Jacobi equation under a 13-parameter group containing C_S.

The invariance of the variational principle (26) under a 12-parameter group, by Noether's theorem, implies the existence of 12 local conservation laws. These will be discussed elsewhere, as will be the corresponding classical laws for the Hamilton–Jacobi equation.[26]

The close correspondence between the Hamilton–Jacobi and the Schrödinger equation has, of course, been known for half a century, and our results further illustrate this correspondence.[27]

The behavior of the Schrödinger equation under arbitrary accelerations, i.e., under the group $C_{L'}$, is more appropriately discussed in connection with a consideration of the equivalence principle, and is the subject of a paper by J. Stachel in preparation.[22,28]

The various extensions of the Galilei group considered here give rise to two-body invariants of importance in a generalized dynamics which will be discussed elsewhere[29] in connection with the two-body invariants of the Galilei group found earlier.[13,14]

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Coping with different languages in the null tetrad formulation of general relativity

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We describe how, in spite of differing conventions regarding spacetime signature, sign of the Riemann tensor, and definition of the Ricci tensor, we were able to construct a precise dictionary relating various notations which are being employed in the null tetrad formulation of general relativity. In addition, we give in appendices the forms assumed by the Newman–Penrose equations and the corresponding abstract structural equations when nontraditional assumptions are made with respect to the three sign conventions.

While I have never worried much about changing conventions, it would appear that many people do, for I am frequency asked "But exactly what does that mean in terms of my favorite language?" The task, of course, would be quite trivial were it not for differing sign conventions, especially with regard to the signature of the metric, for then one could more easily construct a dictionary of notation for the petitioner.

If you tend toward desperation each time you are faced with the desire to translate results from one language to another, where the signatures as well as the notations happen to differ in the two languages, then this paper may bring you some relief.

I. EFFECT OF CHANGE OF SIGNATURE UPON THE NEWMAN-PENROSE VARIABLES

Under a change of spacetime signature the coordinate components of the metric tensor $g_{\alpha\beta}$, the inverse metric tensor $g^{\alpha\beta}$, the Riemann tensor[1] $R_{\alpha\beta\gamma\delta}$ and the curvature scalar R *change sign*, while the coordinate components of the Ricci tensor $R_{\alpha\beta}$ and the traceless part of the Ricci tensor

$$S_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{4}Rg_{\alpha\beta}$$

remain unchanged.

If $(l_\alpha, n_\alpha, m_\alpha, \bar{m}_\alpha)$ constitute a null tetrad, one[2] real null vector, say l_α, *changes sign* under a change of spacetime signature, while the others, n_α, m_α, and $\bar{m}_\alpha$, remain unchanged. On the other hand, l^α remains unchanged, while n^α, m^α and $\bar{m}^\alpha$ *change sign*. One may infer that the Newman—Penrose[3] D, κ, ϵ, and π remain unchanged under a change of spacetime signature, while Δ, δ, $\bar{\delta}$ and all the other spin coefficients *change sign*. These inferences are based upon the observation that those languages which assume signature $+2$ employ $l_{\beta;\alpha}m^\beta$, $l_{\beta;\alpha}n^\beta+m_{\beta;\alpha}\bar{m}^\beta$, $n_{\beta;\alpha}\bar{m}^\beta$, where those languages which assume signature -2 employ $l_{\beta;\alpha}m^\beta$, $l_{\beta;\alpha}n^\beta+\bar{m}_{\beta;\alpha}m^\beta$, and $\bar{m}_{\beta;\alpha}n^\beta$, respectively. As far as the curvature quantities are concerned, Λ, ψ_0, ψ_1, ψ_2, ψ_3 and ψ_4 *change sign*, while all the Φ's remain unchanged. Of course, we must assume here that in the unlikely event Newman and Penrose were actually to change to signature $+2$ they would continue to use the notation $\psi_0, \ldots, \psi_4$ *for the corresponding bivector components* of the Weyl conform tensor, and that they would continue to use Φ_{00}, $\Phi_{01}, \ldots, \Phi_{22}$ *for the corresponding bivector components* of the traceless Ricci part of the Riemann tensor.

It should be pointed out that there are other conventions besides spacetime signature which effect the result. The conventions regarding over-all sign of the Riemann tensor may differ, as may the conventions concerning which indices are contracted in forming the Ricci tensor from the Riemann tensor.

In Table I we identify the conventions currently being employed by various research groups. The spacetime signature will be denoted by $2\varepsilon_1$. Thus, $(+++-)$ corresponds to $\varepsilon_1=+1$, while $(---+)$ corresponds to $\varepsilon_1=-1$. Hence, for example, Eisenhart's convention[4] corresponds to $\varepsilon_1=-1$.

The over-all sign of the Riemann tensor is determined by $\varepsilon_2=\pm 1$, where

$$\xi_{\gamma;\alpha\beta}-\xi_{\gamma;\beta\alpha}=\varepsilon_2 R_{\alpha\beta\gamma\delta}\xi^\delta.$$

For Eisenhart, $\varepsilon_2=-1$.

We shall say $\varepsilon_3=+1$ if the Ricci tensor is formed by contracting over the second and fourth (or first and third) indices of the Riemann tensor. On the other hand, $\varepsilon_3=-1$ if the contraction is over the first and fourth (or second and third) indices. For Eisenhart, $\varepsilon_3=-1$.

While it will not effect the considerations of this paper, it should be remarked that the introduction of the stress tensor obliges one to decide upon the sign ε_4 of $\xi^\mu T_{\mu\nu}\xi^\nu$ for a timelike unit vector ξ^μ and ordinary matter. Once ε_4 is chosen, the relative sign of $T_{\mu\nu}$ and $R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}$ is automatically equal to $-\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. Plebański and I have used $\varepsilon_4=-1$, but it is my intention to switch to $\varepsilon_4=+1$ in the future in order to get complete agreement with Hauser's language.

TABLE I. Values of $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ implicit in four languages currently used by researchers in general relativity.

Language	ε_1	ε_2	ε_3	ε_4
Hauser[5] (IIT)	+	+	+	+
Newman–Penrose[3]	−	−	−	+
Debever[6]	−	−	−	+
Plebański[7]	+	−	−	−

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TABLE II. Notational correspondences which would hold if conventions were identical, i.e., if $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ were the same.

Hauser (IIT)	Newman—Penrose	Debever (original)	Debever (new)	Plebański	Hauser (IIT)	Newman—Penrose	Debever (original)	Debever (new)	Plebański
k_α	l_α	l_α	l_α	$e^3{}_\alpha$	v_k	κ	$-\frac{1}{2}\sigma^2{}_3$	κ	$-\Gamma_{424}$
m_α	n_α	n_α	n_α	$e^4{}_\alpha$	v_m	τ	$-\frac{1}{2}\sigma^2{}_0$	τ	$-\Gamma_{423}$
t_α	m_α	m_α	m_α	$e^1{}_\alpha$	v_t	σ	$\frac{1}{2}\sigma^2{}_2$	σ	$-\Gamma_{422}$
t^*_α	$\bar{m}_\alpha$	$\bar{m}_\alpha$	$\bar{m}_\alpha$	$e^2{}_\alpha$	v_{t^*}	ρ	$\frac{1}{2}\sigma^2{}_1$	ρ	$-\Gamma_{421}$
k^α	l^α	l^α	l^α	$e_4{}^\alpha$	u_k	2ϵ	$-\sigma^3{}_3$	2ϵ	$\Gamma_{124}+\Gamma_{344}$
m^α	n^α	n^α	n^α	$e_3{}^\alpha$	u_m	2γ	$-\sigma^3{}_0$	2γ	$\Gamma_{123}+\Gamma_{343}$
t^α	m^α	m^α	m^α	$e_2{}^\alpha$	u_t	2β	$\sigma^3{}_2$	2β	$\Gamma_{122}+\Gamma_{342}$
$t^{*\alpha}$	$\bar{m}^\alpha$	$\bar{m}^\alpha$	$\bar{m}^\alpha$	$e_1{}^\alpha$	u_{t^*}	2α	$\sigma^3{}_1$	2α	$\Gamma_{121}+\Gamma_{341}$
d_k	D	$,3$	$,1$	$,4$	w_k	π	$-\frac{1}{2}\sigma^1{}_3$	π	$-\Gamma_{314}$
d_m	Δ	$,0$	$,2$	$,3$	w_m	ν	$-\frac{1}{2}\sigma^1{}_0$	ν	$-\Gamma_{313}$
d_t	δ	$-,2$	$,3$	$,2$	w_t	μ	$\frac{1}{2}\sigma^1{}_2$	μ	$-\Gamma_{312}$
d_{t^*}	$\bar{\delta}$	$-,1$	$,4$	$,1$	w_{t^*}	λ	$\frac{1}{2}\sigma^1{}_1$	λ	$-\Gamma_{311}$
P_k	$\epsilon+\bar{\epsilon}$				iQ_k	$\epsilon-\bar{\epsilon}$			
P_m	$\gamma+\bar{\gamma}$				iQ_m	$\gamma-\bar{\gamma}$			
P_t	$\beta+\bar{\alpha}$				iQ_t	$\beta-\bar{\alpha}$			
P_{t^*}	$\alpha+\bar{\beta}$				iQ_{t^*}	$\alpha-\bar{\beta}$			
C_2	$-\psi_0$	$\frac{1}{2}C_{11}$	$-C_{11}$	$\frac{1}{2}C^{(5)}$	S_{kk}	$-2\Phi_{00}$	$E_{1\bar{1}}$	$-2E_{1\bar{1}}$	C_{44}
C_1	ψ_1	$\frac{1}{4}C_{13}$	C_{12}	$\frac{1}{2}C^{(4)}$	S_{kt}	$-2\Phi_{01}$	$-E_{1\bar{3}}$	$-2E_{1\bar{2}}$	C_{42}
C_0	$-\psi_2$	$\frac{1}{2}C_{12}$	$-C_{13}$	$\frac{1}{2}C^{(3)}$	S_{tt}	$-2\Phi_{02}$	$-E_{1\bar{2}}$	$-2E_{1\bar{3}}$	C_{22}
C_{-1}	ψ_3	$\frac{1}{4}C_{23}$	C_{32}	$\frac{1}{2}C^{(2)}$	S_{kt^*}	$-2\Phi_{10}$	$-E_{3\bar{1}}$	$-2E_{2\bar{1}}$	C_{41}
C_{-2}	$-\psi_4$	$\frac{1}{2}C_{22}$	$-C_{33}$	$\frac{1}{2}C^{(1)}$	S_{tt^*}	$-2\Phi_{11}$	$\frac{1}{4}E_{3\bar{3}}$	$-2E_{2\bar{2}}$	C_{21}
					S_{mt}	$-2\Phi_{12}$	$E_{3\bar{2}}$	$-2E_{2\bar{3}}$	C_{32}
R	24Λ	R	R	R	$S_{t^*t^*}$	$-2\Phi_{20}$	$-E_{2\bar{1}}$	$-2E_{3\bar{1}}$	C_{11}
					S_{mt^*}	$-2\Phi_{21}$	$-E_{2\bar{3}}$	$-2E_{3\bar{2}}$	C_{31}
v		$-\frac{1}{2}\sigma^2$	σ_1	$-\Gamma_{42}$	S_{mm}	$-2\Phi_{22}$	$E_{2\bar{2}}$	$-2E_{3\bar{3}}$	C_{33}
u		$-\sigma^3$	$2\sigma_2$	$\Gamma_{12}+\Gamma_{34}$					
w		$-\frac{1}{2}\sigma^1$	σ_3	$-\Gamma_{31}$	k		θ^0	θ_1	e^3
					m		θ^3	θ_2	e^4
B_+		Z^2	Z^3	$(\frac{1}{2}S_{11})$	t		θ^1	θ_3	c^1
B_0		$2Z^3$	$-Z^2$	$(-S_{12})$	t^*		θ^2	θ_4	e^2
B_-		Z^1	Z^1	$(\frac{1}{2}S_{22})$					

Clearly a change of ε_2 effects all curvature quantities, while a change of ε_3 effects just the Ricci tensor and the Ricci scalar. Knowing how all quantities transform under changes of ε_1, ε_2, and ε_3, it is rather easy to re-express the Newman—Penrose equations in terms of general values of ε_1, ε_2, and ε_3. (See Appendix A.) It is accordingly easy to deduce the correspondence between notations both as they would be if all conventions agreed (see Table II) and as they actually are in view of the differing conventions (see Table III). In our tables we compare the Illinois Institute of Technology (IIT) notation, developed by I. Hauser,[5] the Newman—Penrose notation, the notation of R. Debever[6] (which appears to be in a state of flux), and the notation of J. F. Plebański.[7]

II. EFFECT OF CHANGE OF SIGNATURE UPON THE MORE ABSTRACT STRUCTURAL EQUATIONS

We at IIT use $(k^\alpha, m^\alpha, t^\alpha, t^{*\alpha})$ in place of $(l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$. This is not the result of innate perversity, but is simply due to the fact that after a number of years in particle theory my interest in general relativity was rekindled about 1966, when I happened to notice R. P. Kerr's famous $1\frac{1}{2}$ page paper.[8] Sheer curiosity concerning how he got his solution prompted me to set up a tetrad formalism using Kerr's notation and conventions. I also managed to interest I. Hauser in my efforts, and he devised an elaborate null tetrad machinery, most of which has never been published, but which our small relativity group at IIT uses. It was considerably later[9] that we were exposed to the Newman—Penrose approach to null tetrads, specifically when we received a copy of the Ph.D. thesis of W. Kinnersley. Today we are much better acquainted with the literature than we were at the time we were getting started, and perhaps if we were developing the IIT formalism today we would choose to employ $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$. However, as a result of working with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +1$ for so long, we do feel

TABLE III. Notational correspondences which actually hold in view of differing conventions, i.e., differing $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$.

Hauser (IIT)	Newman–Penrose	Debever (original)	Debever (new)	Plebański	Hauser (IIT)	Newman–Penrose	Debever (original)	Debever (new)	Plebański
k_α	$-l_\alpha$	$-l_\alpha$	$-l_\alpha$	$e^3{}_\alpha$	v_k	κ	$-\frac{1}{2}\sigma^2{}_3$	κ	$-\Gamma_{424}$
m_α	n_α	n_α	n_α	$e^4{}_\alpha$	v_m	$-\tau$	$\frac{1}{2}\sigma^2{}_0$	$-\tau$	$-\Gamma_{423}$
t_α	m_α	m_α	m_α	$e^1{}_\alpha$	v_t	$-\sigma$	$-\frac{1}{2}\sigma^2{}_2$	$-\sigma$	$-\Gamma_{422}$
t^*_α	$\overline{m}_\alpha$	$\overline{m}_\alpha$	$\overline{m}_\alpha$	$e^2{}_\alpha$	v_{t^*}	$-\rho$	$-\frac{1}{2}\sigma^2{}_1$	$-\rho$	$-\Gamma_{421}$
k^α	l^α	l^α	l^α	$e_4{}^\alpha$	u_k	2ϵ	$-\sigma^3{}_3$	2ϵ	$\Gamma_{124}+\Gamma_{344}$
m^α	$-n^\alpha$	$-n^\alpha$	$-n^\alpha$	$e_3{}^\alpha$	u_m	-2γ	$\sigma^3{}_0$	-2γ	$\Gamma_{123}+\Gamma_{343}$
t^α	$-m^\alpha$	$-m^\alpha$	$-m^\alpha$	$e_2{}^\alpha$	u_t	-2β	$-\sigma^3{}_2$	-2β	$\Gamma_{122}+\Gamma_{342}$
$t^{*\alpha}$	$-\overline{m}^\alpha$	$-\overline{m}^\alpha$	$-\overline{m}^\alpha$	$e_1{}^\alpha$	u_{t^*}	-2α	$-\sigma^3{}_1$	-2α	$\Gamma_{121}+\Gamma_{341}$
d_k	D	$,3$	$,1$	$,4$	w_k	π	$-\frac{1}{2}\sigma^1{}_3$	π	$-\Gamma_{314}$
d_m	$-\Delta$	$-,0$	$-,2$	$,3$	w_m	$-\nu$	$\frac{1}{2}\sigma^1{}_0$	$-\nu$	$-\Gamma_{313}$
d_t	$-\delta$	$,2$	$-,3$	$,2$	w_t	$-\mu$	$-\frac{1}{2}\sigma^1{}_2$	$-\mu$	$-\Gamma_{312}$
d_{t^*}	$-\bar{\delta}$	$,1$	$-,4$	$,1$	w_{t^*}	$-\lambda$	$-\frac{1}{2}\sigma^1{}_1$	$-\lambda$	$-\Gamma_{311}$
P_k	$\epsilon+\bar{\epsilon}$				iQ_k	$\epsilon-\bar{\epsilon}$			
P_m	$-(\gamma+\bar{\gamma})$				iQ_m	$-(\gamma-\bar{\gamma})$			
P_t	$-(\beta+\bar{\alpha})$				iQ_t	$-(\beta-\bar{\alpha})$			
P_{t^*}	$-(\alpha+\bar{\beta})$				iQ_{t^*}	$-(\alpha-\bar{\beta})$			
C_2	$-\psi_0$	$\frac{1}{2}C_{11}$	$-C_{11}$	$-\frac{1}{2}C^{(5)}$	S_{kk}	$-2\Phi_{00}$	$E_{1\bar{1}}$	$-2E_{1\bar{1}}$	C_{44}
C_1	ψ_1	$\frac{1}{4}C_{13}$	C_{12}	$-\frac{1}{2}C^{(4)}$	S_{kt}	$2\Phi_{01}$	$E_{1\bar{3}}$	$2E_{1\bar{2}}$	C_{42}
C_0	$-\psi_2$	$\frac{1}{2}C_{12}$	$-C_{13}$	$-\frac{1}{2}C^{(3)}$	S_{tt}	$-2\Phi_{02}$	$-E_{1\bar{2}}$	$-2E_{1\bar{3}}$	C_{22}
C_{-1}	ψ_3	$\frac{1}{4}C_{23}$	C_{32}	$-\frac{1}{2}C^{(2)}$	S_{kt^*}	$2\Phi_{10}$	$E_{3\bar{1}}$	$2E_{2\bar{1}}$	C_{41}
C_{-2}	$-\psi_4$	$\frac{1}{2}C_{22}$	$-C_{33}$	$-\frac{1}{2}C^{(1)}$	s_{tt^*}	$-2\Phi_{11}$	$\frac{1}{4}E_{3\bar{3}}$	$-2E_{2\bar{2}}$	C_{21}
					S_{mt}	$-2\Phi_{12}$	$E_{3\bar{2}}$	$-2E_{2\bar{3}}$	C_{32}
R	$-24\ \Lambda$	$-R$	$-R$	R	$S_{t^*t^*}$	$-2\Phi_{20}$	$-E_{2\bar{1}}$	$-2E_{3\bar{1}}$	C_{11}
					S_{mt^*}	$-2\Phi_{21}$	$-E_{2\bar{3}}$	$-2E_{3\bar{2}}$	C_{31}
v		$-\frac{1}{2}\sigma^2$	σ_1	$-\Gamma_{42}$	S_{mm}	$-2\Phi_{22}$	$E_{2\bar{2}}$	$-2E_{3\bar{3}}$	C_{33}
u		$-\sigma^3$	$2\sigma_2$	$\Gamma_{12}+\Gamma_{34}$					
w		$-\frac{1}{2}\sigma^1$	σ_3	$-\Gamma_{31}$	k		$-\theta^0$	$-\theta_1$	e^3
					m		θ^3	θ_2	e^4
B_+		$-Z^2$	$-Z^3$	$(\frac{1}{2}S_{11})$	t		θ^1	θ_3	e^1
B_0		$-2Z^3$	Z^2	$(-S_{12})$	t^*		θ^2	θ_4	e^2
B_-		$-Z^1$	$-Z^1$	$(\frac{1}{2}S_{22})$					

more secure using those conventions. The present paper should facilitate translating our results as well as our equations into other languages.

We shall denote the basic 1-vectors by $(\mathbf{k}, \mathbf{m}, \mathbf{t}, \mathbf{t}^*)$ and the corresponding 1-forms by (k, m, t, t^*). Under a change of signature $\mathbf{k}$ remains unchanged while $\mathbf{m}$, $\mathbf{t}$, and $\mathbf{t}^*$ *change sign*. On the other hand, k *changes sign*, while m, t, and t^* remain unchanged. Because the metric tensor changes sign, any inner product of 1-vectors or 1-forms undergoes *an additional sign change*.

The connection 1-forms,

$$v = d\mathbf{k}\cdot\mathbf{t},\quad u = d\mathbf{k}\cdot\mathbf{m} + d\mathbf{t}\cdot\mathbf{t}^*,\quad w = d\mathbf{m}\cdot\mathbf{t}^*,$$

appropriate for signature $+2$ are transformed into the connection 1-forms,

$$v = d\mathbf{k}\cdot\mathbf{t},\quad u = d\mathbf{k}\cdot\mathbf{m} + d\mathbf{t}^*\cdot\mathbf{t},\quad w = d\mathbf{t}^*\cdot\mathbf{m},$$

appropriate for signature -2, and vice versa. Thus, if you have an explicit expression for v, u, or w in terms of coordinate differentials, you can adapt it without change to the opposite signature.

The null tetrad components of v, u, and w we denote by subscripts. $v_k = \mathbf{k}\cdot v$, $u_k = \mathbf{k}\cdot u$, $w_k = \mathbf{k}\cdot w$ do not change sign under change of signature, but all other null tetrad components of v, u, and w *change sign*. Similarly $d_k = \mathbf{k}\cdot d$ remains unchanged, but $d_m = \mathbf{m}\cdot d$, $d_t = \mathbf{t}\cdot d$, and $d_{t^*} = \mathbf{t}^*\cdot d$ all *change sign*. We expressed all these results earlier in terms of the Newman–Penrose notation.

Under a change of signature the basis[10] for 2-forms,

$$B_+ = k\wedge t,\quad B_0 = k\wedge m + t\wedge t^*,\quad B_- = m\wedge t^*,$$

appropriate for $\varepsilon_1 = +1$ is transformed into the *negative* of the basis for 2-forms.

$$B_+ = k\wedge t,\quad B_0 = k\wedge m + t^*\wedge t,\quad B_- = t^*\wedge m,$$

appropriate for $\varepsilon_1 = -1$. Thus, if you have an explicit expression for B_+, B_0, or B_-, in terms of coordinate differentials, you can adapt it to the opposite signature simply by changing its sign.

At IIT (where $\varepsilon_1=\varepsilon_2=\varepsilon_3=+1$) we write the structural equations, which constitute a more abstract form of the eighteen Newman–Penrose equations, as follows:

$$dv - u\wedge v = C_2B_- + C_2B_0 + (C_0+\tfrac{1}{12}R)B_+ + \tfrac12 S_{kk}B_-^* + \tfrac12 S_{kt}B_0^* + \tfrac12 S_{tt}B_+^*,$$
$$du - 2w\wedge v = -2[C_1B_- + (C_0-\tfrac{1}{24}R)B_0 + C_-B_+ + \tfrac12 S_{kt^*}B_-^* + \tfrac12 S_{tt^*}B_0^* - \tfrac12 S_{mt}B_+^*],$$
$$dw - w\wedge u = (C_0+\tfrac{1}{12}R)B_- + C_{-1}B_0 + C_{-2}B_+ + \tfrac12 S_{t^*t^*}B_-^* - \tfrac12 S_{mt^*}B_0^* + \tfrac12 S_{mm}B_+^*.$$

We have already stated that under a change of signature v, u, and w remain unchanged, while B_+, B_0, and B_- *change sign*. In addition, R, S_{kt}, S_{kt^*}, C_2, C_1, C_0, C_{-1}, and C_{-2} *change sign*, while all other components of the traceless part of the Ricci tensor remain unchanged.[11]

Knowing how all quantities transform under changes of ε_1, ε_2, and ε_3, one may easily deduce the form of the structural equations for arbitrary ε_1, ε_2, and ε_3. (See Appendix B.) In particular, we suggest how the structural equations should be written in order that they be most compatible with the Newman–Penrose notation ($\varepsilon_1=\varepsilon_2=\varepsilon_3=-1$). Since, to our knowledge, Newman and Penrose have not introduced symbols for $(v,\tfrac12 u,w)$ and $(B_+,\tfrac12 B_0,B_-)$ we have employed our symbols[12] there.

We sincerely hope that the reader finds this guide to changing languages informative and useful. We were tempted to include a discussion of the more sophisticated aspects of IIT formalism such as (p,q)-forms in general, the $(2,2)$-forms $\mathbb{R}$ and $\mathbb{D}$ in particular, and the two Grassmann inner products $\ulcorner$ and $\urcorner$, but in the interest of maintaining simplicity of presentation we successfully resisted the temptation.

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APPENDIX A: NEWMAN-PENROSE EQUATIONS FOR ARBITRARY $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

The eighteen Newman–Penrose equations:

(1a) $D\rho-\bar\delta\kappa=-\varepsilon_1(\rho^2+\sigma\bar\sigma)+(\epsilon+\bar\epsilon)\rho-\bar\kappa\tau-\kappa(3\alpha+\bar\beta+\varepsilon_1\pi)-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{00}$,

(1b) $D\sigma-\delta\kappa=-\varepsilon_1(\rho+\bar\rho)\sigma+(3\epsilon-\bar\epsilon)\sigma-(\tau+\varepsilon_1\bar\pi+\bar\alpha+3\beta)\kappa-\varepsilon_2\Psi_0$,

(1c) $D\tau-\Delta\kappa=(-\varepsilon_1\tau+\bar\pi)\rho+(-\varepsilon_1\bar\tau+\pi)\sigma+(\epsilon-\bar\epsilon)\tau-(3\gamma+\bar\gamma)\kappa-\varepsilon_2\Psi_1-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{01}$,

(1d) $D\alpha-\bar\delta\epsilon=(-\varepsilon_1\rho+\bar\epsilon-2\epsilon)\alpha-\varepsilon_1\beta\sigma-\bar\beta\epsilon-\kappa\lambda-\bar\kappa\gamma+(-\varepsilon_1\epsilon+\rho)\pi-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{10}$,

(1e) $D\beta-\delta\epsilon=(-\varepsilon_1\alpha+\pi)\sigma+(-\varepsilon_1\bar\rho-\bar\epsilon)\beta-(\mu+\gamma)\kappa-(\bar\alpha+\varepsilon_1\bar\pi)\epsilon-\varepsilon_2\Psi_1$,

(1f) $D\gamma-\Delta\epsilon=(-\varepsilon_1\tau+\bar\pi)\alpha+(-\varepsilon_1\bar\tau+\pi)\beta-(\epsilon+\bar\epsilon)\gamma-(\gamma+\bar\gamma)\epsilon+\tau\pi-\nu\kappa-\varepsilon_2\psi_2-\varepsilon_2\varepsilon_3\Lambda-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{11}$,

(1g) $D\lambda-\bar\delta\pi=-\varepsilon_1(\rho\lambda+\bar\sigma\mu)-\varepsilon_1\pi^2+(\alpha-\bar\beta)\pi-\nu\bar\kappa-(3\epsilon-\bar\epsilon)\lambda-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{20}$,

(1h) $D\mu-\delta\pi=-\varepsilon_1(\bar\rho\mu+\sigma\lambda)-\varepsilon_1\pi\bar\pi-(\epsilon+\bar\epsilon)\mu-\pi(\bar\alpha-\beta)-\nu k-\varepsilon_2\psi_2+2\varepsilon_2\varepsilon_3\Lambda$,

(1i) $D\nu-\Delta\pi=(\pi-\varepsilon_1\bar\tau)\mu+(\bar\pi-\varepsilon_1\tau)\lambda+(\gamma-\bar\gamma)\pi-(3\epsilon+\bar\epsilon)\nu-\varepsilon_2\psi_3-\varepsilon_1\varepsilon_2\varepsilon_3\Phi_{21}$,

(1j) $\Delta\lambda-\bar\delta\nu=-(\mu+\bar\mu)\lambda-(3\gamma-\bar\gamma)\lambda+(3\alpha+\bar\beta-\varepsilon_1\pi-\bar\tau)\nu-\varepsilon_1\varepsilon_2\psi_4$,

(1k) $\delta\rho-\bar\delta\sigma=\rho(\bar\alpha+\beta)-\sigma(3\alpha-\bar\beta)+(\rho-\bar\rho)\tau-\varepsilon_1(\mu-\bar\mu)\kappa-\varepsilon_1\varepsilon_2\psi_1+\varepsilon_2\varepsilon_3\Phi_{01}$,

(1l) $\delta\alpha-\bar\delta\beta=(\mu\rho-\lambda\sigma)+\alpha\bar\alpha+\beta\bar\beta-2\alpha\beta+\gamma(\rho-\bar\rho)-\varepsilon_1\epsilon(\mu-\bar\mu)-\varepsilon_1\varepsilon_2\psi_2-\varepsilon_1\varepsilon_2\varepsilon_3\Lambda+\varepsilon_2\varepsilon_3\Phi_{11}$,

(1m) $\delta\lambda-\bar\delta\mu=(\rho-\bar\rho)\nu-\varepsilon_1(\mu-\bar\mu)\pi+\mu(\alpha+\bar\beta)+\lambda(\bar\alpha-3\beta)-\varepsilon_1\varepsilon_2\psi_3+\varepsilon_2\varepsilon_3\Phi_{21}$,

(1n) $\delta\nu-\Delta\mu=(\mu^2+\lambda\bar\lambda)+(\gamma+\bar\gamma)\mu+\varepsilon_1\bar\nu\pi+(\tau-3\beta-\bar\alpha)\nu+\varepsilon_2\varepsilon_3\Phi_{22}$,

(1o) $\delta\gamma-\Delta\beta=(\tau-\bar\alpha-\beta)\gamma+\mu\tau-\sigma\nu+\varepsilon_1\epsilon\bar\nu-\beta(\gamma-\bar\gamma-\mu)+\alpha\bar\lambda+\varepsilon_2\varepsilon_3\Phi_{12}$,

(1p) $\delta\tau-\Delta\sigma=(\mu\sigma+\bar\lambda\rho)+(\tau+\beta-\bar\alpha)\tau-(3\gamma-\bar\gamma)\sigma+\varepsilon_1\kappa\bar\nu+\varepsilon_2\varepsilon_3\Phi_{02}$,

(1q) $\Delta\rho-\bar\delta\tau=-(\rho\bar\mu+\sigma\lambda)+(\bar\beta-\alpha-\bar\tau)\tau+(\gamma+\bar\gamma)\rho-\varepsilon_1\nu\kappa-\varepsilon_1\varepsilon_2\psi_2+2\varepsilon_1\varepsilon_2\varepsilon_3\Lambda$,

(1r) $\Delta\alpha-\bar\delta\gamma=(\rho-\varepsilon_1\epsilon)\nu-(\tau+\beta)\lambda+(\bar\gamma-\bar\mu)\alpha+(\bar\beta-\bar\tau)\gamma-\varepsilon_1\varepsilon_2\psi_3$.

Commutation relations:

(2a) $(\Delta D-D\Delta)\phi=[(\gamma+\bar\gamma)D+(\epsilon+\bar\epsilon)\Delta-(-\varepsilon_1\tau+\bar\pi)\bar\delta-(-\varepsilon_1\bar\tau+\pi)\delta]\phi$,

(2b) $(\delta D-D\delta)\phi=[(\bar\alpha+\beta+\varepsilon_1\bar\pi)D+\kappa\Delta+\varepsilon_1\sigma\bar\delta-(-\varepsilon_1\bar\rho+\epsilon-\bar\epsilon)\delta]\phi$,

(2c) $(\delta\Delta-\Delta\delta)\phi=[\varepsilon_1\nu D+(\tau-\bar\alpha-\beta)\Delta+\bar\lambda\bar\delta+(\mu-\gamma+\bar\gamma)\delta]\phi$,

(2d) $(\bar\delta\delta-\delta\bar\delta)\phi=[-\varepsilon_1(\bar\mu-\mu)D+(\bar\rho-\rho)\Delta-(\bar\alpha-\beta)\bar\delta-(\bar\beta-\alpha)\delta]\phi$.

Vacuum Bianchi identities:

(3a) $D\psi_1+\varepsilon_1\bar\delta\psi_0=-3\kappa\psi_2+(2\epsilon-4\varepsilon_1\rho)\psi_1-(-\pi-4\varepsilon_1\alpha)\psi_0$,

(3b) $D\psi_2+\varepsilon_1\bar\delta\psi_1=-2\kappa\psi_3-3\varepsilon_1\rho\psi_2-(-2\pi-2\varepsilon_1\alpha)\psi_1+\varepsilon_1\lambda\psi_0$,

(3c) $D\psi_3+\varepsilon_1\bar\delta\psi_2=-\kappa\psi_4-(2\epsilon+2\varepsilon_1\rho)\psi_3+3\pi\psi_2+2\varepsilon_1\lambda\psi_1$,

(3d) $D\psi_4+\varepsilon_1\bar\delta\psi_3=-(4\epsilon+\varepsilon_1\rho)\psi_4+(4\pi-2\varepsilon_1\alpha)\psi_3+3\varepsilon_1\lambda\psi_2$,

(3e) $\Delta\psi_0-\delta\psi_1=(4\gamma-\mu)\psi_0-(4\tau+2\beta)\psi_1+3\sigma\psi_2$,

(3f) $\Delta\psi_1-\delta\psi_2=\nu\psi_0+(2\gamma-2\mu)\psi_1-3\tau\psi_2+2\sigma\psi_3$, (3g) $\Delta\psi_2 - \delta\psi_3 = 2\nu\psi_1 - 3\mu\psi_2 + (-2\tau + 2\beta)\psi_3 + \sigma\psi_4$,

(3h) $\Delta\psi_3 - \delta\psi_4 = 3\nu\psi_2 - (2\gamma + 4\mu)\psi_3 + (-\tau + 4\beta)\psi_4$.

APPENDIX B: STRUCTURAL EQUATIONS FOR ARBITRARY $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

Differentials of 1-forms:

(1a) $dk = P\wedge k + \varepsilon_1 v^*\wedge t + \varepsilon_1 v\wedge t^*$,

(1b) $dm = -P\wedge m + w\wedge t + w^*\wedge t^*$,

(1c) $dt = -\varepsilon_1 w\wedge k - v\wedge m + iQ\wedge t$,

$$P = (u + u^*)/2, \quad Q = (u - u^*)/2i.$$

Definitions of 2-forms:

(2) $B_+ = k\wedge t, \quad B_0 = k\wedge m + \varepsilon_1 t\wedge t^*, \quad B_- = \varepsilon_1 m\wedge t^*$.

Differentials of 2-forms:

(3a) $dB_+ = u\wedge B_+ - v\wedge B_0$,

(3b) $dB_0 = 2w\wedge B_+ - 2v\wedge B_-$,

(3c) $dB_- = w\wedge B_0 - u\wedge B_-$.

2-form analog of 18-Newman–Penrose equations:

(4a) $dv - u\wedge v = \varepsilon_2(C_2B_- + C_1B_0 + C_0B_+) - \frac{1}{12}\varepsilon_2\varepsilon_3 RB_+ + \frac{1}{2}\varepsilon_1\varepsilon_2\varepsilon_3(S_{kk}B_-^* + \varepsilon_1 S_{kt}B_0^* + S_{tt}B_+^*)$,

(4b) $du - 2w\wedge v = -2\varepsilon_2(C_1B_- + C_0B_0 + C_{-1}B_+) - \frac{1}{12}\varepsilon_2\varepsilon_3 RB_0 + \frac{1}{2}\varepsilon_1\varepsilon_2\varepsilon_3(\varepsilon_1 S_{kt^*}B_-^* + S_{tt^*}B_0^* - S_{mt}B_+^*)$,

(4c) $dw - w\wedge u = \varepsilon_2(C_0B_- + C_{-1}B_0 + C_{-2}B_+) - \frac{1}{12}\varepsilon_2\varepsilon_3 RB_- + \frac{1}{2}\varepsilon_1\varepsilon_2\varepsilon_3(S_{t^*t^*}B_-^* - S_{mt^*}B_0^* + S_{mm}B_t^*)$.

Thus, for example, if one desires to write the structural equations in a form most compatible with the traditional Newman–Penrose equations $(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1)$, one should write the following:

(5a) $dl = P\wedge l - \bar{v}\wedge m - v\wedge\bar{m}$,

(5b) $dn = -P\wedge n + w\wedge m + \bar{w}\wedge\bar{m}$,

(5c) $dm = \bar{w}\wedge l - v\wedge n + iQ\wedge m$,

(6) $B_+ = l\wedge m, \quad B_0 = l\wedge n + \bar{m}\wedge m, \quad B_- = \bar{m}\wedge n$,

(7a) $v = \kappa n + \tau l - \sigma\bar{m} - \rho m$,

(7b) $\frac{1}{2}u = \epsilon n + \gamma l - \beta\bar{m} - \alpha m$,

(7c) $w = \pi n + \nu l - \mu\bar{m} - \lambda m$,

(8a) $dv - u\wedge v = \psi_0 B_- - \psi_1 B_0 + (\psi_2 - 2\Lambda)B_+ + \Phi_{00}\bar{B}_- - \Phi_{01}\bar{B}_0 + \Phi_{02}\bar{B}_+$,

(8b) $\frac{1}{2}(du - w\wedge v) = \psi_1 B_- - (\psi_2 + \Lambda)B_0 + \psi_3 B_+ - \Phi_{10}\bar{B}_- + \Phi_{11}\bar{B}_0 - \Phi_{12}\bar{B}_-$

(8c) $dw - w\wedge u = (\psi_2 - 2\Lambda)B_- - \psi_3 B_0 + \psi_4 B_+ + \Phi_{20}\bar{B}_- - \Phi_{21}\bar{B}_0 + \Phi_{22}\bar{B}_-$.

Probably one should select NP-type names for the basic 2-forms $(B_+, \frac{1}{2}B_0, B_-)$ and the connection 1-forms $(v, \frac{1}{2}u, w)$, but I shall leave that choice up to Newman and Penrose.

[1]All the languages with which we are concerned regard $R_{\alpha\beta\gamma\delta}$ as skew-symmetric in α and β as well as in γ and δ.

[2]It is convenient to select l_α in order to preserve the form of $D = l^\alpha\partial_\alpha$.

[3]E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962); **4**, 998 (1963). In determining $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ we took note of Sec. II and Eqs. (2.7) and (2.8).

[4]L.P. Eisenhart, *Riemannian Geometry* (Princeton U.P., Princeton, N.J., 1926).

[5]I. Hauser and R.J. Malhiot, J. Math. Phys. **15**, 816 (1974); **16**, 150, 1625 (1975). Our identifications of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 may be based upon Sec. 1 of the last paper, Footnote 10 of the second paper, and Eq. (39) of the first paper.

[6]R. Debever, Cah. Phys. **168–169**, 303 (1964). In determining $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 we took note of Sec. I.1, Eq. (5.6), and Sec. I.6. Debever's notation is currently in a state of flux! Compare, for example, Bull. Cl. Sci. Acad. Roy. Belg. **60**, 998 (1974).

[7]J.F. Plebański, *Spinors, Tetrads and Forms*, a proto-book representing lecture notes from a course on advanced relativity given at the Centro de Investigación y de Estudios Avanzados del IPN (México), 1974. In determining $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 we took note of Sec. I.1 and Eqs. (V.2.21), (V.2.12), (II.1.11), and (II.1.12). The reader should note that at an earlier time Plebański employed signature -2, but currently uses signature $+2$.

[8]R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[9]I thank R. Isaacson for suggesting that I look into the relationship between the IIT language and the Newman–Penrose language when he and I were colleagues at IIT.

[10]At IIT we generally suppress the symbol $\wedge$ between differential forms.

[11]We would continue to use $C_2, \ldots, C_{-2}$ for the corresponding bivector components of the Weyl conform tensor if we were to switch signature.

[12]Our symbols generally have subscripts associated with spin weight. However, the symbols (v, u, w) are used in order to avoid having two different types of subscripts on spin-coefficients.

Analytic continuation of an operator-valued H-function with applications to neutron transport theory

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An operator-valued generalization of Chandrasekhar's H-function satisfies a nonlinear integral equation. A bifurcation analysis of this equation gives an analytic continuation of the H-function. This result is applied to a criticality problem in neutron transport theory, and asymptotic results are obtained.

I. INTRODUCTION

Operator values analogs of the Chandrasekhar H-function[1] have been discussed by Mullikin[2] and Kelley.[3] The H operators in Ref. 3 satisfy a coupled system of nonlinear equations:

$$\begin{pmatrix} H_l(\mu,\zeta) & 0 \\ 0 & H_r'(\mu,\zeta)\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I'\end{pmatrix} + \mu\zeta\int_0^1 \begin{pmatrix} D(\nu) & 0 \\ 0 & D'(\nu)\end{pmatrix}\begin{pmatrix} H_r(\nu,\zeta) & 0 \\ 0 & H_l'(\nu,\zeta)\end{pmatrix} \times \frac{d\nu}{\mu+\nu}\begin{pmatrix} H_l(\mu,\zeta) & 0 \\ 0 & H_r'(\mu,\zeta)\end{pmatrix}. \qquad (1.1)$$

In (1.1), H and D are operator valued and Bochner-intergrable on $[0,1]$, the prime denotes Banach space adjoint, and ζ is a complex parameter. It was shown in Ref. 3 that if

$$|\zeta| < \tfrac{1}{2}(\|\int_0^1 D(\nu)\,d\nu\|)^{-1}, \qquad (1.2)$$

the system (1.1) can be solved by an interative scheme. In the scalar[4] and matrix[5-7] cases, it has been shown, under certain positivity assumptions on D, that the point

$$\zeta = 2\|\int_0^1 D(\nu)\,d\nu\|_{sp} \qquad (1.3)$$

is a branch point of order two for Eq. (1.1). In (1.3), $\|\circ\|_{sp}$ denotes spectral radius. In this paper as in Refs. 4 and 5, we normalize $\|\int_0^1 D(\nu)\,d\nu\|_{sp}$ to be $\frac{1}{2}$. The branch point for (1.1) then is at $\zeta=1$. This gives an analytic continuation of H past the point $\zeta=1$. In the scalar case this result has been applied to probability by Mullikin,[4] and in the matrix case to criticality problems in neutron transport theory by Victory,[6] Mullikin and Victory,[5] and Bowden, Greenberg, and Zweifel.[8]

In the present paper we show that, under certain positivity and continuity assumptions on $D(\nu)$, $\zeta=1$ is a branch point of order two for (1.1). This result allows one to continue the H-operators analytically to a cut neighborhood of $\zeta=1$. As an application we indicate how the criticality results[5,6] may be generalized.

In this paper, for $\mathcal{B}$ a Banach space, $\mathcal{L}(\mathcal{B})$ (resp. $\mathrm{Com}(\mathcal{B})$) will denote the spaces of bounded (resp. compact) linear maps on $\mathcal{B}$. $\mathbb{B}_p((a,b),\mathcal{B})$ will denote the space of Bochner p-integrable functions[9] on (a,b) having values in $\mathcal{B}$. $\mathcal{B}^2$ will denote the space of two vectors having components in $\mathcal{B}$. $\mathbb{R}$ (resp. $\mathbb{C}$) will denote the real (resp. complex) numbers. θ will denote the Heaviside function.

II. A BIFURCATION ANALYSIS OF EQ. (1.1)

Let $E_0 < E_1$ be real. Let $\mathcal{N}$ be the Banach algebra of integral operators on $\mathcal{C} = \mathcal{C}([E_0, E_1])$ with continuous kernels with the following norm: for $A\in\mathcal{N}$, let $k_A(E,E')$ be the associated kernel, define

$$\|A\|_{\mathcal{N}} = \max_{E,E'}|k_A(E,E')|. \qquad (2.1)$$

The multiplication on $\mathcal{N}$ is given by

$$k_{AB}(E,E') = \int_{E_0}^{E_1} k_A(E,E'')k_B(E'',E')\,dE'' \qquad (2.2)$$

For $A\in\mathcal{N}$, define A' by

$$k_{A'}(E,E') = k_A(E',E) \qquad (2.3)$$

We note that $(AB)' = B'A'$.

Note also that A' is not the operator adjoint of A; it is that integral operator defined on $\mathcal{C}$ having as its kernel the transpose kernel of A.

Let $\mathcal{N}_0$ be the algebra formed by adjoining the identity to $\mathcal{N}$. We define $I'_{\mathcal{N}_0} = I_{\mathcal{N}_0}$.

Let $N_0(N)$ be the algebra of 2×2 diagonal matrices of the form

$$A = \begin{pmatrix} A_l & 0 \\ 0 & A_r'\end{pmatrix}, \quad A_l, A_r \in \mathcal{N}_0(\mathcal{N}). \qquad (2.4)$$

We define

$$\|A\|_N = \max(\|A_l\|_{\mathcal{N}}, \|A_r\|_{\mathcal{N}}), \qquad (2.5)$$

$$AB = \begin{pmatrix} A_l E_l & 0 \\ 0 & A_r' B_r'\end{pmatrix}, \qquad (2.6)$$

$$I = \begin{pmatrix} I_{\mathcal{N}_0} & 0 \\ 0 & I_{\mathcal{N}_0}\end{pmatrix}, \qquad (2.7)$$

and

$$A^* = \begin{pmatrix} A_r & 0 \\ 0 & A_l'\end{pmatrix} \qquad (2.8)$$

We note $(AB)^* = B^*A^*$.

For $A\in N$, let $\mathbf{k}_A$ be the 2-vector of continuous functions

$$\begin{pmatrix} k_{A_l}(E,E') \\ k_{A_r}(E',E)\end{pmatrix}.$$

If $\mathbf{g} = \binom{g_1}{g_2}$ and $\mathbf{f} = \binom{f_1}{f_2}$ are 2-vectors of continuous functions, we say $g \geq f$ if $g_j \geq f_j$, $j=1,2$. We note that elements of N may be considered as operators on the Banach space $\mathcal{C}^2$ of 2-vectors of continuous functions on $[E_0, E_1]$. We write

$$\|A\|_{L(C^2)} \overset{\text{def}}{=} \|A\|_2. \tag{2.9}$$

Note that

$$\|A\|_N \geq (E_1 - E_0)^{-1}\|A\|_2. \tag{2.10}$$

The norm on $\mathcal{C}^2$ is given by

$$\|\binom{u}{v}\|_{\mathcal{C}^2} = \max(\|u\|_{\mathcal{C}}, \|v\|_{\mathcal{C}}).$$

We consider the algebra $\mathcal{X} = \mathbb{B}_1(\mathbb{R}; N)$ with multiplication and norm given by

$$T * S(x) = \int_{-\infty}^{\infty} T(x-y)S(y)\,dy, \tag{2.11}$$

$$\|T\|_{\mathcal{X}} = \int_{-\infty}^{\infty} \|T(x)\|_N\,dx. \tag{2.12}$$

In $\mathcal{X}$ we may define a Fourier transform by

$$\hat{T}(\lambda) = \int_{-\infty}^{\infty} T(x)\exp(i\lambda x)\,dx. \tag{2.13}$$

We have

$$\widehat{T * S(\lambda)} = \hat{T}(\lambda)\hat{S}(\lambda). \tag{2.14}$$

We define a projector $\mathcal{P}$ by

$$\mathcal{P}T(x) = T(x)\theta(x), \tag{2.15}$$

and let $\mathbb{P}\mathcal{T}$ and $\mathcal{T}$ be defined in $\mathcal{L}(\mathcal{P}\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ by

$$(\mathbb{P}\mathcal{T})S(x) = \mathcal{P}\int_0^{\infty} T(x-y)S(y)\,dy,$$
$$\mathcal{T}S(x) = \int_{-\infty}^{\infty} T(x-y)S(y)\,dy. \tag{2.16}$$

We have

$$\|\mathbb{P}\mathcal{T}\|_{\text{sp}} \leq \|\mathbb{P}\mathcal{T}\|_{\mathcal{X}(\mathcal{P}\mathcal{X})} \leq \|\mathcal{T}\|_{L(\mathcal{X})} \leq \int_{-\infty}^{\infty} \|T(x)\|_2\,dx. \tag{2.17}$$

If $\|\mathbb{P}\mathcal{T}\|_{\text{sp}} \leq 1$, as an operator on $\mathcal{P}\mathcal{X}$, then for each ζ, $|\zeta| < 1$, we can find a unique $\Gamma \in \mathcal{P}\mathcal{X}$ so that

$$\Gamma(x,\zeta) - \zeta\int_0^{\infty} T(x-y)\Gamma(y,\zeta)\,dy = \zeta T(x), \quad x > 0. \tag{2.18}$$

We define

$$\mathcal{H}(\lambda,\zeta) = I + \hat{\Gamma}^*(\lambda,\zeta). \tag{2.19}$$

Then $\mathcal{H}$ is an analytic, N_0-valued function of λ for $\mathrm{Im}\lambda > 0$, and continuous for $\mathrm{Im}\lambda \geq 0$.

We have, as in Ref. 3,

Theorem 2.1: Let $T \in \mathcal{X}$. Assume that

(i) $\|\mathbb{P}\mathcal{T}\|_{\text{sp}} \leq 1$,

(ii) $T^*(-x) = T(x)$.

Then for $\lambda \in \mathbf{R}$, $|\zeta| < 1$, we have

$$[I - \zeta\hat{T}(\lambda)]\mathcal{H}^*(\lambda,\zeta)\mathcal{H}(-\lambda,\zeta) = I. \tag{2.20}$$

Now let $C \in \mathcal{N}$ be such that k_C is strictly positive. Let $\sigma \in \mathcal{C}$ be bounded from below by 1. For $\phi \in \mathcal{C}$ we define a multiplication operator $\Sigma \in \mathcal{L}(\mathcal{C})$ by

$$\Sigma\phi(E) = \sigma(E)\phi(E). \tag{2.21}$$

We assume that

$$\|\Sigma^{-1}C\|_{\text{sp}} = \tfrac{1}{2}. \tag{2.22}$$

Note that (2.22) implies that $\|\mathbb{P}\mathcal{T}\|_{\text{sp}} = 1$ as an operator on $\mathcal{P}\mathcal{X}$. We define an operator-valued function $D(\nu)$ by

$$D(\nu)\phi(E) = \int_{E_0}^{E_1} k_C(E,E')\theta(\sigma(E')^{-1} - \nu)\phi(E')\,dE' \tag{2.23}$$

and an $\mathcal{N}$-valued function K by

$$K(x) = \begin{cases} \int_0^1 \exp(-|x|/\nu)D(\nu)(d\nu/\nu), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{2.24}$$

We set

$$T(x) = \begin{pmatrix} K(x) & 0 \\ 0 & K'(x) \end{pmatrix}.$$

As in Ref. 3, $T \in \mathcal{X}$, $T^*(x) = T(x) = T(-x)$, and $\|\mathbb{P}\mathcal{T}\|_{\text{sp}} = 1$.

Let $H(\mu,\zeta)$ and $\mathbf{D}(\mu)$ be given by

$$H(\mu,\zeta) = \mathcal{H}(i/\mu,\zeta) \tag{2.25}$$

$$\mathbf{D}(\mu) = \begin{pmatrix} D(\mu) & 0 \\ 0 & D'(\mu) \end{pmatrix}, \quad 0 \leq \mu \leq 1.$$

As in Ref. 3, we obtain

Theorem 2.2: $H(\mu,\zeta)$ is a continuous N_0-valued function of μ for $0 \leq \mu \leq 1$; it is a meromorphic N_0-valued function of μ for $\mu \in C/[-1,0]$, and analytic for $\mathrm{Re}\mu > 0$. Moreover,

$$H(\mu,\zeta) = I + \mu\zeta\int_0^1 \mathbf{D}(\nu)H^*(\nu,\zeta)\frac{d\nu}{\mu+\nu}H(\mu,\zeta). \tag{2.26}$$

As H commutes with itself and the identity, we may also write Eq. (2.26) as

$$H(\mu,\zeta) = I + \mu\zeta H(\mu,\zeta)\int_0^1 \mathbf{D}(\nu)H^*(\nu,\zeta)\frac{d\nu}{\mu+\nu}. \tag{2.26'}$$

Theorem 2.2 implies that $k_{H(\mu,\zeta)-I}(E,E')$ is a continuous $\mathbb{R}^2$-valued function of (μ,E,E') for $0 \leq |\zeta| < 1$. As in Ref. 7, if $0 \leq \zeta < \eta < 1$, we have

$$0 \leq k_{H(\mu,\zeta)-I}(E,E') \leq k_{H(\mu,\eta)-I}(E,E'). \tag{2.27}$$

We require

Theorem 2.3: The limit

$$\lim_{\zeta\to1^-} H(\mu,\zeta) = H(\mu) \tag{2.28}$$

exists in N_0, uniformly in μ, and therefore

$$k_{H(\mu)-I}(E,E') \in \mathcal{C}([0,1]\times[E_0,E_1]\times[E_0,E_1])^2.$$

Proof: As in Ref. 7 the assumptions on C and Σ imply that there is $\mathbf{u} = \binom{u_r}{u_l} > 0$, so that u_r and u_l are in $\mathcal{C}$ and

$$[I - \hat{T}(0)]\mathbf{u} = 0, \tag{2.29}$$

$$(u_r, u_l) = \int_{E_0}^{E_1} u_r(E)u_l(E)\,dE = 1. \tag{2.30}$$

From Eq. (2.26) we have

$$H^*(\mu,\zeta)[I - \zeta\int_0^1 H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu]$$
$$= I - \zeta H^*(\mu,\zeta)\int_0^1 [\nu/(\mu+\nu)]H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu. \tag{2.31}$$

By theorem 2.1 we have

$$I - \zeta\int_0^1 H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu = H(\infty,\zeta)(I - \zeta\hat{T}(0)). \tag{2.32}$$

We apply both sides of (2.31) to $\mathbf{u}$ and use (2.32) to get

$$(1-\zeta)H^*(\mu,\zeta)H(\infty,\zeta)\mathbf{u}$$
$$= \left[I - \zeta H^*(\mu,\zeta)\int_0^1 \frac{\nu}{\mu+\nu}H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu\ \mathbf{u}\right]. \tag{2.33}$$

Now as $k_{H(\mu,\zeta)-I}(E,E') \geq \binom{0}{0}$ and $H(\mu,\zeta)\mathbf{u} \geq \mathbf{u}$, we have

$$(1-\zeta)H^*(\mu,\zeta)\mathbf{u}$$
$$\leq \left[I-\zeta H^*(\mu,\zeta)\int_0^1 \frac{\nu}{\mu+\nu}H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu\right]\mathbf{u}.$$

Hence

$$H^*(\mu,\zeta)\left[(1-\zeta)I+\zeta\int_0^1\frac{\nu}{\mu+\nu}H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu\right]\mathbf{u}\leq\mathbf{u}. \quad (2.34)$$

Now the vector function f, given by

$$\mathbf{f}(\mu,\zeta,E,E')=\left[(1-\zeta)I+\zeta\int_0^1\frac{\nu}{\mu+\nu}H(\nu,\zeta)\mathbf{D}(\nu)\,d\nu\right]\mathbf{u},$$

is bounded away from zero in the sense that

$$\inf_{\substack{0\leq\zeta<1\\0\leq\mu\leq1\\E_0\leq E,E'\leq E_1}}\mathbf{f}(\mu,\zeta,E,E')=A>0.$$

Hence

$$\int_{E_0}^{E_1}k_{H^*(\mu,\zeta)-I}(E,E')\,dE'\leq(1/A)[\max(\|u_l\|_\infty,\|u_r\|_\infty)]. \quad (2.35)$$

Therefore,

$$\lim_{\zeta\to1^-}k_{H^*(\mu,\zeta)-I}(E,E')=k_{H(\mu)-I}(E,E')$$

exists in $(L_1([0,1]\times[E_0,E_1]\times[E_0,E_1]))^2$. Moreover, by Eq. (2.35), $H(\mu)\in\mathbb{B}_1([0,1];\mathcal{L}_1(L_1([E_0,E_1])))$, and $H^*(\mu)\in\mathbb{B}_1([0,1];\mathcal{L}(L_\infty([E_0,E_1])))$. We may therefore take limits in Eq. (2.26′) to get

$$H(\mu)=I+\mu H(\mu)\int_0^1\mathbf{D}(\nu)H^*(\nu)\frac{d\nu}{\mu+\nu}. \quad (2.36)$$

Now for each $\mu\in[0,1]$, $H(\mu)-I\in\mathrm{Com}(L_1)$ and $H(\mu)^{-1}=I-\mu\int_0^1\mathbf{D}(\nu)H^*(\nu)\,d\nu/(\mu+\nu)$ is a right inverse of $H(\mu)$. Hence $H(\mu)^{-1}$ must also be a left inverse. Therefore, we may rewrite Eq. (2.36) as

$$H(\mu)=I+\mu\int_0^1\mathbf{D}(\nu)H^*(\nu)\frac{d\nu}{\mu+\nu}H(\mu). \quad (2.36')$$

(2.36′) implies that $k_{H(\mu)-I}(E,E')$ is continuous in E for each fixed E'. Hence $k_{H^*(\mu)-I}(E,E')$ is continuous in E'. Equation (2.36) and the dominated convergence theorem imply that $k_{H(\mu)-I}(E,E')$ is continuous on $[E_0,E_1]\times[E_0,E_1]$ for each fixed μ.

Now $H(\mu)^{-1}$ is an analytic, $\mathcal{L}(\mathcal{C}^2)$-valued function of μ for $\mu\in C/[-1,0]$, and continuous on $[0,1]$. Hence $H(\mu)$ and $H^*(\mu)$ are meromorphic $\mathcal{L}(\mathcal{C}^2)$-valued functions, continuous on $[0,1]$ since $H(\mu)-I$ is compact. Hence $k_{H(\mu)-I}(E,E')$ is an analytic $\mathcal{C}^2$-valued function for μ near $(0,1]$, $\mathrm{Re}\mu>0$. Especially, $k_{H(\mu)-I}$ is a continuous $\mathcal{C}^2$-valued function on $(0,1]$. But $\lim_{\mu\to0^+}k_{H(\mu)-I}=0$, by Eq. (2.36), and hence $k_{H(\mu)-I}$ is continuous on $[0,1]\times[E_0,E_1]\times[E_0,E_1]$. Dini's Theorem will therefore imply Eq. (2.28).

Now set $\epsilon=1-\zeta$ and $G(\mu,\epsilon)=[H(\mu)-H(\mu,\zeta)]H(\mu)^{-1}$. G is a continuous, N-valued function of μ for $0\leq\mu\leq1$. We have, as in the scalar[4] and matrix[3] cases,

Lemma 2.1: For $0\leq\epsilon\leq1$, G satisfies

$$G(\mu,\epsilon)-\mu H(\mu)\int_0^1\mathbf{D}(\nu)H^*(\nu)G^*(\nu,\epsilon)\frac{d\nu}{\mu+\nu}$$
$$=\frac{\epsilon}{1-\epsilon}\{[I-G(\mu,\epsilon)]H(\mu)-I\}$$
$$-\mu G(\mu,\epsilon)H(\mu)\int_0^1\mathbf{D}(\nu)H(\nu)G^*(\nu,\epsilon)\frac{d\nu}{\mu+\nu}. \quad (2.37)$$

The Frechet derivative of the map given by Eq. (2.37) at $\epsilon=0$, $G=0$ is $I-\mathcal{L}$, where, for $K\in\mathcal{C}([0,1];N)$,

$$\mathcal{L}K(\mu)=\mu H(\mu)\int_0^1\mathbf{D}(\nu)H^*(\nu)K^*(\nu)\frac{d\nu}{\mu+\nu}. \quad (2.38)$$

As in Ref. 7 we consider the alternate operator, $\mathcal{M}$ given by

$$\mathcal{M}K(\mu)=H(\mu)\int_0^1\frac{\nu}{\mu+\nu}\mathbf{D}(\nu)H^*(\nu)K^*(\nu)\,d\nu. \quad (2.38')$$

We note that $(I-\mathcal{M})K=0$ iff $F(\mu)=\mu K(\mu)$ satisfies $(I-\mathcal{L})F=0$.

Now the assumptions on Σ and C together with Eq. (2.32) imply that there is $\binom{w_r}{w_l}=\mathbf{w}\in\mathcal{C}^2$ so that

$$\left(I-\int_0^1\mathbf{D}(\nu)H^*(\nu,\zeta)\,d\nu\right)\mathbf{w}=0. \quad (2.39)$$

for $\mathbf{f}=\binom{f_r}{f_l}\in\mathcal{C}^2$ set

$$Pf=\begin{pmatrix}(f_r,w_l)w_r\\(f_l,w_r)w_l\end{pmatrix}=\begin{pmatrix}P_l&0\\0&P_r\end{pmatrix}\mathbf{f}. \quad (2.40)$$

We have $P^*=P$ and $(I-\mathcal{M})P=0$.

If K is such that $(I-\mathcal{M})K=0$, then $(I-\mathcal{M}^2)K=0$, i.e.,

$$K(\mu)=\mathcal{M}^2K(\mu)$$
$$=H(\mu)\int_0^1\frac{\nu}{\mu+\nu}\mathbf{D}(\nu)H^*(\nu)$$
$$\times\int_0^1K(\alpha)H(\alpha)\mathbf{D}(\alpha)\frac{\alpha}{\alpha+\nu}\,d\alpha H^*(\nu)\,d\nu. \quad (2.41)$$

Hence the associated kernels, k_{K_l} and k_{K_r}, satisfy equations of the form

$$k_{K_l}=\mathcal{M}_lk_{K_l},\quad k_{K_r}=\mathcal{M}_rk_{K_r}. \quad (2.42)$$

It is easy to see that the operators $\mathcal{M}_r$ and $\mathcal{M}_l$ are compact on $\mathcal{C}([0,1]\times[E_0,E_1]\times[E_0,E_1])$ and strictly positive in the sense of Karlin.[10] Hence, if there is a solution to Eq. (2.42), it is unique up to a scalar multiple. We then must have

$$K(\mu)=P. \quad (2.41')$$

Hence $(I-\mathcal{L})F=0$ has, up to a scalar multiple, the unique solution

$$F(\mu)=\mu P. \quad (2.42')$$

Also there is $\Delta\in(\mathcal{C}^2[[0,1]\times[E_0,E_1]^2])'$ so that if $k\in\mathcal{C}^2([0,1]\times[E_0,E_1]^2)$ is nonnegative and not identically zero, $\Delta(k)>0$ and for every $F\in\mathcal{C}([0,1];N)$

$$\Delta(k_{(I-\mathcal{L})F})=0. \quad (2.43)$$

For $F\in\mathcal{C}([0,1];N)$ we set $\Lambda(F)=\Delta(k_F)$; Eq. (2.37) then becomes

$$G(\mu,\epsilon,\alpha)=\alpha\mu P+Q(\mathcal{N}(\epsilon,G)),$$
$$\Lambda(\mathcal{N}(\epsilon,G))=0. \quad (2.44)$$

We describe Δ in more detail in the Appendix.

Here, α is a complex parameter, $\mathcal{N}(\epsilon,G)$ is the right side of Eq. (2.37), and Q is a pseudo-inverse of $(I-\mathcal{L})$.

For $\|G\|_{L([0,1],N)}$, $|\epsilon|$, and $|\alpha|$ small, Eq. (2.44) has a unique solution, if α is considered as a variable. We put this solution into the condition

$$E(\epsilon,\alpha)=\Lambda(\mathcal{N}(\epsilon, G(\mu,\epsilon,\alpha)))=0 \qquad (2.45)$$

and differentiate with respect to α and ϵ to get

$$E(0,0)=0,\quad \frac{\partial}{\partial\alpha}E(0,0)=0,$$

$$\frac{\partial E}{\partial\epsilon}(0,0)=\Lambda(H(\mu)-I)>0, \qquad (2.46)$$

$$\frac{\partial E}{\partial\alpha^2}(0,0)=-\Lambda(\mu^2 P)<0.$$

As $\mathcal{N}(\epsilon, G)$ is analytic in ϵ near $\epsilon=0$, $E(\epsilon,\alpha)$ is analytic in ϵ and α for small $|\epsilon|$ and $|\alpha|$. Hence α is a two-valued function of ϵ, near $\epsilon=0$. So $G=0$, $\epsilon=0$ is a branch point of order 2 of Eq. (3.37). We may then write

$$G(\mu,\epsilon)=\epsilon^{1/2}G_1(\mu)+\epsilon G_2(\mu)+O(\epsilon^{3/2}). \qquad (2.47)$$

Where $\epsilon^{1/2}$ is the positive square root.

We substitute Eq. (2.47) into Eq. (2.38) to get

$$G_1(\mu)=\alpha\mu P,\quad \alpha^2[\Lambda(\mu^2 P)]=\Lambda(H(\mu)-I). \qquad (2.48)$$

We have

Theorem 2.4: If $D(\nu)$ is as in Eq. (2.23), the point $\zeta=1$ is a branch point of order two for the nonlinear system (1.1).

The reader should note that we have normalized $\|\mathbb{P}\mathcal{T}\|_{sp}$ to be one. Had we not done this, the branch point would have occurred at the point $\zeta=\|\mathbb{P}\mathcal{T}\|_{sp}^{-1}$. It should also be pointed out that we have shown that $H(\mu,\zeta)$ is in the space N_0 for ζ near 1, $\arg(1-\zeta)\neq\pi$. Therefore, $H_l(\mu,\zeta)-I$, $H_r(\mu,\zeta)-I$ are integral operators with continuous kernels for such ζ. It should also be pointed out that the assumption $k_c>0$ may be weakened. One need only assume that there is a positive integer p so that $k_{cp}>0$ on $[E_0,E_1]\times[E_0,E_1]$.

III. CRITICALITY IN NEUTRON TRANSPORT

Let K be given by Eq. (2.24). For $0<\tau<\infty$ consider the operator $\mathcal{K}_\tau\in \mathrm{Com}(\mathcal{C}([0,\tau],\mathcal{C}))$ given by

$$\mathcal{K}_\tau\phi(x)=\int_0^\tau K(x-y)\phi(y)\,dy,\quad 0\le x\le\tau. \qquad (3.1)$$

It is clear that $(I-\zeta\mathcal{K}_\tau)^{-1}$ exists for $|\zeta|<1$. For $|\zeta|<1$, define $\Gamma(x;\zeta,\tau)$ by

$$\Gamma(x,\zeta,\tau)-\zeta\int_0^1 T(x-y)\Gamma(y;\zeta,\tau)\,dy$$
$$=\zeta T(x),\quad 0\le x\le\tau. \qquad (3.2)$$

As in Refs. 5 and 6, we have

Theorem 3.1: $(I-\zeta\mathcal{K}_\tau)^{-1}\phi(x)=\phi(x)+\int_0^\tau R(x,y;\zeta,\tau)\times\phi(y)\,dy$, for $0\le x\le\tau$, where

$$R(x,y;\zeta,\tau)=\begin{Bmatrix}\Gamma_l(x-y;\zeta,\tau), & 0\le y\le x\le\tau\\ \Gamma_r(y-x;\zeta,\tau), & 0\le x\le y\le\tau\end{Bmatrix}$$
$$+\int_0^{\min(x,y)}[\Gamma_l(x-r;\zeta,\tau)\Gamma_r(y-r;\zeta,\tau)$$
$$-\Gamma_l(\tau-x+r;\zeta,\tau)\Gamma_r(\tau-y+r;\zeta,\tau)]\,dr. \qquad (3.3)$$

We note that formula (3.3) holds if $K\in\mathbb{B}_l(\mathbf{R};\mathrm{Com}(\mathcal{B}))$ and ϕ is $\mathcal{B}$-valued for any Banach space $\mathcal{B}$ if $\int_{-\infty}^{\infty}\|K(x)\|_{L(\mathcal{B})}\,dx\le 1$.

Now we let $\tilde{\Gamma}$ be the extension of Γ to all of $\mathbf{R}$ by the defining equation (3.2). We define Γ^1 and Γ^2 with support in $(0,\infty)$ by

$$\Gamma^1(-x)+\Gamma_l(x)+\Gamma^2(x-\tau)$$
$$=\zeta K(x)+\zeta\mathcal{K}_\tau\Gamma_l(x)=\tilde{\Gamma}_l(x). \qquad (3.4)$$

For $\mathrm{Im}z\ge 0$, we set

$$P(z;\zeta,\tau)=\mathcal{H}_l(z,\zeta)[I-\hat{\Gamma}^1(z;\zeta,\tau)], \qquad (3.5)$$

$$Q(z;\zeta,\tau)=\mathcal{H}_l(z,\zeta)\hat{\Gamma}^2(z;\zeta,\tau). \qquad (3.6)$$

As in Refs. 5 and 6, we have

$$\Gamma_l(x;\zeta,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta\hat{K}(z)[I-\zeta\hat{K}(z)]^{-1}$$
$$\times\mathcal{H}_l^{-1}(z,\zeta)F(x,z,\zeta,\tau)\,dz, \qquad (3.7)$$

where $F(x,z,\zeta,\tau)=P(z;\zeta,\tau)\exp(ixz)-Q(z;\zeta,\tau)\times\exp[i(\tau-x)z]$. The convergence of the integral in Eq. (3.7) is in the sense of $\mathbb{B}_2(\mathbf{R},\mathcal{N})$.

We make the following assumption on the distribution of zeros of $[I-\zeta\hat{K}(z)]$.

(I) There is $\beta>0$ such that, for $|\mathrm{Im}z|<\beta$ and ζ sufficiently close to 1, $[I-\zeta\hat{K}(z)]^{-1}$ exists for every $z\neq\pm z_0$, where $|\mathrm{Im}z_0|<\beta$.

We note that assumption (I) implies that $[I-\hat{K}(z)]^{-1}$ exists for every $z\neq 0$, $|\mathrm{Im}z|<\beta$, since $z_0=0$ if $\zeta=1$.

As in Refs. 5 and 6, we have

Theorem 3.2: For $\epsilon=1-\zeta$ small, $z_0=z_0(\epsilon)$ is an analytic function of $\epsilon^{1/2}$ for $|\arg\epsilon|<\pi$. For ϵ real, z_0 is pure imaginary and

$$z_0(\epsilon)=ci\epsilon^{1/2}+O(\epsilon). \qquad (3.8)$$

In (3.8) c is given by

$$c=[2/(\hat{K}''(0)u_r,u_l)]^{1/2}, \qquad (3.9)$$

where the double prime denotes second derivative and u_r and u_l are given by (2.29).

Moreover, eigenvectors $u_r(\epsilon)$ and $u_l(\epsilon)$ may be chosen so that $[I-\zeta\hat{T}(z_0)]\binom{u_r}{u_l}=0$, $(u_r,u_l)=1$, and

$$u(\epsilon)=\begin{pmatrix}u_r(\epsilon)\\ u_l(\epsilon)\end{pmatrix}=\begin{pmatrix}u_r\\ u_l\end{pmatrix}+O(\epsilon). \qquad (3.10)$$

At this point we note that Theorem 2.4 implies that there are two solutions to (1.1). Only one of these, namely $H(\mu,\zeta)=I+\hat{\Gamma}^*(i/\mu,\zeta)$, is of physical importance. The other, which we denote by $H^{(1)}$, is given by

$$H^{(1)}(\mu,\zeta)=\left[I+\frac{2z_0(\epsilon)}{z-z_0(\epsilon)}S_\epsilon\right]H(\mu,\zeta). \qquad (3.11)$$

S_ϵ is a projector in $\mathcal{L}(\mathcal{C}^2)$ given by

$$S_\epsilon\begin{pmatrix}f_r\\ f_l\end{pmatrix}=\frac{1}{(v_l,v_r)}\begin{pmatrix}(f_r,v_r)v_l\\ (f_l,v_l)v_r\end{pmatrix}, \qquad (3.12)$$

where

$$\mathbf{v}(\epsilon)=\begin{pmatrix}v_r(\epsilon)\\ v_l(\epsilon)\end{pmatrix}=H^*(z_0,\zeta)^{-1}\mathbf{u}(\epsilon). \qquad (3.13)$$

Note that the physical solution H is analytic for $|\zeta| < 1$ and for real ζ, $0 \leq \zeta \leq 1$, $H_l - I$, and $H_r - I$ are integral operators with positive kernels. $H^{(1)}$ has neither of these properties. Also the iterative scheme discussed in Ref. 3 will yield the physical solution. It has recently been shown[11] that the iterative scheme converges to $H(\mu, \zeta)$ for $|\zeta| \leq 1$ in the multigroup case; the proof in Ref. 11 generalizes directly to the case of continuous energy dependence discussed here.

We chose ζ sufficiently close to 1 so that $|\text{Im} z_0| < \beta/2$. We move contours in Eq. (3.7) and obtain

$$\Gamma_l(x, \zeta, \tau) = iR(\zeta)\hat{K}(z_0)\mathcal{H}_l^{-1}(z_0, \zeta)F(x, z_0, \zeta, \tau) + (1/2\pi i)\int_{-\infty+i\beta/2}^{\infty+i\beta/2} \zeta\hat{K}(z)[I - \zeta\hat{K}(z)] \times \mathcal{H}_l^{-1}(z, \zeta)F(x, z, \zeta, \tau)\, dz. \quad (3.14)$$

In Eq. (3.14) we have, as in Ref. 8,

$$R(\zeta) = \operatorname*{Res}_{z=z_0}[I - \zeta\hat{K}(z)]^{-1} = \frac{-1}{\zeta\langle\hat{K}'(z_0)u_r(\epsilon), u_l(\epsilon)\rangle} Q_\epsilon, \quad (3.15)$$

where, for $x \in \mathcal{C}$,

$$Q_\epsilon x = \langle x, u_r(\epsilon)\rangle u_l(\epsilon). \quad (3.16)$$

We define, for $\mathcal{N}$-valued $G(z)$,

$$T_\tau(G)(z) = (-1/2\pi i)\int_{-\infty+i\beta/2}^{\infty+i\beta/2} \mathcal{H}_r^{-1}(t, \zeta)[I - \zeta\hat{K}(t)]^{-1} \times \mathcal{H}_l^{-1}(t, \zeta)[G(t)/(t+z)]\exp(i\tau t)\, dt, \quad (3.17)$$

the integral in (3.17) being understood as a Cauchy principle value with respect to the infinite limits.

If $G(z)$ is analytic in $\{z \mid \text{Im} z \geq \beta/2\}$ and $\|G(z)\|_{\mathcal{N}_0}$ is bounded there, we have, for $n \geq 0$, $T_\tau^n(G) = S_\tau^n(G)$, where

$$S_\tau(G)(z) = (-1/2\pi i)\int_{-\infty+i\beta/2}^{\infty+i\beta/2} \mathcal{H}_r^{-1}(t, \zeta)\zeta\hat{K}(t)[I - \zeta\hat{K}(t)]^{-1} \times \mathcal{H}_l^{-1}(t, \zeta)[G(t)/(t+z)]\exp(i\tau t)\, dt. \quad (3.18)$$

Moreover, there is $C > 0$ so that

$$\sup_{\text{Im} z \geq \beta/2} \|S_\tau^n(G)(z)\|_{\mathcal{N}_0} \leq C\exp(-n\tau\beta/2) \times (\sup_{\text{Im} z \geq \beta/2} \|F(z)\|_{\mathcal{N}_0}).$$

Hence, for τ sufficiently large, we may define function P_0 and P_1

$$P_0(z) = \sum_{n=0}^{\infty} T_\tau^{2n}(I)(z), \qquad P_1(z) = \sum_{n=0}^{\infty} T_\tau^n\left(\frac{I}{(\cdot) + z_0} I\right)(z), \quad (3.19)$$

where the convergence is uniform in $\{z \mid \text{Im} z \geq \beta/2\}$. Now P_0 and P_1 are analytic and bounded in $\{z \mid \text{Im} z \geq \beta/2\}$. We define Q_0 by

$$Q_0 = T_\tau P_0. \quad (3.20)$$

If $A(z) = \exp(i\tau z_0)P_1(z)\mathcal{H}_r^{-1}(z_0, \zeta)R(\zeta)\mathcal{H}_l^{-1}(z_0, \zeta)$, we have, as in Ref. 5,

$$P(z) = P_0(z) - A(z)Q(z_0), \qquad Q(z) = Q_0(z) - A(z)P(z_0). \quad (3.21)$$

The proof of this is a direct generalization of that found in Ref. 5. An important role in that proof is played by the following lemma, which we state for completeness

Lemma 3.1: For $\text{Im}\lambda > 0$,

$$\mathcal{H}(\lambda, \zeta) = I + \frac{\zeta}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathcal{H}(z, \zeta)\hat{T}(z)\, dz}{z - \lambda}, \quad (3.22)$$

$$\mathcal{H}^{-1}(\lambda, \zeta) = I + \frac{\zeta}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{T}(z)\mathcal{H}^*(z, \zeta)}{z + \lambda}\, dz. \quad (3.22')$$

The lemma follows from the Parseval relation as in Ref. 5.

We let $\zeta \to \zeta^+ > 1$, $\arg(1 - \zeta) \to -\pi$. Theorem 2.4 implies that for $\zeta^+ - 1$ small,

$$\mathcal{H}^{-1}(t, \zeta^+) = \lim_{\substack{\zeta \to \zeta^+ \\ \arg(1-\zeta) \to -\pi}} \mathcal{H}^{-1}(t, \zeta)$$

is a well-defined, $\mathcal{N}_0$-valued function for $\text{Im} z \geq 0$. Theorem 3.2 implies that

$$z^+ = \lim_{\substack{\zeta \to \zeta^+ \\ \arg(1-\zeta) \to -\pi}} z_0(1 - \zeta)$$

exists. For τ sufficiently large, P_0, P_1, and Q_0 have limits at ζ^+. We apply the bounded convergence theorem to Eq. (3.14) and obtain

$$\Gamma_l(x; \zeta^+, \tau) = iR(\zeta^+)\hat{K}(z^+)\mathcal{H}_l^{-1}(z^+, \zeta^+)F(x, z^+, \zeta^+, \tau) + (1/2\pi i)\int_{-\infty+i\beta/2}^{\infty+i\beta/2} \zeta^+\hat{K}(z)[I - \zeta^+\hat{K}(z)] \times \mathcal{H}_l^{-1}(z; \zeta^+)F(x, z, \zeta^+, \tau)\, dz. \quad (3.23)$$

If $(I - \zeta^+ K_\tau)\phi = 0$ has a solution $\phi(x, E)$ which is positive for $0 \leq x \leq \tau$, $E_0 \leq E \leq E_1$, one can show, as in Ref. 5, that ϕ must be even about $\tau/2$. Recently Victory[12] has shown that, for $\zeta^+ > 1$ and sufficiently close to 1, there is $\tau < \infty$ so that $(I - \zeta^+ K_\tau)\phi = 0$ has a positive solution, and $\zeta^+ \to 1$ as $\tau > \infty$.

We apply (3.23) in a way similar to that of Ref. 5 to get the following asymptotic result:

Theorem 3.3: If $(I - \zeta^+ K_\tau)\phi = 0$ has a positive solution ϕ, then there is $c > 0$ so that for τ large and $|x - \tau/2| < \tau/4$,

$$\phi(x) = c\cos[z^+(\tau/2 - x)]u_r(\epsilon) + O[\exp(-\tau\beta/4)] \quad (3.24)$$

as $\tau \to \infty$. τ and ζ^+ are related asymptotically, as $\zeta^+ \to 1$, by

$$\tau = \pi/z^+ - 2\alpha_0 + O(|1 - \zeta^+|), \quad (3.25)$$

where

$$\alpha_0 = \int_0^1 \alpha(\mu)\mu^2\, d\mu \Big/ \int_0^1 \alpha(\mu)\mu\, d\mu \quad (3.26)$$

and

$$\alpha(\mu) = \langle u_l(0), \int_0^1 \nu D(\nu)H_r(\nu)\, d\nu H_l(\mu)D(\mu)u_r(0)\rangle. \quad (3.27)$$

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APPENDIX

In this appendix we give an explicit formula for the functional Λ which is given uniquely up to a scalar multiple by

$$\Lambda((I-\mathcal{L})F)=0, \quad \text{all } F\in \mathcal{C}([0,1],N). \tag{A1}$$

Our formula is a direct generalization of that given in Appendix B of Ref. 7.

For $A, B\in N$ define

$$\langle A,B\rangle = \int_{E_0}^{E_1}\int_{E_0}^{E_1} k_{A_l}(E,E')k_{B_l}(E,E') + k_{A_r}(E,E')k_{B_r}(E,E')\,dE\,dE'.$$

Note that for $A, B, C\in N$, we have

$$\langle A,B\rangle = \langle A^*,B^*\rangle, \quad \langle AC,B\rangle = \langle A,CB\rangle = \langle A,BC\rangle. \tag{A2}$$

For $G, K\in \mathbb{B}_\infty([0,1],N)$ define

$$[K,G]=\int_0^1 \langle K(\mu),G(\mu)\rangle\,d\mu. \tag{A3}$$

Now let y_r and y_l be positive functions on $[E_0,E_1]$ satisfying

$$\left(I-\int_0^1 H(\nu)\mathbf{D}^*(\nu)\,d\nu\right)\begin{pmatrix} y_r \\ y_l\end{pmatrix}=0. \tag{A4}$$

For $f=\binom{f_r}{f_l}\in \mathcal{C}^2$ define

$$Af=\begin{pmatrix}(f_r,y_l)y_r\\ (f_l,y_r)y_l\end{pmatrix}. \tag{A5}$$

We have $A^*=A$ and

$$A\left[I-\int_0^1 \mathbf{D}(\nu)H^*(\nu)\,d\nu\right]=0. \tag{A6}$$

Now define $G\in \mathbb{B}_\infty([0,1],N)$ by

$$G(\mu)=H(\mu)\mathbf{D}^*(\mu)AH^{-1}(\mu). \tag{A7}$$

Then for each $F\in \mathcal{C}([0,1],N)$

$$[\mathcal{L}F,G]=[(\mathcal{L}F)^*,G^*]$$

$$=\int_0^1\left\langle \int_0^1 \frac{\mu}{\mu+\nu}F(\nu)H(\nu)\mathbf{D}^*(\nu)\,d\nu H^*(\nu), G^*(\nu)\right\rangle d\nu$$

$$=\int_0^1\left\langle F(\nu),H(\nu)\mathbf{D}^*(\nu)\int_0^1 \frac{\mu}{\mu+\nu}H^*(\mu)G^*(\mu)\,d\mu\right\rangle d\nu$$

$$=\int_0^1 \langle F(\nu),G(\nu)\rangle\,d\nu=[F,G].$$

Hence $[(I-\mathcal{L})F,G]=0$ for all $F\in \mathcal{C}([0,1],N)$. We have for $K\in \mathcal{C}([0,1],N)$

$$\Lambda(K)=[K,G]. \tag{A8}$$

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Solution by iteration of H-equations in multigroup neutron transport

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The Chandrasekhar H-equations for matrix-valued functions are solved by an iterative method. Complex variables and positivity techniques are used to obtain convergence. This approach may be applied to subcritical neutron transport in a slab with isotropic scattering.

I. INTRODUCTION

Certain problems in the theory of radiative transfer were made computationally tractable by Chandrasekhar[1] in the forties. He introduces X and Y functions as solutions to nonlinear integral equations arising in an analysis of radiative transfer in plane-parallel atmospheres of finite thickness. For half-space problems these equations reduce to a single equation for the scalar H-function satisfying the nonlinear equation

$$H(\mu)=1+\mu H(\mu)\int_0^1\frac{H(\nu)\psi(\nu)}{\mu+\nu}\,d\nu. \tag{1.1}$$

In (1.1) ψ is nonnegative and

$$\int_0^1\psi(\nu)\,d\nu\leq\tfrac{1}{2}. \tag{1.2}$$

In recent years, matrix-valued and operator-valued H-functions have been used in the study of multigroup[2–5] and continuous energy[6,7] neutron transport, and in scattering models in radiative transfer.[8–10] Operator equations analogus to Eq. (1.1) are derived, and methods for their solution become of interest.

It is known that, in certain cases, the nonlinear H-equations do not have a unique solution, there being one solution of physical significance and perhaps other extraneous solutions. Iterative methods of solution must be investigated both with respect to convergence and to convergence to the desired physical solution.

Numerical methods for solving scalar H-equations have been investigated for some time.[9–15] Recently Bowden, Menikoff, and Zweifel[14,15] have used contraction mapping arguments for the scalar and matrix H-equations to show convergence of iteration sequences with respect to certain norms.

In this note we show convergence of an iterative sequence to the desired solution under conditions less restrictive than those of Bowden, Menikoff, and Zweifel. We use concepts of positivity and analyticity rather than that of contraction mapping.

We give proofs for the models of N-group neutron transport in a subcritical half-space with isotropic scattering and fission. These models are considered for simplicity, the proofs extend to more general models with attention to mathematical technicalities.

II. SUBCRITICAL N-GROUP TRANSPORT

We consider the integral operator with $N\times N$ matrix kernel

$$k(x-y)=C\int_{|x-y|}^{\infty}\exp(-t\Sigma)\frac{dt}{2t}. \tag{2.1}$$

In Eq. (2.1) C is a nonnegative matrix and Σ is diagonal with $1=\sigma_{11}\geq\sigma_{22}\geq\cdots\geq\sigma_{NN}$. This is related to the transport equation in a half—space ($x\geq 0$),

$$\mu\frac{\partial\psi}{\partial x}+\Sigma\psi=\frac{C}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu'. \tag{2.2}$$

It was shown by Burniston, Mullikin, and Siewert[16] that a sufficient condition for the half—space to be subcritical is that the spectral radius of $\Sigma^{-1}C$ satisfy $\|\Sigma^{-1}C\|_{sp}<1$. This is also known to be a necessary condition.[17]

Assuming that $\|\Sigma^{-1}C\|_{sp}<1$ for given matrices Σ and C, we introduce a one-parameter family of kernels

$$k(x,y,\omega)=\omega C\int_{|x-y|}^{\infty}\exp(-t\Sigma)\frac{dt}{2t}. \tag{2.3}$$

In Eq. (2.3), ω is a complex parameter restricted by $|\omega|\,\|\Sigma^{-1}C\|_{sp}<1$. For real ω, $0\leq\omega<\|\Sigma^{-1}C\|_{sp}^{-1}$, the kernels are associated with a physically reasonable transport problem. We have imbedded the transport equation (2.3) in a one parameter family with $\omega=1$ giving the original equation.

As in Ref. 2, for real ω, $0\leq\omega<\|\Sigma^{-1}C\|^{-1}_{sp}$, the H-functions of physical interest are defined, for $0\leq\mu\leq 1$, by

$$H_l(\mu,\omega)=J_r(0^+,\mu,\omega)\,,\quad H_r(\mu,\omega)=J_l(0^+,\mu,\omega). \tag{2.4}$$

The J-functions are solutions, with I the $N\times N$ identity matrix, to

$$J_r(x,\mu,\omega)=\exp(-x/\mu)I+\omega\int_0^{\infty}k(x-y)J_r(y,\mu,\omega)\,dy,$$
$$J_l(x,\mu,\omega)=\exp(-x/\mu)I+\omega\int_0^{\infty}J_l(y,\mu,\omega)k(x-y)\,dy. \tag{2.5}$$

Solutions are unique[2] in $\mathcal{C}$, the space of $N\times N$ matrices of bounded continuous functions on $[0,\infty)$. The norm in $\mathcal{C}$ is given by

$$\|M\|=\max_{1\leq i,j\leq N}\{\sup_{0\leq x<\infty}|m_{ij}(x)|\}.$$

We make use of the tools of positivity and analyticity.

Lemma 2.1: H_r and H_l are, for $0<\mu\leq 1$, defined for complex ω as analytic functions in the disc $|\omega|<\|\Sigma^{-1}C\|_{sp}^{-1}$. For real ω, $0\leq\omega<\|\Sigma^{-1}C\|_{sp}^{-1}$, these are nonnegative $N\times N$ matrix-valued functions of μ, $0\leq\mu\leq 1$, where $H_r(0,\omega)=H_l(0,\omega)=I$.

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Proof: The Neumann series for J_r and J_l converge in $\mathcal{C}$ to represent J_r and J_l as convergent power series in ω with values in $\mathcal{C}$. Hence the H-functions are analytic in ω. For real ω, the Neumann series has matrix coefficients with nonnegative entries.

It follows from Theorem 2 of Ref. 2 that H_r and H_l also satisfy the coupled system of nonlinear integral equations,

$$H_r(\mu,\omega)=I+\frac{\mu\omega}{2}H_r(\mu,\omega)\int_0^1 H_l(\nu,\omega)\frac{CD(\nu)}{\mu+\nu}\,d\nu,$$
$$H_l(\mu,\omega)=I+\frac{\mu\omega}{2}\int_0^1 \frac{CD(\nu)}{\mu+\nu}H_r(\nu,\omega)\,d\nu\, H_l(\mu,\omega). \tag{2.6}$$

In Eq. (2.6), D is a diagonal matrix of characteristic functions, $d_{ii}=\chi[0,1/\sigma_{ii}]$. For convenience of notation, as in Ref. 4, we define, with ~ denoting transpose.

$$H=\begin{pmatrix} H_l & 0\\ 0 & \tilde{H}_r\end{pmatrix},\quad H^*=\begin{pmatrix} H_r & 0\\ 0 & \tilde{H}_l\end{pmatrix},\quad \psi=\begin{pmatrix} CD & 0\\ 0 & \widetilde{DC}\end{pmatrix}. \tag{2.7}$$

Then the system (2.6) becomes

$$H(\mu,\omega)=I+\frac{\mu\omega}{2}H(\mu,\omega)\int_0^1 \psi(\nu)H^*(\nu,\omega)\frac{d\nu}{\mu+\nu} \tag{2.8}$$

or, more briefly, with L a linear integral operator,

$$H=I+\omega HL(H^*). \tag{2.9}$$

For matrices A and B, we write $A\geq B$ if the matrix $A-B$ has nonnegative entries.

Theorem 2.1: The physical solution to Eq. (2.9) for $0\leq\omega\leq\|\Sigma^{-1}C\|_{sp}^{-1}$ is the limit of the sequence $\{H_n\}$ given by

$$H_0=I,\quad H_{n+1}=I+\omega H_n L(H_n^*). \tag{2.10}$$

In particular this is true for $\omega=1$ if $\|\Sigma^{-1}C\|_{sp}\leq 1$. The sequence converges uniformly in μ and ω for $0\leq\mu\leq 1$, $|\omega|\leq\|\Sigma^{-1}C\|_{sp}^{-1}$.

Proof: For $|\omega|<\|\Sigma^{-1}C\|_{sp}^{-1}$ a solution to Eq. (2.9) is given by Eq. (2.4). It is known[3] that this physical solution $H(\mu,\omega)$ has a limit $H(\mu,\|\Sigma^{-1}C\|_{sp}^{-1})$ as ω in increases to $\|\Sigma^{-1}C\|_{sp}^{-1}$. This limit satisfies (2.9) and, if $|H|$ denotes the matrix $(|h_{ij}|)$, it follows from Eqs. (2.4) and (2.5) that, for $|\omega|<\|\Sigma^{-1}C\|_{sp}^{-1}$

$$|H(\mu,\omega)|\leq H(\mu,|\omega|)\leq H(\mu,\ \|\Sigma^{-1}C\|_{sp}^{-1})$$

For real ω, $0\leq\omega\leq\|\Sigma^{-1}C\|_{sp}^{-1}$, the physical solution H to Eq. (2.9) satisfies $H\geq I=H_0$. Hence

$$H_0\leq H_1=I+\omega H_0L(H_0^*)\leq I+\omega HL(H^*)=H.$$

If follows by induction that

$$H_n(\mu,\omega)\leq H_{n+1}(\mu,\omega)\leq H(\mu,\omega)\leq H(\mu,\|\Sigma^{-1}C\|_{sp}^{-1}).$$

It was shown in Ref. 2 that the entries of H are continuous functions of μ, $0\leq\mu\leq 1$. The matrices H_n are monotone increasing in n for $0\leq\omega\leq\|\Sigma^{-1}C\|_{sp}^{-1}$ and bounded. We denote the limit, pointwise in μ, by

$$K(\mu,\omega)=\lim_{n\to\infty}H_n(\mu,\omega).$$

We obtain L_1 convergence in μ of $\{H_n\}$ to K. As K satisfies Eq. (2.9), K is continuous for $0\leq\mu\leq 1$. Dini's theorem gives uniform convergence of $\{H_n\}$ to K.

For complex ω and $n>m$, $H_n(\mu,\omega)-H_m(\mu,\omega)$ is a polynomial in ω with nonnegative matrix coefficients. Hence

$$|H_n(\mu,\omega)-H_m(\mu,\omega)|$$
$$\leq H_n(\mu,|\omega|)-H_m(\mu,|\omega|)$$
$$\leq H_n(\mu,\|\Sigma^{-1}C\|_{sp}^{-1})-H_m(\mu,\|\Sigma^{-1}C\|_{sp}^{-1}). \tag{2.11}$$

The right-hand side of the inequality (2.11) converges to zero as n,m approach infinity. Hence $K(\mu,\omega)$ is analytic in ω for $|\omega|<\|\Sigma^{-1}C\|_{sp}^{-1}$ and continuous in ω for $|\omega|\leq\|\Sigma^{-1}C\|_{sp}^{-1}$.

We obtain agreement of $K(\mu,\omega)$ with $H(\mu,\omega)$ by showing that these analytic matrices agree on an interval for real ω. For $0\leq\omega\leq\alpha<\|\Sigma^{-1}C\|_{sp}^{-1}$, we have

$$H=I+\omega HL(H^*),\quad K=I+\omega KL(K^*),\quad K\leq H.$$

Hence,

$$H-K=\omega HL(H^*-K^*)+\omega(H-K)L(K^*)$$
$$\leq\omega HL(H^*-K^*)+\omega(H-K)L(H^*).$$

This implies that

$$(H-K)(I-\omega H(H^*))H\leq\omega HL(H^*-K^*)H.$$

Hence, by Eq. (2.9)

$$H(\omega)-K(\omega)\leq\omega H(\alpha)L(H^*(\omega)-K^*(\omega))H(\alpha).$$

For $0\leq\omega\leq\beta\leq\alpha$ and β sufficiently small and positive, we get $H(\omega)=K(\omega)$. This completes the proof.

The condition of the above theorem that $\|\Sigma^{-1}C\|_{sp}\leq 1$ is less restrictive that those of Refs. 6, 14, and 15.

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Phase-space approach to relativistic quantum mechanics. II. Geometrical aspects

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The formalism introduced in Paper I [J. Math. Phys. **18**, 952 (1977)] is made manifestly covariant by including as an admissible phase space any $2n$-dimensional submanifold of the forward tube $\mathcal{T}=\{x-iy\in C^{n+1}\,|\,y_0>|\mathbf{y}|\}$ which is of the "product" form $\sigma=S-i\Omega_\lambda$, where $\Omega_\lambda=\{y\in R^{n+1}|\,y_0=(\lambda^2+y^2)^{1/2}\}$, $\lambda>0$, and S is any space-or-lightlike submanifold of space–time R^{n+1}. The σ's have natural symplectic structures covariant with respect to the Poincaré group, and a norm $\|\cdot\|_\sigma$ on the space $\mathcal{K}$ of solutions is defined by integrating with respect to the Liouville measure on σ. This automatically gives $\|f\|_\sigma^2$ as the total flux of a conserved space–time vector field, implying that $\|f\|_\sigma$ is independent of σ. Some inconsistencies encountered in the space–time theory of Klein–Gordon particles appear to be resolved in the phase-space framework.

1. INTRODUCTION

In a previous paper,[1] hereafter refered to as I, a phase-space formulation of relativistic quantum mechanics was initiated. A "coherent-state" representation of the Poincaré group $\mathcal{P}_+^\uparrow$ for massive scalar particles was constructed on the space $\mathcal{K}$ of positive-energy solutions of the Klein–Gordon equation in $(n+1)$-dimensional space–time. The elements of $\mathcal{K}$ extend as holomorphic functions to the *forward tube*[2] $\mathcal{T}$:

$$f(z)=(2\pi)^{-n/2}\int_\Omega \exp(-izp)\hat{f}(p)\,d\Omega(p), \tag{1.1}$$

where

$$z\equiv(\mathbf{z},z_0)\in\mathcal{T}\equiv\{x-iy\in C^{n+1}\,|\,y_0>|\mathbf{y}|\}\equiv R^{n+1}-iV_+,$$

$$\Omega\equiv\{p\in R^{n+1}\,|\,p_0=(m^2c^2+\mathbf{p}^2)^{1/2}\equiv\omega(\mathbf{p})\},\quad m>0,$$

$$d\Omega(p)=dp_1\cdots dp_n/\omega,\quad zp=z_0p_0-\mathbf{z}\cdot\mathbf{p},$$

and $\hat{f}(p)=\omega[f(\mathbf{x},0)]\hat{}\,(\mathbf{p})$ where $\hat{}$ denotes the Euclidean Fourier transform in R^n. It was shown that the functions

$$e_z(p)=(2\pi)^{-n/2}\exp(i\bar{z}p) \tag{1.2}$$

[defined so that $f(z)=\langle e_z|\hat{f}\rangle_{L^2(\Omega)}$; $\bar{z}$ is the complex conjugate of $z\in\mathcal{T}$] belong to $L^2(\Omega)$ and represent optimal wavepackets (in the sense of Theorem 4 of I) centered in space–time about x (that is, focused at $\mathbf{x}$ at time[3] x_0) and traveling with expected energy–momentum proportional to y. Hence the submanifold

$$P_\lambda=\{x-iy\in\mathcal{T}\,|\,x_0=0,\,y^2=\lambda^2\},\quad \lambda>0, \tag{1.3}$$

can be interpreted as a classical "initial phase space" for the particle. The space of restrictions of $f\in\mathcal{K}$ to P_λ is denoted by $\mathcal{K}_\lambda$. It was shown that the expression

$$\|f\|_\lambda^2\equiv\int_{P_\lambda}|f(z)|^2\,d\mu_\lambda(z), \tag{1.4}$$

$$d\mu_\lambda(z)=C_\lambda\,dx_1\cdots dx_n\,dy_1\cdots dy_n \tag{1.5}$$

(where C_λ is a certain constant) defines a norm on $\mathcal{K}$ and $\mathcal{K}_\lambda$ which satisfies $\|f\|_\lambda=\|\hat{f}\|_{L^2(\Omega)}$ (Theorem 2 of I). This implies that $\mathcal{K}$ and $\mathcal{K}_\lambda$ are Hilbert spaces under the corresponding inner product $\langle\cdot|\cdot\rangle_\lambda$ and tnat the map $\hat{f}\mapsto f$ is unitary from $L^2(\Omega)$ onto $\mathcal{K}$ and $\mathcal{K}_\lambda$. We have a continuous resolution of the identity[4] in terms of the e_z, $z\in P_\lambda$,

$$\int_{P_\lambda}|e_z\rangle\langle e_z|\,d\mu_\lambda(z)=\mathbb{I}. \tag{1.6}$$

Now $L^2(\Omega)$ carries an irreducible unitary representation U of $\mathcal{P}_+^\uparrow$ (characterized by mass m and spin zero) under which the e_z are *covariant*,

$$U_g e_z=e_{gz},\quad g\in\mathcal{P}_+^\uparrow,\quad z\in\mathcal{T}. \tag{1.7}$$

Hence the corresponding representation on $\mathcal{K}$ (also denoted by U) is given by

$$(U_g f)(z)=f(g^{-1}z). \tag{1.8}$$

The set P_λ is not invariant under the action of $\mathcal{P}_+^\uparrow$ on $\mathcal{T}$, hence the above formalism is not manifestly covariant. In this paper we construct a natural class $\mathcal{S}_1$ of "phase spaces" $\sigma\subset\mathcal{T}$ and associated measures μ_σ to which the main results of I extend. $\mathcal{S}_1$ includes P_λ (the corresponding measure being μ_λ) and is invariant under $\mathcal{P}_+^\uparrow$, hence the formalism is freed from its dependence on P_λ and becomes manifestly covariant.

We begin in Sec. 2 by regarding $\mathcal{T}$ as an extended phase space,[5] on which $\mathcal{P}_+^\uparrow$ acts by canonical transformations. Candidates for phase space are $2n$-dimensional symplectic submanifolds[6,7] $\sigma\subset\mathcal{T}$, and $\mathcal{P}_+^\uparrow$ transforms different σ's into one another by canonical transformations. A $2n$-submanifold of the "product" form $S-i\Omega_\lambda$, where S is an n-submanifold of space–time and Ω_λ is a "mass hyperboloid," turns out to be symplectic (with respect to the induced structure) if and only if S is *space-or-lightlike*. Such σ's form a family $\mathcal{S}_1$ which is invariant under $\mathcal{P}_+^\uparrow$.

In Sec. 3 we extend the results of I to arbitrary $\sigma\in\mathcal{S}_1$. Each σ carries a canonically associated (Liouville) measure μ_σ. We show that for each $f\in\mathcal{K}$, $\|f\|_\sigma^2\equiv\|f\|_{L^2(\mu_\sigma)}^2$ is the total flux of a conserved vector field, hence independent of σ.

In Sec. 4 we show how the phase-space formalism can be used to resolve certain inconsistencies in the usual theory of Klein–Gordon particles.

2. SYMPLECTIC STRUCTURE

The Poincaré group acts on $\mathcal{K}$ by simply transforming the underlying space $\mathcal{T}$: $(U_g f)(z)=f(g^{-1}z)$, where $g=(a,\Lambda)\in\mathcal{P}_+^\uparrow$ and $gz=\Lambda z+a$. We wish to supply $\mathcal{T}$ with a symplectic structure[6,7] such that the map $z\mapsto gz$ is a canonical transformation for each $g\in\mathcal{P}_+^\uparrow$. That is, we need a 2-form α on $\mathcal{T}$ which is (a) closed ($d\alpha=0$),

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(b) nondegenerate [the $(n+1)$-fold exterior product α^{n+1} is never zero], and (c) invariant under $\mathcal{P}_{+}^{\uparrow}$. The last condition means that $g^*\alpha=\alpha$, where g^* is the pullback map on forms induced by g (a brief description of which is given in the Appendix). Since every Poincaré-invariant function $\varphi(z)$ on $\mathcal{T}$ depends on z only through y^2, the most general invariant 2-form is given by

$$\alpha=\varphi(y^2)\,dy_\mu\,dx^\mu+\psi(y^2)y_\mu y_\nu\,dy^\mu\,dx^\nu. \tag{2.1}$$

(We are suppressing the wedge notation; thus, e.g., $dy^\mu\,dx^\nu=-dx^\nu\,dy^\mu$.) Now the action of $\mathcal{P}_{+}^{\uparrow}$ on $\mathcal{T}$ is not transitive, and $\mathcal{T}$ decomposes into a union of orbits [Eq. (3.8) in I]

$$\mathcal{T}=\bigcup_{\lambda>0}P_\lambda',$$
$$P_\lambda'=\{z=x-iy\in\mathcal{T}\mid y^2=\lambda^2\}\approx\mathcal{P}_{+}^{\uparrow}/\mathrm{So}(n). \tag{2.2}$$

As shown in I, each P_λ' gives rise to an equivalent representation of $\mathcal{P}_{+}^{\uparrow}$. Our main results in this paper (Sec. 3) will be confined to P_λ' for a fixed λ. Since the second term in (2.1) contains $d(y^2)=2y_\mu\,dy^\mu$ as a factor, its restriction to P_λ' vanishes. Hence we will confine our attention to

$$\alpha=dy_\mu\,dx^\mu \tag{2.3}$$

without essential loss of generality. This form is symplectic as well as invariant, thus each $g\in\mathcal{P}_{+}^{\uparrow}$ acts on $\mathcal{T}$ by canonical transformations.

$\mathcal{T}$ is an extended phase space, containing the time x^0 and the "energy" y_0 as a pair of free canonical variables. A $2n$-submanifold σ of $\mathcal{T}$ will be a candidate for phase space only if the pullback α_σ of α to σ is a symplectic form. Let σ be given by

$$\sigma=\{z\in\mathcal{T}\mid s(z)=h(z)=0\}, \tag{2.4}$$

where s and h are two real-valued, C^∞ functions on $\mathcal{T}$ such that $ds\wedge dh\neq 0$ on σ. For example, $\sigma=P_\lambda$ can be obtained from $s(z)=x_0$ and $h(z)=\sqrt{y^2}-\lambda$. The pullback α_σ does not depend on s and h.

Proposition 1: The form α_σ is symplectic if and only if the Poisson bracket

$$\{s,h\}\equiv\frac{\partial s}{\partial x^\mu}\frac{\partial h}{\partial y_\mu}-\frac{\partial s}{\partial y_\mu}\frac{\partial h}{\partial x^\mu}\neq 0 \tag{2.5}$$

everywhere on σ.

Proof: α_σ is closed since α is closed. Hence α_σ is symplectic iff it is nondegenerate, i.e., if and only if the nth exterior power α_σ^n of α_σ vanishes nowhere on σ. Now α_σ^n equals the pullback of α^n to σ, and

$$\alpha^n=n!\,d\hat{y}_\mu\,d\hat{x}^\mu, \tag{2.6}$$

where

$$d\hat{y}_\mu=(-)^\mu\,dy_0\cdots dy_{\mu-1}\,dy_{\mu+1}\cdots dy_n,$$
$$d\hat{x}^\mu=(-)^\mu\,dx^n\cdots dx^{\mu+1}\,dx^{\mu-1}\cdots dx^0. \tag{2.7}$$

Let $\{u_1,\ldots,u_{2n},v_1,v_2\}$ be a basis for the tangent space $\mathcal{T}_z$ of $\mathcal{T}$ at $z\in\sigma$, with $u_1,\ldots,u_{2n}$ a basis for the subspace σ_z. Then since ds and dh vanish on the u_j,

$$(\alpha^n\wedge ds\wedge dh)(u_1,\ldots,u_{2n},v_1,v_2)$$
$$=\alpha^n(u_1,\ldots,u_{2n})(ds\wedge dh)(v_1,v_2)$$
$$=\alpha_\sigma^n(u_1,\ldots,u_{2n})(ds\wedge dh)(v_1,v_2).$$

By assumption $ds\wedge dh\neq 0$ at z, hence $(ds\wedge dh)(v_1,v_2)\neq 0$. Thus α_σ is nondegenerate at z if and only if $\alpha^n\wedge ds\wedge dh\neq 0$ at z. But by (2.6),

$$\alpha^n\wedge ds\wedge dh=n!\{s,h\}\,dy\,dx,$$

where

$$dy=dy_0\cdots dy_n,\quad dx=dx^n\cdots dx^0.$$

Hence $\alpha_\sigma^n\neq 0$ at z if and only if $\{s,h\}\neq 0$ at z. ∎

We denote the family of all symplectic $2n$-submanifolds of $\mathcal{T}$ by $\mathcal{S}_0$.

Proposition 2: Let $\sigma\in\mathcal{S}_0$ and $g\in\mathcal{P}_{+}^{\uparrow}$. Then $g\sigma\in\mathcal{S}_0$ and the restriction $g:\sigma\to g\sigma$ is a canonical transformation from (σ,α_σ) onto $(g\sigma,\alpha_{g\sigma})$.

Proof: Let g^* denote the pullback map defined by g, taking forms on $g\sigma$ to forms on σ. Then the invariance of α implies

$$g^*\alpha_{g\sigma}=\alpha_\sigma, \tag{2.8}$$

thus $\alpha_{g\sigma}$ is nondegenerate, hence symplectic (it is automatically closed since α is closed). Thus $g\sigma\in\mathcal{S}_0$. To say that $g:\sigma\to g\sigma$ is canonical means precisely that α_σ and $\alpha_{g\sigma}$ are related by (2.8). ∎

We will be mainly interested in the special case where $h(z)=(y^2)^{1/2}-\lambda$ for some $\lambda>0$ and $s(z)$ depends only on x. Then $S\equiv\{x\in R^{n+1}\mid s(x)=0\}$ is an n-submanifold of space—time R^{n+1}, hence a candidate for configuration space, and $\sigma=S-i\Omega_\lambda$ where Ω_λ is the hyperboloid with $y_0=(\lambda^2+\mathbf{y}^2)^{1/2}$. The following theorem is physically significant in that it relates the pseudo-Euclidean geometry of space—time and the symplectic geometry of phase space.

Theorem 1: Let $\sigma=S-i\Omega_\lambda$ be as above. Then (σ,α_σ) is symplectic if and only if

$$\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x^\mu}\geq 0,$$

that is, S is *space-or-lightlike*.

Proof: On σ, we have

$$\{s,h\}=\frac{\partial s}{\partial x^\mu}\frac{y^\mu}{\lambda}\neq 0, \tag{2.9}$$

and we may assume $\{s,h\}$ to be positive without loss. For fixed $x\in S$, (2.9) must hold for all $y\in\Omega_\lambda$, hence for all $y\in V_+$. This implies that the vector $(\partial s/\partial x^\mu)$ is in the closure $\bar{V}_+$ of V_+, that is,

$$\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x^\mu}\geq 0.$$ ∎

We denote the class of $\sigma=S-i\Omega_\lambda$ with

$$\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x^\mu}\geq 0$$

by $\mathcal{S}_1$. $\mathcal{S}_1$ is a subfamily of $\mathcal{S}_0$ and is clearly invariant under $\mathcal{P}_{+}^{\uparrow}$.

3. REPRESENTATION IN $\mathcal{K}_\sigma$, $\sigma\in\mathcal{S}_1$

Every symplectic manifold σ has a canonically associated measure μ_σ, hence an associated complex Hilbert space $L^2(\mu_\sigma)$. Given $\sigma\in\mathcal{S}_0$, we denote the vector space of restrictions of $f \in \mathcal{K}$ to σ by $\mathcal{K}_\sigma$. In general $\mathcal{K}_\sigma$ may not be contained in $L^2(\mu_\sigma)$. In this section we prove that when $\sigma \in \mathcal{S}_1$, then $\|f\|_{L^2(\mu_\sigma)} \equiv \|f\|_\sigma = \|f\|_\lambda$ for all $f \in \mathcal{K}$ [in particular, $\mathcal{K}_\sigma$ is a closed subspace of $L^2(\mu_\sigma)$ and we have the counterparts of Theorem 2 of I and corollary 1 of I for $\mathcal{K}_\sigma$]. Hence each $\sigma \in \mathcal{S}_1$ is as good as P_λ. The proof suggests that $\mathcal{S}_1$ is the natural class of "phase spaces" for our approach.

It is remarkable, and somewhat surprising, that so large a class of phase spaces are admissible, in particular those with lightlike S. A possible application is suggested in Sec. 4.

Thus let $\sigma = S - i\Omega_\lambda \in \mathcal{S}_1$ and define the forms μ (on $\mathcal{T}$) and μ_σ (on σ) by

$$\mu = \frac{1}{n!} C_\lambda \alpha^n = C_\lambda d\hat{y}_\mu d\hat{x}^\mu, \quad \mu_\sigma = \frac{1}{n!} C_\lambda \alpha_\sigma^n, \tag{3.1}$$

where C_λ is given by (3.12) of I. We shall give a concrete expression for μ_σ. Since

$$\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x^\mu} \geq 0$$

and $ds \neq 0$ on σ, we can solve $ds = 0$ (satisfied by the *restriction* of ds to σ) for dx^0 and substitute this into $d\hat{x}^k$. This (and a similar procedure for y) gives

$$\left.\begin{aligned} d\hat{x}^\mu &= \left(\frac{\partial s}{\partial x^0}\right)^{-1} \frac{\partial s}{\partial x^\mu} d\hat{x}^0 \\ d\hat{y}_\mu &= \left(\frac{\partial h}{\partial y_0}\right)^{-1} \frac{\partial h}{\partial y_\mu} d\hat{y}_0 = y_0^{-1} y^\mu d\hat{y}_0 \end{aligned}\right\} \text{on } \sigma, \tag{3.2}$$

hence[8]

$$\mu_\sigma = C_\lambda \left(\frac{\partial s}{\partial x^0} y_0\right)^{-1} \left(\frac{\partial s}{\partial x^\mu} y^\mu\right) d\hat{y}_0 d\hat{x}^0. \tag{3.3}$$

We identify σ with R^{2n} by solving $s(x) = 0$ for $x^0 = t(\mathbf{x})$ and mapping $(\mathbf{x} - i\mathbf{y}, t(\mathbf{x}) - i(\lambda^2 + \mathbf{y}^2)^{1/2})$ to $(\mathbf{x}, \mathbf{y})$. We further identify $d\hat{y}_0 d\hat{x}^0$ with Lebesgue measure $d^n y\, d^n x$ on R^{2n} (this amounts to choosing the nonstandard orientation[9] $dy_1 \cdots dy_n dx^n \cdots dx^1$ of R^{2n}). This gives μ_σ as a measure on R^{2n}. Note that when $s(x) = x^0$ we obtain $\sigma = P_\lambda$ and $\mu_\sigma = \mu_\lambda$ [Eq. (3.11) of I]. Now $s(x) \equiv 0$ on σ implies

$$0 = \frac{\partial}{\partial x^k} s(\mathbf{x}, t(\mathbf{x})) = \frac{\partial s}{\partial x^k} + \frac{\partial s}{\partial x^0}\frac{\partial t}{\partial x^k}, \tag{3.4}$$

which can be substituted into (3.3) to give

$$\begin{aligned} \mu_\sigma &= C_\lambda \left(1 - \frac{\partial t}{\partial x^k}\frac{y^k}{y_0}\right) d^n y\, d^n x \\ &= C_\lambda \left(1 - \nabla t \cdot \frac{\mathbf{y}}{y_0}\right) d^n y\, d^n x. \end{aligned} \tag{3.5}$$

But

$$\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x^\mu} \geq 0$$

means $|\nabla t| \leq 1$, hence $\lambda > 0$ ($|\mathbf{y}/y_0| < 1$) implies that μ_σ is nondegenerate as expected. Equation (3.5) also shows that if $|\nabla t| = 1$ for some $\mathbf{x}$, μ_σ becomes "asymptotically" degenerate at $(\mathbf{x}, \mathbf{y})$ as $|\mathbf{y}| \to \infty$ in the direction of ∇t. That is, if S is lightlike at $(\mathbf{x}, t(\mathbf{x}))$, then μ_σ becomes small as the velocity $\mathbf{y}/y_0$ approaches the speed of light in the direction of ∇t. This means that functions in $L^2(\mu_\sigma)$—and in particular, as we will show, in $\mathcal{K}_\sigma$—are allowed high velocities in the direction of $\nabla t(\mathbf{x})$ at $(\mathbf{x}, t(\mathbf{x})) \in S$.

Let $\sigma \in \mathcal{S}_1$ and denote the Hilbert space of all complex-valued, measurable functions on σ with

$$\|f\|_\sigma^2 \equiv \int_\sigma |f|^2 \mu_\sigma < \infty \tag{3.6}$$

by $L^2(\mu_\sigma)$. If f is a C^∞ function on $\mathcal{T}$, we restrict it to σ and define $\|f\|_\sigma$ by (3.6). To prove that $\|f\|_\sigma = \|f\|_\lambda$ for $f \in \mathcal{K}$ we first show that each $f \in \mathcal{K}$ defines a conserved (probability) current on space–time. Let

$$j^\mu(x) = C_\lambda \int_{\Omega_\lambda} |f(x - iy)|^2 d\hat{y}_\mu, \tag{3.7}$$

where Ω_λ has the orientation defined by $d\hat{y}_0$, so that $j^0(x)$ is positive. Then

$$\|f\|_\sigma^2 = \int_S j^\mu(x)\, d\hat{x}^\mu, \tag{3.8}$$

where S is oriented by $d\hat{x}^0$ (the restriction of $d\hat{x}^0$ to S does not vanish since $|\nabla t| \leq 1$).

Theorem 2: Let $\hat{f}(\mathbf{p})$ be C^∞ with compact support. Then $j^\mu(x)$ is C^∞ and

$$\frac{\partial j^\mu}{\partial x^\mu} = 0. \tag{3.9}$$

Proof: By (3.2),

$$j^\mu(x) = C_\lambda \int_{\Omega_\lambda} y^\mu |f(x - iy)|^2 d\Omega_\lambda(y), \tag{3.10}$$

where $d\Omega_\lambda(y) = d\hat{y}_0/y_0$. The function

$$F_x^\mu(y, p, q) \equiv y^\mu \exp[ix(p - q) - y(p + q)]\overline{\hat{f}(p)}\hat{f}(q)$$

is in $L^1(\Omega_\lambda \times \Omega \times \Omega)$, hence by Fubini's theorem,

$$\begin{aligned} j^\mu(x) &= (2\pi)^{-n} C_\lambda \int_{\Omega_\lambda} d\Omega_\lambda(y) \int_{\Omega\times\Omega} d\Omega(p)\, d\Omega(q) F_x^\mu(y, p, q) \\ &= (2\pi)^{-n} C_\lambda \int_{\Omega\times\Omega} d\Omega(p)\, d\Omega(q) \exp[ix(p - q)] \\ &\quad \times \overline{\hat{f}(p)}\hat{f}(q) \int_{\Omega_\lambda} d\Omega_\lambda(y) y^\mu \exp[-y(p + q)] \\ &= (2\pi)^{-n} C_\lambda \int_{\Omega\times\Omega} d\Omega(p)\, d\Omega(q) \exp[ix(p - q)] \\ &\quad \times \overline{\hat{f}(p)}\hat{f}(q) [(p^\mu + q^\mu)/\pi] (2\pi\lambda/\eta)^{\nu+1} K_{\nu+1}(\lambda\eta), \end{aligned} \tag{3.11}$$

where $\eta = [(p + q)^2]^{1/2} \geq 2m$ and we have used (A6) of I. Differentiation under the integral sign to any order in x still gives an absolutely convergent integral since $\hat{f}$ has compact support; hence j^μ is C^∞. Differentiation with respect to x^μ brings down $i(p_\mu - q_\mu)$ from the exponent, hence (3.9) follows from $p^2 = q^2 = m^2$. ■

Remark: Equation (3.9) can also be given a geometrical argument. Let $B_\lambda = \{y \in V_+ \mid y_0 > (\lambda^2 + \mathbf{y}^2)^{1/2}\}$, oriented by $dy = dy_0 \cdots dy_n$. Then $\Omega_\lambda = -\partial B_\lambda$ (Ω_λ is oriented by $d\hat{y}_0$), hence by Stokes' theorem[9]

$$\begin{aligned} j^\mu(x) &= -C_\lambda \int_{\partial B_\lambda} |f(x - iy)|^2 d\hat{y}_\mu \\ &= -C_\lambda \int_{B_\lambda} d(|f|^2 d\hat{y}_\mu) \\ &= -C_\lambda \int_{B_\lambda} \frac{\partial |f|^2}{\partial y_\mu} dy. \end{aligned} \tag{3.12}$$

To justify the use of Stokes' theorem it must be shown that the contribution from $|\mathbf{y}| \to \infty$ to the first integral vanishes. Then (3.9) is obtained by differentiating under the integral sign (which must also be justified) and using

$$\frac{\partial^2 |f|^2}{\partial x^\mu \, \partial y_\mu} = 0, \tag{3.13}$$

which holds for $f \in \mathcal{K}$. Equation (3.13) depends upon both the holomorphy (or antiholomorphy) of f and the fact that f satisfies the Klein–Gordon equation—that is, it holds for positive-energy (or negative-energy) solutions only. For such solutions (3.13) states that $(\partial |f|^2/\partial y_\mu)$ is a "microlocal" (local in phase space) conserved space–time probability current for each fixed $y \in V_+$. Hence the scalar function $|f(z)|^2$ is a kind of "potential" for the probability current. We can now prove our main result.

Theorem 3: Let $\sigma = S - i\Omega_\lambda \in \mathcal{S}_1$ and $f \in \mathcal{K}$. Then $\|f\|_\sigma = \|f\|_\lambda$.

Remarks:

1. Properties (a)–(c) of Theorem 2 of I have counterparts for $\mathcal{K}_\sigma$.

2. As before, we obtain a resolution of the identity by polarization.

3. Let $\hat{f} \in L^2(\Omega)$, let f be the corresponding function in $\mathcal{K}$, and let f_σ be its restriction to $\sigma \in \mathcal{S}_1$. Then $\|\hat{f}\| = \|f\|_\sigma = \|f_\sigma\|_\sigma$. We will always regard $\mathcal{K}$ and $\mathcal{K}_\sigma$ as *Hilbert* spaces and *identity* $\mathcal{K}_\sigma \approx \mathcal{K} \approx L^2(\Omega)$.

Proof: We will prove that $\|f\|_\sigma = \|f\|_\lambda$ when $\hat{f}(\mathbf{p}) \in \mathcal{D}(R^n)$, which implies the result for arbitrary $\hat{f} \in L^2(\Omega)$ by continuity. Let S be given by $x_0 = t(\mathbf{x})$, and let

$$D_R = \{x \in R^{n+1} \mid |\mathbf{x}| < R,\ x_0 \in [0, t(\mathbf{x})]\},$$

$$E_R = \{x \in R^{n+1} \mid |\mathbf{x}| = R,\ x_0 \in [0, t(\mathbf{x})]\},$$

$$S_{0R} = \{x \in R^{n+1} \mid |\mathbf{x}| < R,\ x_0 = 0\},$$

$$S_R = \{x \in R^{n+1} \mid |\mathbf{x}| < R,\ x_0 = t(\mathbf{x})\},$$

where $[0, t(\mathbf{x})]$ means $[t(\mathbf{x}), 0]$ if $t(\mathbf{x}) < 0$. We orient S_{0R} and S_R by $d\hat{x}_0$, E_R by the "outward normal"

$$\hat{r} = \frac{1}{R} \sum_{k=1}^{n} x^k \, d\hat{x}^k, \tag{3.14}$$

and D_R so that $\partial D_R = S_R - S_{0R} + E_R$. Now let $\hat{f}(\mathbf{p}) \in \mathcal{D}(R^n)$. Then j^μ is C^∞, hence by Stokes' theorem,

$$\left[\int_{S_R} - \int_{S_{0R}} + \int_{E_R}\right] j^\mu \, d\hat{x}^\mu = \int_{D_R} d(j^\mu \, d\hat{x}^\mu) = (-)^n \int \frac{\partial j^\mu}{\partial x^\mu} \, dx = 0.$$

We will show that

$$\Delta(R) \equiv \int_{E_R} j^\mu \, d\hat{x}^\mu \to 0 \quad \text{as } R \to \infty, \tag{3.15}$$

which implies that

$$\|f\|_\sigma^2 \equiv \lim_{R\to\infty} \int_{S_R} j^\mu \, d\hat{x}^\mu = \lim_{R\to\infty} \int_{S_{0R}} j^\mu \, d\hat{x}^\mu \equiv \|f\|_\lambda^2.$$

To prove (3.15), note that on E_R, $d\hat{x}^0 = 0$ and

$$d\hat{x}^k = x^k \frac{d\hat{x}^1}{x^1} = x^k \frac{d\hat{x}^2}{x^2} = \cdots = x^k \frac{d\hat{x}^n}{x^n}, \tag{3.16}$$

each form being defined except on a set of measure zero; hence $\hat{r} = R\, d\hat{x}^1/x^1$. By (3.10), $|j^k(x)| \le j^0(x)$, hence

$$|\Delta(R)| = \left|\sum_{k=1}^{n} \int_{E_R} j^k \, d\hat{x}^k\right| = \left|\sum_{k=1}^{n} \int_{E_R} j^k x^k \frac{d\hat{x}^1}{x^1}\right| \le n \int_{E_R} j^0(x) R \frac{d\hat{x}^1}{x^1} = n \int_{E_R} j^0(x) \hat{r} \equiv a(R). \tag{3.17}$$

Now by (3.11)

$$j^0(x) = \int_{R^{2n}} d^n p \, d^n q \, \exp[ix(p-q)] \phi(\mathbf{p}, \mathbf{q}), \tag{3.18}$$

where

$$\phi(\mathbf{p}, \mathbf{q}) = (2\pi)^{-n} C_\lambda \overline{\hat{f}(\mathbf{p})} \hat{f}(\mathbf{q}) \left(\frac{p_0 + q_0}{\pi p_0 q_0}\right) \left(\frac{2\pi\lambda}{\eta}\right)^{\nu+1} K_{\nu+1}(\lambda\eta),$$

with $\eta \equiv [(p+q)^2]^{1/2} \ge 2m$. Let D be the operator $\hat{x} \cdot \nabla_p$, where $\hat{x} = \mathbf{x}/R$, and observe that (for $x \in E_R$)

$$D \exp(ixp) = -iR\left(1 - \frac{x_0}{R} \hat{x} \cdot \mathbf{v}\right) \exp(ixp) = -iR \xi(x, \mathbf{p}) \exp(ixp), \tag{3.19}$$

where $\mathbf{v} = \mathbf{p}/p_0$. Since $\phi \in \mathcal{D}(R^{2n})$, there is a constant $\alpha < 1$ such that $|\mathbf{v}| \le \alpha$ for all $\mathbf{p} \in \operatorname{supp}\phi$. Furthermore, since $|\nabla t| \le 1$, given any $\epsilon > 0$ we have $|x_0| < R(1+\epsilon)$ for $x \in E_R$ with R large enough; hence

$$|\xi(x, \mathbf{p})| \ge 1 - \alpha(1+\epsilon), \quad x \in E_R, \quad \mathbf{p} \in \operatorname{supp}\phi. \tag{3.20}$$

Choose $0 < \epsilon < 1/\alpha - 1$, substitute

$$\exp(ixp) = \frac{i}{R\xi(x, p)} D \exp(ixp), \quad x \in E_R \tag{3.21}$$

into (3.18) and integrate by parts:

$$j^0(x) = \frac{1}{iR} \int_{R^{2n}} d^n p \, d^n q \, \exp[ix(p-q)] D\left(\frac{\phi(\mathbf{p}, \mathbf{q})}{\xi(x, \mathbf{p})}\right) \equiv \frac{1}{iR} \int_{R^{2n}} d^n p \, d^n q \, \exp[ix(p-q)] \phi_x^1(\mathbf{p}, \mathbf{q}). \tag{3.22}$$

This process can be continued, giving (for $x \in E_R$)

$$j^0(x) = (iR)^{-N} \int_{R^{2n}} d^n p \, d^n q \, \exp[ix(p-q)] \phi_x^N(\mathbf{p}, \mathbf{q}), \quad N = 1, 2, \cdots, \tag{3.23}$$

where

$$\phi_x^N(\mathbf{p}, \mathbf{q}) = \left(D \frac{1}{\xi}\right)^N \phi(\mathbf{p}, \mathbf{q}) \equiv \left(\hat{x} \cdot \nabla_p \left(1 - \frac{x_0}{R} \hat{x} \cdot \mathbf{v}\right)^{-1}\right)^N \phi(\mathbf{p}, \mathbf{q}). \tag{3.24}$$

Now $[D(1/\xi)]^N$ is a partial differential operator in $\mathbf{p}$ whose coefficients are polynomials in $D^k(1/\xi)$, $k = 0, 1, \ldots, N$. We will show that for $x \in E_R$ with R sufficiently large, there are constants b_k such that

$$|D^k(1/\xi)| < b_k, \quad k = 0, 1, 2, \cdots \tag{3.25}$$

which implies that

$$\|\phi_x^N\|_{L^1(R^{2n})} < C_N, \quad x \in E_R, \quad N = 1, 2, \cdots \tag{3.26}$$

for constants C_N, so that by (3.17) and (3.23),

$$a(R) = n \int_{E_R} j^0 \hat{r} \le nR^{-N} \int_{E_R} \|\phi_x^N\|_{L^1(R^{2n})} \hat{r}(x) \le nR^{-N} C_N \cdot \frac{2\pi^{n/2}}{\Gamma(n/2)} R^{n-1} \int_0^{R(1+\epsilon)} dx_0 = \frac{2n\pi^{n/2}}{\Gamma(n/2)} C_N R^{n-N}(1+\epsilon) \to 0 \quad \text{as } R \to \infty$$

if we choose $N > n$. To prove (3.25), note that it holds for $k=0$ by (3.20) and let $u = \hat{x} \cdot \mathbf{v}$. Then

$$Du \equiv (\hat{x} \cdot \nabla_p)(\hat{x} \cdot \mathbf{p}/p_0) = \frac{1-u^2}{p_0},$$

and if

$$D^k u = P_k(u)/p_0^k, \tag{3.27}$$

where P_k is a constant-coefficient polynomial, then

$$D^{k+1}u = \frac{P_k'(u)Du}{p_0^k} - \frac{kP_k(u)}{p_0^{k+1}} \cdot u \equiv \frac{P_{k+1}(u)}{p_0^{k+1}};$$

hence (3.27) holds for $k = 1, 2, \cdots$ by induction. Thus

$$D^k \xi = -\frac{x_0}{R} D^k u = -\frac{x_0}{R} \frac{P_k(u)}{p_0^k}, \quad k = 1, 2, \cdots,$$

which implies

$$|D^k \xi| \leq \frac{1+\epsilon}{m^k} \max_{|u| \leq 1} |P_k(u)|. \tag{3.28}$$

But $D^k(1/\xi)$ is a polynomial in $1/\xi$ and $D\xi, D^2\xi, \ldots, D^k\xi$, hence (3.25) follows from (3.20) and (3.28). ∎

4. DISCUSSION

1. The phase-space approach appears to resolve some difficulties[10,11] encountered in the usual (space—time) theory of Klein—Gordon particles, which goes as follows: The counterpart of $\mathcal{K}$ is the space $\mathcal{H}$ of *boundary values* $f(x)$ of $f(x-iy)$, $f \in \mathcal{K}$, as $y \to 0$ in V_+. For a given spacelike surface $S \subset R^{n+1}$, the norm in $\mathcal{H}$ is defined by[12]

$$\|f\|_S^2 = \int_S J^\mu(x)\, d\hat{x}^\mu, \tag{4.1}$$

$$J^\mu(x) = -\operatorname{Im}\left\{\overline{f(x)} \frac{\partial f(x)}{\partial x_\mu}\right\}. \tag{4.2}$$

The current J^μ satisfies the continuity equation (3.9), ensuring that $\|f\|_S$ is independent of S. Now the Newton—Wigner postulates[13] for "localized states" uniquely determine these states (at time $x_0 = 0$) to be

$$\psi_{\mathbf{x}}(\mathbf{p}) = (2\pi)^{-n/2}\sqrt{\omega}\exp(-i\mathbf{x}\cdot\mathbf{p}) \tag{4.3}$$

[these are the generalized eigenvectors of the position operators X_k of Eq. (4.1) of I]. Hence the configuration-space probability density at time $x_0 = 0$ is given by

$$\rho(\mathbf{x}) = |\langle \psi_{\mathbf{x}} | \hat{f} \rangle_{L^2(\Omega)}|^2 = (2\pi)^{-n} \iint \frac{d^n p\, d^n p'}{\sqrt{\omega\omega'}} \overline{\hat{f}(p)} \hat{f}(p') \exp[i\mathbf{x}\cdot(\mathbf{p}'-\mathbf{p})]. \tag{4.4}$$

This expression does not coincide with

$$J^0(x) = \tfrac{1}{2}(2\pi)^{-n} \iint d^n p\, d^n p' \left(\frac{1}{\omega} + \frac{1}{\omega'}\right) \overline{\hat{f}(p)} \hat{f}(p') \times \exp[i\mathbf{x}\cdot(\mathbf{p}'-\mathbf{p})], \tag{4.5}$$

which is the probability density associated with the current $J^\mu(x)$. In fact, $J^0\rho(x)$ cannot be the time component of *any* space—time vector field (which shows once more[13] that sharp localization, in the sense of Newton and Wigner, is incompatible with relativistic covariance). This is the first difficulty. The second difficulty is that even if one gives up the notion of localized states as a fundamental concept, $J^\mu(x)$ cannot be accepted as the probability current since it turns out that $J^0(x)$ can be negative.[11] (Even for positive-energy solutions, i.e., $f \in \mathcal{H}$, for which $\|f\|_S$ is actually positive definite!)

By contrast, our expression $j^0(x)$ is nonnegative as well as being the time component of a vector field; hence the second difficulty is clearly resolved in the phase-space framework. As for the first, note that if "sharp" localization in space is replaced with "soft" localization in phase space—i.e., replace the $\psi_{\mathbf{x}}$, which satisfy $\langle \psi_{\mathbf{x}'} | \psi_{\mathbf{x}} \rangle_{L^2(\Omega)} = \delta(\mathbf{x}' - \mathbf{x})$, by the e_z, $z \in \sigma$ for some $\sigma \in \mathcal{S}_1$—then we obtain

$$\tilde{\rho}(z) \equiv |\langle e_z | \hat{f} \rangle_{L^2(\Omega)}|^2 = |f(z)|^2, \quad f \in \mathcal{K} \tag{4.6}$$

as the probability density (with respect to μ_σ) in phase space. This, as we have seen, *is* compatible with (in fact, gives rise to!) the current $j^\mu(x)$. The price of this compatibility is that $j^0(x)$ can no longer be regarded as a *sharp* probability density but, rather, is the average of the "soft" density $\tilde{\rho}(z)$ over the "mass shell" Ω_λ. Accordingly, $j^\mu(x)$ depends on the parameter λ which measures, very roughly, the extent of spatial smearing associated with the continuous basis e_z, $z \in \sigma \subset P_\lambda'$.

2. In I we have seen that $\lambda = 0$ can be admitted as a limiting value in the relation $\|f\|_\lambda = \|\hat{f}\|$, provided f is interpreted as a boundary-value function. Similar considerations apply to general $\sigma \subset P_\lambda'$, $\sigma \in \mathcal{S}_1$, when $\lambda \to 0$. Thus the family $\mathcal{S}_1$ could be slightly enlarged. Note that the expression (3.5) for μ_σ shows that even when $\lambda = 0$ (i.e., $|\mathbf{y}/y_0| = 1$), μ_σ is nondegenerate except at those $(\mathbf{x}, \mathbf{y})$ for which (a) $|\nabla t| = 1$ (i.e., S is lightlike at $\mathbf{x}$) and (b) $\mathbf{y}/y_0 = \nabla t$. This set is of measure zero in R^{2n}, and on it $f \in \mathcal{K}$ may develop singularities (caustics).

3. In recent years there has been progress in the quantization of field theories on surfaces other than $x_0 = \text{const.}$[14,15] In particular, the so-called "lightlike quantization" uses surfaces which are everywhere lightlike, and appears to have practical applications.[14] The transition from $x_0 = \text{const}$ to lightlike surfaces appears to present mathematical problems[15]; possibly the present formalism, when extended to quantum field theory, can be of help.

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APPENDIX

We give here a brief description of the pullback map, used throughout this paper.

Given two manifolds M and N of dimensions m and n respectively, a differentiable mapping $g: M \to N$ can be expressed locally as $g: U \to V$ where U and V are open subsets in R^m and R^n, respectively. Then the differential map g_* maps each tangent space M_x, $x \in M$, to the tangent space N_{gx} at $gx \in N$ and is given in local coordinates by

$$g_*^i(\xi)=A_j^i(x)\xi^j,\quad A_j^i\equiv\frac{\partial g^i}{\partial x^j}. \tag{A1}$$

The pullback map g^* takes the dual N_{gx}^* of N_{gx} to the dual M_x^* of M_x as follows: The linear form $p:N_{gx}\to R$ is mapped to the linear form $g^*p:M_x\to R$ defined by

$$(g^*p)(\xi)=p(g_*\xi). \tag{A2}$$

A 2-form on N is a bilinear, skew-symmetric mapping $\alpha:N_y\times N_y\to R$ on each tangent space N_y, $y\in N$. The map g defines a map (also denoted by g^*) taking 2-forms on N to 2-forms on M, as follows,

$$(g^*\alpha)(\xi,\xi')=\alpha(g_*\xi,g_*\xi'),\quad \xi,\xi'\in M_x. \tag{A3}$$

In case M is a submanifold of N and g denotes the inclusion map, we have $A_j^i(x)\equiv\delta_j^i$, $x\in M$; hence $g^*\alpha$ is the restriction of α to vectors tangent to M.

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Unitary analytic representations of $\overline{SL(3,R)}$ and hadronic Regge sequences[a)]

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The unitary, analytic representations of the covering group $\overline{SL(3,R)}$ are determined in the space of functions $H(z)$ which are derived from homogeneous functions of degree $-b_1$. Four Regge trajectories, $\{N\}$, $\{\rho\}$, $\{\pi\}$, and $\{\Delta\}$, are associated with these representations.

I. INTRODUCTION

This paper is a continuation of a previous work,[1] in which two-different principal series of representations of the noncompact group SL(3, R) were obtained. The role of the unitary representations of SL(3, R) in physics was mentioned there. One of the interesting applications was achieved by Biedenharn, Cusson, Han, and Weaver.[2] They used the Jordan–Schwinger boson construction to obtain four primitive representations of SL(3, R), and they associated these four representations with four known Regge trajectories $\{\Pi\}=(j=0,2,4,\cdots)$ $\{\rho\}=(j=1,3,5,\cdots)$, $\{N\}=(j=\frac{1}{2},\frac{5}{2},\frac{9}{2},\cdots)$, and $\{\Delta\}=(j=\frac{3}{2},\frac{7}{2},\frac{11}{2},\cdots)$. Besides, Šijački[3] obtained three Regge trajectories $\{\Pi\}$, $\{\rho\}$, and $\{N\}$ as the multiplicity-free representations of SL(3, R). The aim of the present paper is to extend the work done in Ref. 1 to obtain boson and fermion Regge trajectories.

II. THE LIE ALGEBRA OF SL(3,R) AND THE CONSTRUCTION OF Z OPERATORS

The Lie algebra sl(3, R) of SL(3, R) consists of 3×3 real traceless matrices. We will use the Cartan decomposition of an sl(3, R) matrix to introduce the maximal compact subalgebra su(2). The Cartan decomposition of an sl(3, R) matrix is the decomposition of a traceless real matrix into the sum of an antisymmetric and a symmetric matrix. Antisymmetric matrices form the maximal compact subalgebra su(2). This decomposition corresponds to the decomposition of an SL(3, R) matrix in to the product of an SU(2) and a symmetric matrix.

In this paper we will use the eight generators of sl(3, R) in the form that they are used in Ref. 3. Matrix representations of the generators in three dimensions are given as

$$J_0=\begin{pmatrix}0&-i&0\\ i&0&0\\ 0&0&0\end{pmatrix},$$

$$J_{\pm}=J_x\pm iJ_y=\begin{pmatrix}0&0&\pm1\\ 0&0&-i\\ \pm1&i&0\end{pmatrix},$$

$$T_0=-i(\tfrac{2}{3})^{1/2}\begin{pmatrix}1&0&0\\ 0&1&0\\ 0&0&-2\end{pmatrix},$$
$$T_{\pm1}=\begin{pmatrix}0&0&\pm i\\ 0&0&1\\ \pm i&1&0\end{pmatrix},\qquad T_{\pm2}=\begin{pmatrix}i&\pm1&0\\ \pm1&-i&0\\ 0&0&0\end{pmatrix} \tag{1}$$

where the generators J_0, $J_\pm$ form the su(2) subalgebra. In this representation commutation relations are

$$\begin{aligned}
&[J_0, J_\pm]=\pm J_\pm, && [T_0, T_{+1}]=6^{1/2}J_+,\\
&[J_+, J_-]=2J_0, && [T_0, T_{-1}]=6^{1/2}J_-,\\
&[J_0, T_\mu]=\mu T_\mu \quad (\mu=0,\pm1,\pm2), && [T_{+1}, T_{-1}]=2J_0,\\
&[J_\pm, T_\mu]=[6-\mu(\pm1)]^{1/2}T_{\mu\pm1}, && [T_{+1}, T_{-2}]=-2J_-,\\
&[T_{-2}, T_{+2}]=4J_0, && [T_{-1}, T_{+2}]=-2J_+.
\end{aligned} \tag{2}$$

Since the representation of the generators is different from those of Ref. 1, [Eq. (1)] one should reconstruct the matrix Ω and check the commutation relations of its elements using the generators J_0, $J_\pm$, and T_μ ($\mu=0,\pm1,\pm2$). This is essential for the construction of Z operators.

The nonzero elements of 8×8 metric matrix F are

$$F_{11}=\tfrac{1}{12},\quad F_{13}=\tfrac{1}{24},\quad F_{44}=-\tfrac{1}{24},\quad F_{56}=\tfrac{1}{24},\quad F_{78}=-\tfrac{1}{24}, \tag{3}$$

Determination of the matrix Ω with operator entries and satisfying the equation

$$U\Omega U^{-1}=\Lambda\Omega\Lambda^{-1} \tag{4}$$

is the starting point of our construction. Here U is any representation of SL(3, R) and Λ is its 3×3 representation.

The matrix Ω defined as

$$\Omega=\gamma_h F_{hl} M_l \quad (h, l=1,\ldots,8) \tag{5}$$

satisfies Eq. (4) in which γ_h are the generators of 3×3 representation of sl(3, R) just as given in (1) and M_l are the generators of any unitary representation of sl(3, R). The method to determine the Z_j ($j=1,2$) operators is completely similar to the one used in Ref. 1. The method will be just summarized for the sake of continuity.

The explicit form of the matrix Ω is

$$\Omega=\begin{pmatrix}i(\sqrt{\tfrac{2}{3}}M_4-M_8-M_7) & -2iM_1+M_8-M_7 & i(M_5-M_6)+M_2-M_3\\ 2iM_1+M_8-M_7 & i(\sqrt{\tfrac{2}{3}}M_4+M_8+M_7) & -i(M_2+M_3)+M_5+M_6\\ i(M_5-M_6)+M_3-M_2 & i(M_2+M_3)+M_5+M_6 & -2i\sqrt{\tfrac{2}{3}}M_4\end{pmatrix}. \tag{6}$$

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The elements Ω_{hj} satisfy the commutation relations

$$[\Omega_{hj}, \Omega_{kl}] = 4(\delta_{hl}\Omega_{kj} - \delta_{kj}\Omega_{hl}). \quad (7)$$

Let

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

be the eigenvector of Ω satisfying the eigenvalue equation

$$\Omega\psi = 2\lambda\psi. \quad (8)$$

The transformation law for the eigenvector ψ is

$$U\psi U^{-1} = \psi' = \Lambda^{-1}\psi\, C(\Lambda), \quad (9)$$

where $C(\Lambda)$ is a diagonal matrix.

Defining two operators Z_1 and Z_2 as

$$Z_1 = \psi_1\psi_3^{-1}, \quad Z_2 = \psi_2\psi_3^{-1} \quad (10)$$

and using the transformation law (9), we deduce

$$UZ_1U^{-1} = Z_1' = (\Lambda_{11}^{-1}Z_1 + \Lambda_{12}^{-1}Z_2 + \Lambda_{13}^{-1})(\Lambda_{31}^{-1}Z_1 + \Lambda_{32}^{-1}Z_2 + \Lambda_{33}^{-1})^{-1},$$

$$UZ_2U^{-1} = Z_2' = (\Lambda_{21}^{-1}Z_1 + \Lambda_{22}^{-1}Z_2 + \Lambda_{23}^{-1})(\Lambda_{31}^{-1}Z_1 + \Lambda_{32}^{-1}Z_2 + \Lambda_{33}^{-1})^{-1}. \quad (11)$$

III. CONSTRUCTION OF THE REPRESENTATION SPACE

Unitary irreducible representations of $\overline{SL(3, R)}$ will be labeled by the real eigenvalues of two Casimir operators C_1 and C_2. Defining C_1 and C_2 as

$$C_1 = \mathrm{Tr}\Omega^2, \quad C_2 = \mathrm{Tr}\Omega^3 \quad (12)$$

and letting the eigenvalues of Ω be $\lambda_1 = \frac{1}{2}(a_1 + ib_1)$, $\lambda_2 = \frac{1}{2}(a_2 + ib_2)$, we are able to label the representations in terms of λ_1 and λ_2. Unitarity condition gives mainly two classes of representations:

$$\begin{aligned} &(a)\ \lambda_1 = \tfrac{1}{2}(a_1 + ib_1), \qquad &(b)\ \lambda_1 = a_1, \\ &\quad\ \lambda_2 = \tfrac{1}{2}(a_1 - ib_1), &\lambda_2 = a_2, \\ &\quad\ \lambda_3 = -a_1, &\lambda_3 = -a_1 - a_2, \end{aligned} \quad (13)$$

where $b_1 = b_1' + j + \frac{1}{2}$ ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$). The choice of b_1 in this form enables us to introduce the Casimir operator $J^2 = j(j+1)$ of the su(2) subalgebra as a part of C_1.

The common eigenstates $|z(2\lambda_j), \lambda_1, \lambda_2, k\rangle$ of commuting operators $Z(2\lambda_j)$, C_1, C_2, and $K = \exp(-2\pi i j_0)$ are taken as the basis of the representation space. The discrete operator K commutes with every generator of $\overline{sl(3, R)}$. This can easily be shown by using the equation

$$KM_jK^{-1} = M_j + [2\pi i J_0, M_j] + \cdots \quad (14)$$

and the commutation relations (2). Besides

$$KZ_1(2\lambda_j)K^{-1} = Z_1(2\lambda_j), \quad KZ_2(2\lambda_j)K^{-1} = Z_2(2\lambda_j). \quad (15)$$

Since the eigenvalues of matrix J_0 are $0, \frac{1}{2}, 1, \cdots$, the eigenvalues k of K are ± 1.

In order to determine the transformation law of the eigenstate $|z(2\lambda_j), \lambda_1, \lambda_2, k\rangle$, all the generators M_j should be written in terms of canonically conjugate operators Π_k and Z_k in a such a way that they will have proper commutation relations.

Commutation relations of elements Ω_{hj} and Z_k enable us to express the generators M_j in terms of Ω_{hj} provided commutation relations (1) hold. The explicit form of Ω gives all M_1 in terms of Ω_{hj}. In fact,

$$\begin{aligned} &M_1 + (1/4i)(\Omega_{21} - \Omega_{12}), \\ &M_2 = (1/4i)[\Omega_{32} - \Omega_{23} + i(\Omega_{13} - \Omega_{31})], \\ &M_3 = (1/4i)[\Omega_{32} - \Omega_{23} - i(\Omega_{13} - \Omega_{31})], \\ &M_4 = (i/2\sqrt{\tfrac{2}{3}})\Omega_{33}, \\ &M_5 = (1/4i)[\Omega_{13} + \Omega_{31} + i(\Omega_{23} + \Omega_{32})], \\ &M_6 = (1/4i)[-(\Omega_{13} + \Omega_{31}) + i(\Omega_{23} + \Omega_{32})], \\ &M_7 = (1/4i)[(\Omega_{22} - \Omega_{11}) - i(\Omega_{12} + \Omega_{21})], \\ &M_8 = (1/4i)[(\Omega_{22} - \Omega_{11}) + i(\Omega_{12} + \Omega_{21})]. \end{aligned} \quad (16)$$

The procedure to determine the transformation law of the eigenstate $|z(2\lambda_j), \lambda_1, \lambda_2, k\rangle$ is completely similar to the procedure which was followed in Ref. 1. Hence the detailed calculations will be omitted, and the transformation law of functions $f(z, \eta)$ will be given:

$$U(\Lambda)f(z, \eta) = \left[\frac{\beta z + \delta}{|\alpha\delta - \beta\gamma|}\right]^{-b_1} |\alpha\delta - \beta\gamma|^{-i\eta_1} f(z', \eta'), \quad (17)$$

where η is a real variable. Analyticity imposes the condition that b_1 is a positive integer n. In the Hilbert space of functions $f(z, \eta)$ an invariant scalar product is given as

$$(f_1, f_2) = c\int f_1(z, \eta) f_2^*(z, \eta)\, |\mathrm{Im} z|^{n-2}\, dx\, dy\, d\eta. \quad (18)$$

Hence, we have shown that the Cartan decomposition of sl(3, R) algebra also allows us to construct Z operators such that the analytic functions $f(z, \eta)$ which form the representation space transform like (17) provided $b_1 = b_1' + j + \frac{1}{2} = n$

IV. HOMOGENEOUS FUNCTIONS AND REGGE SEQUENCES

Now, let us consider the space of functions[4] $H(\psi_1, \psi_2, \psi_3)$ homogeneous of degree $-b_1$. By definition

$$H(\xi\psi_1, \xi\psi_2, \xi\psi_3) = \xi^{-b_1} H(\psi_1, \psi_2, \psi_3), \quad (19)$$

where ξ, ψ_1, ψ_2, ψ_3 are real. The homogeneous functions $H(\psi_1, \psi_2, \psi_3)$ are the eigenfunctions of the discrete operator K with eigenvalues ± 1. Hence

$$KH(\psi_1, \psi_2, \psi_3) = H(-\psi_1, -\psi_2, -\psi_3) = H(\psi_1, \psi_2, \psi_3) \quad (20)$$

or

$$KH(\psi_1, \psi_2, \psi_3) = H(-\psi_1, -\psi_2, -\psi_3) = -H(\psi_1, \psi_2, \psi_3); \quad (21)$$

letting $\xi = |\xi|\ \mathrm{sgn}\xi$ and using Eq. (19), we obtain

$$H(\xi\psi_1, \xi\psi_2, \xi\psi_3) = |\xi|^{-b_1} H(\mathrm{sgn}\xi\psi_1, \mathrm{sgn}\xi\psi_2, \mathrm{sgn}\xi\psi_3). \quad (22)$$

For $\mathrm{sgn}\xi < 0$ the above equation becomes

$$H(\xi\psi_1, \xi\psi_2, \xi\psi_3) = |\xi|^{-b_1}(\mathrm{sgn}\xi)^{\epsilon} H(\psi_1, \psi_2, \psi_3), \quad (23)$$

where $\epsilon = 0, 1$ are the eigenvalues of the operator $P = (1-K)/2$; letting $\xi = 1/\psi_3$, we obtain function

$H(z_1, z_2) \equiv H(z)$ in terms of homogeneous functions $H(\psi_1, \psi_2, \psi_3)$. In fact

$$H(\psi_1, \psi_2, \psi_3) = |\psi_3|^{-b_1} (\operatorname{sgn}\psi_3)^{\epsilon} H(z). \tag{24}$$

Transformation properties of ψ_1, ψ_2, and ψ_3 makes it possible to obtain the transformation of $H(z)$:

$$H(\psi_1', \psi_2', \psi_3') = |\psi_3'|^{-b_1} (\operatorname{sgn}\psi_3') H(z'). \tag{25}$$

Noting $\Lambda_{31}^{-1} z_1 + \Lambda_{32}^{-1} z_2 + \Lambda_{33}^{-1} = \beta z + \delta$ and doing necessary calculations, we obtain

$$U(\Lambda^{-1}) H(z) = (\beta z + \delta)^{-b_1} \operatorname{sgn}(\beta z + \delta)^{\epsilon - b_1} H(z'). \tag{26}$$

Since ψ_1, ψ_2, and ψ_3 are taken real, $H(z)$ is a real variable function. But, assuming analytic continuation is possible, $H(z)$ should be defined on the complex space.

The use of the Hilbert space of homogeneous functions as the representation space introduces additional $[\operatorname{sgn}(\beta z + \delta)]^{-b_1}$ term into the transformation law which destroys the analyticity. Therefore, we will remove the sign function by taking

$$[\operatorname{sgn}(\beta z + \delta)]^{b_1} = [\operatorname{sgn}(\beta z + \delta)]^{\epsilon}. \tag{27}$$

Removal of the sign function imposes two conditions on b_1: for $\epsilon = 0, 1$

(a) $b_1 = b_1' + j + \frac{1}{2}$

$= \text{even integers except zero} \quad \text{(fermion)}$ (28)

(b) $b_1 = b_1' + j + \frac{1}{2}$

$= \text{odd integers} \quad \text{(boson)}.$

Since J can take values $0, \frac{1}{2}, 1, \cdots$, b_1' can only take the values $0, \frac{1}{2}, 1, \cdots$. Therefore, conditions (28) give the following four sets of J values which are associated with four Regge trajectories.

(1) $J = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$ for $\epsilon = 0$, $b_1' = 1$, $\{N\}$ trajectory,

(2) $J = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \cdots$ for $\epsilon = 0$, $b_1' = 0$, $\{\Delta\}$ trajectory,

(3) $J = 0, 2, 4, \cdots$ for $\epsilon = 1$, $b_1' = \frac{1}{2}$, $\{\pi\}$ trajectory,

(4) $J = 1, 3, 5, \cdots$ for $\epsilon = 1$, $b_1' = \frac{3}{2}$, $\{\rho\}$ trajectory.

V. CONCLUDING REMARKS

Unitary, analytic representations of the covering group $\overline{\mathrm{SL}(3, R)}$ are determined using operator formalism. Representations in the space of analytic functions $H(z)$ which are derived from homogeneous functions $H(\psi_1, \psi_2, \psi_3)$ coincide with special representations $(a_1 = 0)$. The Cartan decomposition of Lie algebra $\mathrm{sl}(3, R)$ enables us to introduce the maximal compact subalgebra su(2). The analyticity requirement of functions $H(z)$ leads to fermion ($\epsilon = 0$) and boson ($\epsilon = 1$) Regge trajectories which were obtained in Refs. 2 and 3 using different methods.

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The SU(4)⊃SU(2) ⊗ SU(2) chain

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We consider the physical SU(4)⊃SU(2) ⊗ SU(2) chain. With the help of the three vectors (S_a, T_i, Q_{ai}), (V_a^S, V_i^T, V_{ai}^Q), (W_a^S, W_i^T, W_{ai}^Q) we can easily write the three fundamental invariants I_2, I_3, I_4; the two operators Ω and Φ (eigenvalues ω and φ) of Moshinsky and Nagel [Phys. Lett. **5**, 173 (1963)] and we are able to construct a new pair of operators playing the same last role. We write them $s = \Sigma_\mu(-)^\mu V_\mu^S V^s_{-\mu}$ and $t = \Sigma_i(-)^i V_i^T V_{-i}^T$ (eigenvalues σ and τ). In any $(pp'p'')$ irreducible representation (IR) we define semireduced matrix elements $\langle\theta' S'T' \| Q \| \theta ST\rangle$ (and the same for V^Q and W^Q), where θ is any of the two pairs $(\omega\varphi)$ or $(\sigma\tau)$. For the unknown $\Sigma_{\theta\theta'}\langle\theta' S'T' \| T_1^Q \| \theta ST\rangle \langle\theta' S'T' \| T_2^Q \| \theta ST\rangle$ (T^Q any one among Q, V^Q or W^Q) we give a set of equations which allows the complete solution of the calculation of these quantities. Besides we give the explicit values of these unknown for $T_1^Q = T_2^Q = Q$ for any $(pp'p'')$ IR and, $S=p$ and any T, or $T=p$ and any S. In the particular case of the (320) IR [with $\theta = (\omega\varphi)$] where the multiplicity reaches 3 for $S=1$, $T=2$ and $S=2$, $T=1$, we have completely solved the problem; i.e., we have calculated the square of the semireduced matrix elements of Q for any value of the label $(\omega\varphi ST)$, together with all the eigenvalues ω and φ.

I. INTRODUCTION

Since 1937, the year when Wigner[1] introduced the SU(4) group in physics, together with the concept of supermultiplet, the unimodular unitary groups SU(n) have accounted for many symmetries in such areas as atomic and nuclear spectroscopy and elementary particles.[2,3] These SU(n) groups are nowadays a fundamental tool in the hands of theorists and they have been the subject of many a work.[4] Their study has led to important results when the labeling of the states is built on the canonical chain of unitary groups SU(n) ⊃ U($n-1$) ⊃ ⋯ ⊃ U(1). Unfortunately—for SU(4)—one often has to use a noncanonical classification of the states if one wants to exhibit the quantities of physical interest. Most of the results previously obtained are no longer adapted and the study of these noncanonical group chains is much poorer and faces numerous difficulties.

In the case of SU(4) the physically interesting group chain is the chain SU(4)⊃SU(2)⊗SU(2)—often called physical chain—which displays explicitly the two spin and isospin SU(2) subgroups. This chain does not allow a complete labeling, as the two subgroups give only four quantum numbers: The spin S with its projection M_S, the isospin T and its projection M_T. We need two more labels to have a complete description of the states of a given $(pp'p'')$ irreducible representation (IR). This problem can be solved in many ways: by a projection technique,[5] by the so-called "elementary multiplet" method,[6] or by the diagonalization of a complete set of commuting operators.[7] This last method is particularly interesting to the physicist for several reasons: It is the only one which gives an orthonormal basis of the states; it allows a certain liberty in the choice of the complete set of commuting operator, it corresponds to a technique appropriate to quantum mechanics and the state labeling problem sometimes corresponds to a search for eigenvalues of quantum mechanics. A lot of work has been devoted these last years to the quest for convenient operators. In the SU(4)⊃SU(2)⊗SU(2) case the number of missing labels is two, and the number of functionally independent missing label operators is twice the number of missing labels, i.e., four.[8] In 1963, Moshinsky and Nagel[9] gave the expression of a pair of such operators which we call, after them, Ω and Φ. We shall see that we can just as well take another pair, which we call s and t (Sec. III).[16]

This paper gives the main results of work concerning the calculation of the matrices of the generators of SU(4) in a given $(pp'p'')$ IR in which the states are labeled by the spin quantum numbers S, M_S, the isospin quantum numbers T, M_T, and the eigenvalues of one of the pairs (Ω, Φ) or (s, t), or some other pair to be discovered. We call this base the physical base in what follows. In order to fulfill this program, one can for example search for a unitary transformation allowing one to go from the Gel'fand basis to the physical one. This method has the defect of being often very technical and not very clear. We have done an algebraic infinitesimal study of the group chain SU(4)⊃SU(2)⊗SU(2) without prior knowledge other than of the physical Lie algebra of SU(4). This process does not require deep knowledge of group theory, and demands no particular calculus technique but the Wigner—Racah SU(2) algebra. This program may appear to be either too ambitious or rather vague; this is only partially true. In fact, in SU(4) one is often limited by the feasibility of the calculation which forbids a complete algebraic solution and obliges one to use computers. Consideration of an adapted base from the beginning makes the calculation simpler, and as it was essentially our goal, this adapted basis has the advantage of emphasizing the particular properties of this group chain.

In Sec. II we define the SU(4) algebra. In Sec. III we discuss the labeling problem of the states; besides this we show that the Racah formula,[10] which gives the multiplicity $N_{ST}(pp'p'')$ of the states of given S and T in a given $(pp'p'')$ IR, can be given in an other form, which facilitates the calculation. In Sec. IV we define the semireduced matrix elements of the Q, V^Q, and W^Q vectors. In Sec. V we give the set of equations satisfied

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by the semireduced matrix elements of Q, V^Q, and W^Q. With these equations we can obtain the square, or the product, of the semireduced matrix elements of Q, V^Q, or W^Q, including a summation on the labels (ω, φ) or (σ, τ). In the particular case $T=p$ for any S (or $S=p$ for any T) we give an explicit formulation of the square of the matrix elements of Q. Finally in Sec. VI we search for a complete determination of each matrix element of Q, labeled by (ω, φ) or (σ, τ). As an example we have completely solved the case of the (320) IR with the (ω, φ) label.

II. THE SU(4) ALGEBRA

The Lie algebra of the SU(n) groups—and particularly of SU(4)—is well known.[4] Usually the algebra of SU(4) is written in its canonical form, i.e., with the generators E_{ij} $(i,j=1,2,3,4)$ which obey the following relations:

(1) the commutation rule

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \tag{II.1}$$

(2) the Hermitian conjugate relation

$$E_{ij}^\dagger = E_{ji}, \tag{II.2}$$

(3) the unimodular condition. We write it with the aid of the number operator N of U(4),

$$N = \sum_{i=1}^{4} E_{ii},$$

and so the number of independent generators is reduced from 16 to 15.

This canonical writing is particularly simple and leads to relatively easy calculations. But for the physicist, it has the major disadvantage that the SU(2) subalgebras of spin and isospin do not appear. In this paper we are interested in the physical aspects of SU(4). So we define new generators, which make these spin and isospin subalgebras appear explicitly. In what follows we shall often denote by X any of these generators:

$$\begin{aligned}
&T_0 = \tfrac{1}{2}[E_{11}+E_{22}-E_{33}-E_{44}], \quad S_0=\tfrac{1}{2}[E_{11}-E_{22}+E_{33}-E_{44}],\\
&T_1 = -\frac{1}{\sqrt{2}}[E_{13}+E_{24}], \quad S_1 = -\frac{1}{\sqrt{2}}[E_{12}+E_{34}],\\
&T_{-1} = \frac{1}{\sqrt{2}}[E_{31}+E_{42}], \quad S_{-1} = \frac{1}{\sqrt{2}}[E_{21}+E_{43}],\\
&Q_{11} = E_{14}, \quad Q_{10} = -\frac{1}{\sqrt{2}}[E_{12}-E_{34}], \quad Q_{1-1} = -E_{32},\\
&Q_{01} = -\frac{1}{\sqrt{2}}[E_{13}-E_{24}], \quad Q_{00} = \tfrac{1}{2}[E_{11}-E_{22}-E_{33}+E_{44}],\\
&Q_{0-1} = \frac{1}{\sqrt{2}}[E_{31}-E_{42}],\\
&Q_{-11} = -E_{23}, \quad Q_{-10} = \frac{1}{\sqrt{2}}[E_{21}-E_{43}], \quad Q_{-1-1} = E_{41}.
\end{aligned} \tag{II.3}$$

These generators obey the following commutation rules:

$$\begin{aligned}
&[S_\mu, S_\nu] = \sqrt{2}(-)^\mu c(111;\mu+\nu,-\mu)S_{\mu+\nu}, \quad [S_\mu, T_i]=0\\
&[T_i, T_j] = \sqrt{2}(-)^i c(111;i+j,-i)T_{i+j},\\
&[S_\mu, Q_{\nu i}] = \sqrt{2}(-)^\mu c(111;\mu+\nu,-\mu)Q_{\mu+\nu\, i},\\
&[T_i, Q_{\mu j}] = \sqrt{2}(-)^i c(111;i+j,-i)Q_{\mu\, i+j},\\
&[Q_{\mu i}, Q_{\nu j}] = \sqrt{2}(-)^{\mu+i}\{\delta_{i+j,0}\, c(111;\mu+\nu,-\mu)S_{\mu+\nu}\\
&\qquad + \delta_{\mu+\nu,0}\, c(111;i+j,-i)T_{i+j}\}.
\end{aligned} \tag{II.4}$$

The indices μ, ν, i, j can take the values $0;\pm 1$. This basis exhibits clearly the $SU(2)_S$ and $SU(2)_T$ spin and isospin subalgebra, through the generators S_μ and T_i respectively, which are here written as canonical tensors of SU(2). On the other hand, the $Q_{\mu i}$ generators are $(1i)$ and (1μ) tensors respectively in relation to the $SU(2)_T$ and $SU(2)_S$ subgroups.

Finally these generators obey the Hermitian conjugate relations

$$S_\mu^\dagger = (-)^\mu S_{-\mu}, \quad T_i^\dagger = (-)^i T_{-i}, \quad Q_{\alpha i}^\dagger = (-)^{\alpha+i}Q_{-\alpha -i}. \tag{II.5}$$

III. THE STATE LABELING PROBLEM

Biedenharn[11] has shown that two more vectors other than the fundamental generators can be constructed in SU(4) with the aid of completely symmetric coefficients. They are easy to calculate; let us call them V and W. They are

$$\begin{aligned}
&V_\alpha^S = \sum_i (-)^i T_i Q_{\alpha -i},\\
&V_i^T = \sum_\alpha (-)^\alpha S_\alpha Q_{-\alpha i},\\
&V_{\alpha i}^Q = S_\alpha T_i + \sum_{\mu j}(-)^{\mu+j} c(111;\alpha,-\mu)\\
&\qquad \times c(111;i,-j)Q_{\mu j}Q_{\alpha-\mu\, i-j},
\end{aligned} \tag{III.1}$$

and

$$\begin{aligned}
&W_\alpha^S = \sum_i (-)^i\{T_i V_{\alpha -i}^Q + V_i^T Q_{\alpha -i}\},\\
&W_i^T = \sum_\alpha (-)^\alpha\{S_\alpha V_{-\alpha i}^Q + V_\alpha^S Q_{-\alpha i}\},\\
&W_{\alpha i}^Q = S_\alpha V_i^T + T_i V_\alpha^S + 2\sum_{\mu j}(-)^{\mu+j} c(111;\alpha,-\mu)\\
&\qquad \times c(111;i,-j)Q_{\mu j}V_{\alpha-\mu\, i-j}^Q.
\end{aligned} \tag{III.2}$$

These vectors satisfy the same Hermitian conjugate relations as the generators (II.5).

Following Biedenharn[11] we can easily construct, with these vectors, the three independent invariants of SU(4):

$$\begin{aligned}
&I_2 = \sum_\mu (-)^\mu S_\mu S_{-\mu} + \sum_i (-)^i T_i T_{-i} + \sum_{\mu i}(-)^{\mu+i}Q_{\mu i}Q_{-\mu -i},\\
&I_3 = \sum_\mu (-)^\mu S_\mu V_{-\mu}^S + \sum_i (-)^i T_i V_{-i}^T + \sum_{\mu i}(-)^{\mu+i}Q_{\mu i}V_{-\mu -i}^Q,\\
&I_4 = \sum_\mu (-)^\mu S_\mu W_{-\mu}^S + \sum_i (-)^i T_i W_{-i}^T + \sum_{\mu i}(-)^{\mu+i}Q_{\mu i}W_{-\mu -i}^Q.
\end{aligned} \tag{III.3}$$

Let us notice the equality of the two first terms of (I_3)

$$\sum_\mu (-)^\mu S_\mu V_{-\mu}^S = \sum_i (-)^i T_i V_{-i}^T = \sum_{\mu i}(-)^{\mu+i}S_\mu Q_{-\mu -i}T_i. \tag{III.4}$$

Furthermore we remark that the vectors V and W allow one to define three other invariant operators:

$$I_5=\sum_{\mu}(-)^{\mu}V_{\mu}^{S}V_{-\mu}^{S}+\sum_{i}(-)^{i}V_{i}^{T}V_{-i}^{T}+\sum_{\mu i}(-)^{\mu+i}V_{\mu i}^{Q}V_{-\mu-i}^{Q},$$
$$I_6=\sum_{\mu}(-)^{\mu}V_{\mu}^{S}W_{-\mu}^{S}+\sum_{i}(-)^{i}V_{i}^{T}W_{-i}^{T}+\sum_{\mu i}(-)^{\mu+i}V_{\mu i}^{Q}W_{-\mu-i}^{Q},$$
$$I_7=\sum_{\mu}(-)^{\mu}W_{\mu}^{S}W_{-\mu}^{S}+\sum_{i}(-)^{i}W_{i}^{T}W_{-i}^{T}+\sum_{\mu i}(-)^{\mu+i}W_{\mu i}^{Q}W_{-\mu-i}^{Q}. \tag{III. 5}$$

These invariants can easily be related to the three fundamental ones above with the result

$$I_5=\tfrac{1}{2}I_4,\quad I_6=\tfrac{2}{3}I_3(I_2+3),\quad I_7=\frac{I_4}{2}(I_2+4)+I_3^2. \tag{III. 6}$$

These three fundamental invariants I_2, I_3, I_4 give a label for an irreducible representation (IR). But we know that six independent operators, commuting with one another, are necessary to give a unique label of each state of a given IR. Four such operators are well known; they are the spin and isospin operators: S_0, $\sum_\mu(-)^\mu S_\mu S_{-\mu}$, T_0, $\sum_i(-)^i T_i T_{-i}$. As for the two missing operators several couples are suitable.[7,9,16] Nagel and Moshinsky[9] have proposed the couple Ω and Φ, which in our notation can be written as follows:

$$\Omega=\sum_{\mu}(-)^{\mu}S_{\mu}V_{-\mu}^{S}=\sum_{i}(-)^{i}T_{i}V_{-i}^{T},$$
$$\Phi=\sum_{\mu}(-)^{\mu}S_{\mu}W_{-\mu}^{S}+\sum_{i}(-)^{i}T_{i}W_{-i}^{T}. \tag{III. 7}$$

We have found another pair of operators, which can play the same role. We call them s and t, and they are:

$$s=\sum_{\mu}(-)^{\mu}V_{\mu}^{S}V_{-\mu}^{S},\quad t=\sum_{i}(-)^{i}V_{i}^{T}V_{-i}^{T}. \tag{III. 8}$$

The demonstration of the commutation rules $[s,t]=0$ and $[\Omega,\Phi]=0$ is greatly simplified if one uses the following equations:

$$\sum_j c(111;i+j,-j)\,T_{-j}V_{\mu\,i+j}^{Q}=\sum_{\alpha}c(111;\alpha+\mu,-\alpha)V_{-\alpha}^{S}Q_{\alpha+\mu\,i},$$

$$\sum_j c(111;i+j,-j)\,V_{-j}^{T}Q_{\mu\,i+j}=\sum_{\alpha}c(111;\alpha+\mu,-\alpha)\times S_{-\alpha}V_{\alpha+\mu\,i}^{Q}, \tag{III. 9}$$

$$\sum_j c(111;i+j,-j)\,T_{-j}W_{\mu\,i+j}^{Q}+\sum_{\alpha}c(111;\alpha+\mu,-\alpha)S_{-\alpha}W_{\alpha+\mu\,i}^{Q}$$
$$=\sum_j c(111;i+j,-j)\,W_{-j}^{T}Q_{\mu\,i+j}+\sum_{\alpha}c(111;\alpha+\mu,-\alpha)\times W_{-\alpha}^{S}Q_{\alpha+\mu\,i}.$$

We now describe a state with the eigenvalues of the operators considered above (or the associated highest weights).

The eigenvalues of the three fundamental invariants [or, what is equivalent, the Wigner partition $(p,p'p'')$ which corresponds to the highest weight of the operators S_0, T_0, Q_{00}] uniquely label an IR of SU(4). The eigenvalues of the invariants, expressed in terms of $(pp'p'')$ are as follows[12]:

$$\langle I_2\rangle=p(p+4)+p'(p'+2)+p''^2,$$
$$\langle I_3\rangle=3p''(p'+1)(p+2), \tag{III. 10}$$
$$\langle I_4\rangle=2\{p''^2[(p+1)^2+(p'+1)^2]+2p''^2(p+1)+p'(p'+2)(p+1)(p+3)\}.$$

Let M_S and M_T be the eigenvalues of the operators S_0 and T_0 respectively; let S and T be the corresponding highest weights relative to the spin and isospin subgroups. The eigenvalues of the operators (Ω,Φ) or (s,t) will be denoted (ω,φ) or (σ,τ) respectively.

In the basis where Ω and Φ are simultaneously diagonal, the states of the IR labeled by $(pp'p'')$ are written

$$|(pp'p'')\varphi\omega SM_STM_T\rangle. \tag{III. 11}$$

In the basis where s and t are simultaneously diagonal, the states are written

$$|(pp'p'')\sigma\tau SM_STM_T\rangle. \tag{III. 12}$$

In order to simplify the notation, when no ambiguity is possible (all the calculations done in a given IR), we write the state without reference to the representation labels

$$|\varphi\omega SM_STM_T\rangle \quad\text{or}\quad |\sigma\tau SM_STM_T\rangle. \tag{III. 13}$$

Furthermore, when the sole multiplicity of the states of given (SM_STM_T) is considered, we use a collective index θ instead of one of the two couples $(\omega\varphi)$ or $(\sigma\tau)$, i.e.,

$$|\theta SM_STM_T\rangle. \tag{III. 14}$$

Let us now recall the conditions satisfied by the Wigner[1] partition $(pp'p'')$

$$|p''|\le p'\le p,$$
$${}^{0}_{1/2}\le S\le p,\quad {}^{0}_{1/2}\le T\le p,\quad {}^{0}_{1/2}\le T+S\le p+p'. \tag{III. 15}$$

In 1949, Racah[10] established an algebraic formulation, which gives the multiplicity of the irreducible representations ST of SU(2)⊗SU(2) in an irreducible representation of SU(4). He called this quantity $N_{TS}(pp'p'')$. It reads

$$N_{TS}(pp'p'')=\omega_{TS}(p+p'',p-p'')-\omega_{TS}(p+p'+1,p-p'-1)-\omega_{TS}(p'+p''-1,\,p'-p''-1), \tag{III. 16}$$

where ω_{TS} is defined by

$$\omega_{TS}(f_1f_2)\ \text{vanishes unless}\ T,S\le\frac{f_1+f_2}{2},$$
$$2T\equiv 2S\equiv f_1+f_2\ (\text{mod}\ 2). \tag{III. 17}$$

When $f_1\ge f_2$, $\omega_{TS}(f_1f_2)\ne 0$ is given by

$$\omega_{TS}(f_1f_2)=\omega_{TS}(f_2f_1)=\varphi(f_2+2-|T-S|)-\varphi(f_2+1-T-S)+\varphi(T+S-f_1-1)-\tfrac{1}{2}\varphi(T+S-|T-S|-f_1+f_2+1), \tag{III. 18}$$

where

$$\varphi(x)=\begin{cases}[x^2/4], & \text{if } x\ge 0,\\ 0 & \text{if } x\le 0,\end{cases}$$

and $[x]$ means the greatest integer contained in x.

Notice that the labels S and T are totally symmetric so that

$$N_{ST}(pp'p'')=N_{TS}(pp'p''). \quad \text{(III.19)}$$

Furthermore, the representations $(pp'p'')$ and $(pp'-p'')$ are contragredient, which implies equal multiplicities,

$$N_{TS}(pp'p'')=N_{TS}(pp'-p''). \quad \text{(III.20)}$$

Despite its generality, this Racah formulation is unfortunately somewhat hard to handle. In a given IR $(pp'p'')$, in order to obtain the multiplicities for all the pairs (T,S), one has to apply the Racah formula in each (T,S) case. This calculation is lengthy, difficult and cumbersome. It is for this reason that many papers[13] have given a new formulation for $N_{TS}(pp'p'')$. We propose here a rewriting of this formula, under a recurrent form, which is not so general as the Racah expression, but leads to much easier and faster calculations.

Observe first that the difference $\varphi(x+2)-\varphi(x)=1/2(x+1+|x+1|)$ does not depend on the parity of the integer x. So we can hope to find a simple expression for the difference,

$$\Lambda_{TS}(pp'p'')=N_{TS}(pp'p'')-N_{T+1\,S-1}(pp'p''). \quad \text{(III.21)}$$

Suppose for simplicity that $T\geq S$ and $p''\geq 0$ which removes nothing of the generality of the problem. We consider the different cases, and find:

(1) $S\neq 0$

$$T\leq p'-1\begin{cases}\Lambda_{TS}(pp'p'')=\frac{1}{2}(p-p''-T+S+1\\ \qquad +|p-p''-T+S+1|)\\ -\frac{1}{2}(p-p'-T+S+|p-p'-T+S|)\\ -\frac{1}{2}(p'-p''-T+S+|p'-p''-T+S|),\end{cases}$$

$$T\geq p'\begin{cases}\Lambda_{TS}(pp'p'')=\frac{1}{2}(p-p''-T+S+1\\ \qquad +|p-p''-T+S+1|)\\ -\frac{1}{2}(p-p'-T+S+|p-p'-T+S|)\\ -\frac{1}{2}(S-p'-1+|S-p'-1|)\\ -\frac{1}{2}(S-p''+|S-p''|),\end{cases} \quad \text{(III.22)}$$

(2) $S=0$

$$\Lambda_{T0}(pp'p'')=\left[\frac{P(p-p''+2-T)}{2}\right]-\left[\frac{P(p-p'+1-T)}{2}\right]-\left[\frac{P(p'-p''+1-T)}{2}\right], \quad \text{(III.23)}$$

where

$$P(y)=\frac{y+|y|}{2}.$$

The $\Lambda_{TS}(pp'p'')$ are easily calculated, and one can straightforwardly derive the $N_{TS}(pp'p'')$ through the recurrent formula

$$N_{TS}(pp'p'')=\Lambda_{TS}(pp'p'')+N_{T+1\,S-1}(pp'p''). \quad \text{(III.24)}$$

In particular one has

$$N_{T0}(pp'p'')=\Lambda_{T0}(pp'p''),$$
$$N_{T(1/2)}(pp'p'')=\Lambda_{T(1/2)}(pp'p''),$$
$$N_{PS}(pp'p'')=\Lambda_{PS}(pp'p'')=\begin{cases}1 & \text{if } S\geq p'',\\ 0 & \text{if } S<p''.\end{cases} \quad \text{(III.25)}$$

IV. THE GENERATOR MATRICES

In a given $(pp'p'')$ IR of SU(4) the matrix elements of S_μ and T_i are given by the Wigner–Racah algebra of the corresponding SU(2) groups,[14]

$$\langle\theta'S'M_S'T'M_T'|S_\mu|\theta SM_STM_T\rangle = \delta_{\theta\theta'}\delta_{SS'}\delta_{TT'}\delta_{M_TM_T'}\sqrt{S(S+1)}\,c(S1S';M_S,\mu,M_S'),$$

$$\langle\theta'S'M_S'T'M_T'|T_i|\theta SM_STM_T\rangle = \delta_{\theta\theta'}\delta_{SS'}\delta_{TT'}\delta_{M_SM_S'}\sqrt{T(T+1)}\,c(T1T';M_T,i,M_T'). \quad \text{(IV.1)}$$

But we are not able to give so simply the matrix elements of the $Q_{\mu i}$ generators. The calculation can be slightly simplified, when one introduces the so-called semireduced matrix element $\langle\theta'S'T'\|Q\|\theta ST\rangle$ i.e., reduced with respect to the spin and isospin SU(2) subgroups[3]

$$\langle\theta'S'M_S'T'M_T'|Q_{\mu i}|\theta SM_STM_T\rangle = c(S1S';M_S,\mu,M_S')c(T1T';M_T,i,M_T')\langle\theta'S'T'\|Q\|\theta ST\rangle. \quad \text{(IV.2)}$$

Now the search for an IR of SU(4) is replaced by the search for the semireduced Q matrix elements.

We define in the same way the semireduced matrix elements for V and W.

The semireduced matrix elements for $V^S, V^T, V^Q, W^S, W^T, W^Q$, can all be expressed in terms of those of Q. For V we have, for example:

$$\langle\theta'S'T'\|V^S\|\theta ST\rangle=\delta_{TT'}\sqrt{T(T+1)}\langle\theta'S'T\|Q\|\theta ST\rangle, \quad \text{(IV.3a)}$$

$$\langle\theta'S'T'\|V^T\|\theta ST\rangle=\delta_{SS'}\sqrt{S(S+1)}\langle\theta'ST'\|Q\|\theta ST\rangle, \quad \text{(IV.3b)}$$

$$\langle\theta'S'T'\|V^Q\|\theta ST\rangle = \delta_{SS'}\delta_{TT'}\delta_{\theta\theta'}\sqrt{S(S+1)T(T+1)} + 3\sum_{\theta_1S_1T_1}\sqrt{(2S_1+1)(2T_1+1)} \times W(SS'11;1S_1)W(TT'11;1T_1) \times\langle\theta'S'T'\|Q\|\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\|Q\|\theta ST\rangle. \quad \text{(IV.3c)}$$

We shall see later that this last expression for V^Q can be simplified, with a summation only on the indices θ_1S_1 (or θ_1T_1).

The Hermitian conjugate relation $Q_{\mu i}^{+}=(-)^{\mu+i}Q_{-\mu-i}$ leads to

$$\langle\theta'S'T'\|Q\|\theta ST\rangle = (-)^{S-S'+T-T'}\left(\frac{(2S+1)(2T+1)}{(2S'+1)(2T'+1)}\right)^{1/2}\langle\theta ST\|Q\|\theta'S'T'\rangle. \tag{IV.4}$$

The vectors V^Q and W^Q fulfill the same relation (just replace Q by V^Q or W^Q).

Let

$$T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}$$

be an irreducible tensor. The adjoint tensor is characterized by

$$T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}^\dagger = b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix} T\begin{pmatrix} pp'-p'' \\ \theta^* \\ S-M_ST-M_T \end{pmatrix}, \tag{IV.5}$$

with $\theta^*=(\omega^*,\varphi^*)$ or $\theta^*=(\sigma^*,\tau^*)$, $\omega^*=-\omega$, $\varphi^*=\varphi$, $\sigma^*=\sigma$, $\tau^*=\tau$.

$b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}$ is a phase factor.

Writing the commutation relation of this tensor with the generator $Q_{\mu i}$ we can establish a relation between the semireduced Q matrix elements in a given $(pp'p'')$ IR and in the contragredient one $(pp'-p'')$. Let us consider the commutator

$$\left[Q_{\alpha i}, T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\right] = \sum_{\substack{\theta' S' T' \\ M_S' M_T'}} \langle (pp'p'')\theta'S'M_S'T'M_T'|Q_{\alpha i}|(pp'p'')\theta SM_STM_T\rangle \times T\begin{pmatrix} pp'p'' \\ \theta' \\ S'M_S'T'M_T' \end{pmatrix} \tag{IV.6}$$

Let us conjugate the two members. One obtains, on the one hand,

$$\left[Q_{\alpha i}, T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\right]^\dagger = (-)^{\alpha+i+1} b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\left[Q_{-\alpha-i}, T\begin{pmatrix} pp'-p'' \\ \theta^* \\ S-M_ST-M_T \end{pmatrix}\right],$$

$$\left[Q_{\alpha i}, T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\right]^\dagger = (-)^{\alpha+i+1} b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\times \sum_{\substack{\theta' S' T' \\ M_S' M_T'}} \langle (pp'-p'')\theta'^*S' - M_S'T'-M_T'|Q_{-\alpha-i}|(pp'-p'')\theta^*S-M_ST-M_T\rangle \times T\begin{pmatrix} pp'-p'' \\ \theta'^* \\ S'-M_S'T'-M_T' \end{pmatrix}. \tag{IV.7}$$

On the other hand, we can also write

$$\left[Q_{\alpha i}, T\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\right]^\dagger = \sum_{\substack{\theta' S' T' \\ M_S' M_T'}} b\begin{pmatrix} pp'p'' \\ \theta' \\ S'M_S'T'M_T' \end{pmatrix}\langle (pp'p'')\theta'S'M_S'T'M_T'|Q_{\alpha i}| \times (pp'p'')\theta SM_STM_T\rangle T\begin{pmatrix} pp'-p'' \\ \theta'^* \\ S'-M_S'T'-M_T' \end{pmatrix}. \tag{IV.8}$$

Identifying the last two relations we can deduce the equation

$$b\begin{pmatrix} pp'p'' \\ \theta' \\ S'M_S'T'M_T' \end{pmatrix}\langle (pp'p'')\theta'S'M_S'T'M_T'|Q_{\alpha i}|(pp'p'')\theta SM_STM_T\rangle + (-)^{\alpha+i} b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix}\langle (pp'-p'')\theta'^*S'-M_S'T'-M_T' \times |Q_{-\alpha-i}|(pp'-p'')\theta^*S-M_ST-M_T\rangle = 0, \tag{IV.9}$$

which can be written otherwise,

$$b\begin{pmatrix} pp'p'' \\ \theta' \\ S'M_S+\alpha\, T'M_T+i \end{pmatrix}\langle (pp'p'')\theta'S'T'\|Q\|(pp'p'')\theta ST\rangle + (-)^{\alpha+i+S-S'+T-T'} b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix} \times\langle (pp'-p'')\theta'^*S'T'\|Q\|(pp'-p'')\theta^*ST\rangle = 0. \tag{IV.10}$$

The same calculation can be done with S_α or T_i in place of $Q_{\alpha i}$. We are led to the following relation between the phase factors,

$$b\begin{pmatrix} pp'p'' \\ \theta \\ S\ M_S+\alpha\ T\ M_{T+i} \end{pmatrix} = (-)^{\alpha+i} b\begin{pmatrix} pp'p'' \\ \theta \\ S M_S\ T M_T \end{pmatrix}. \tag{IV.11}$$

If we define

$$b\begin{pmatrix} pp'p'' \\ \theta \\ S\ M_S=S\ T\ M_T=T \end{pmatrix} = \begin{pmatrix} b & pp'p'' \\ & \theta S T \end{pmatrix}, \tag{IV.12}$$

we get

$$b\begin{pmatrix} pp'p'' \\ \theta \\ SM_STM_T \end{pmatrix} = (-)^{S-M_S+T-M_T}\, b\begin{pmatrix} pp'p'' \\ \theta S T \end{pmatrix}, \tag{IV.13}$$

and the relation (IV.14) above can now be written

$$b\begin{pmatrix} pp'p'' \\ \theta'S'T' \end{pmatrix}\langle (pp'p'')\theta'S'T'\|Q\|(pp'p'')\theta ST\rangle + b\begin{pmatrix} pp'p'' \\ \theta ST \end{pmatrix} \times\langle (pp'-p'')\theta'^*S'T'\|Q\|(pp'-p'')\theta^*ST\rangle = 0. \tag{IV.14}$$

This last relation has two forms depending on whether or $\theta=(\varphi\omega)$ or $\theta=(\sigma\tau)$:

(a) $\theta=(\varphi\omega)$

$$b\begin{pmatrix} pp'p'' \\ \varphi'\omega'S'T' \end{pmatrix}\langle (pp'p'')\varphi'\omega'S'T'\|Q\|(pp'p'')\varphi\omega ST\rangle + b\begin{pmatrix} pp'p'' \\ \varphi\omega ST \end{pmatrix}$$

$$\times \langle (pp'-p'')\,\varphi'-\omega'S'T'\,\|Q\|\,(pp'-p'')\,\varphi-\omega ST\rangle=0;$$

(b) $\theta=(\sigma\tau)$

$$b\begin{pmatrix} pp'p'' \\ \sigma'\tau'S'T' \end{pmatrix}\langle (pp'p'')\,\sigma'\tau'S'T'\,\|Q\|\,(pp'p'')\,\sigma\tau ST\rangle + b\begin{pmatrix} pp'p'' \\ \sigma\tau ST \end{pmatrix}$$

$$\times \langle (pp'-p'')\,\sigma'\tau'S'T'\,\|Q\|\,(pp'-p'')\,\sigma\tau ST\rangle=0.$$

In the particular case of IR with $p''=0$ these relations lead to the following prominent equations:

On the one hand,

$$\langle (pp'0)\,\varphi\omega ST\,\|Q\|\,(pp'0)\,\varphi\omega ST\rangle + \langle (pp'0)\,\varphi-\omega ST\,\|Q\|\,(pp'0)\,\varphi-\omega ST\rangle=0. \qquad \text{(IV.15)}$$

On the other hand,

$$\langle (pp'0)\,\sigma\tau ST\,\|Q\|\,(pp'0)\,\sigma\tau ST\rangle=0. \qquad \text{(IV.16)}$$

The semireduced matrix elements we have considered above, do not exhibit, in the Wigner—Eckart theorem, the problem of inner multiplicity, the SU(2) group being multiplicity free. On the contrary, if in a given IR $(pp'p'')$, we want to reduce completely under SU(4), the matrix elements of the vectors X, V, or W; we need an inner multiplicity index ρ. Using the notation of Hecht and Pang,[3] in the particular case of the labeling by (φ,ω), and Q (which has $\varphi=8$, $\omega=0$), we can write

$$\langle (pp'p'')\,\varphi'\omega'S'T'\left\|Q\begin{pmatrix}110\\80\\11\end{pmatrix}\right\|(pp'p'')\,\varphi\omega ST\rangle$$

$$=\sum_{\rho}\langle (pp'p'')\,\|X\|\,(pp'p'')\rangle_\rho$$

$$\times\langle (pp'p'')\,\varphi\omega ST;\ (110)\ 8011\|\,(pp'p'')\,\varphi'\omega'S'T'\rangle_\rho. \qquad \text{(IV.17)}$$

We have the same form for V^Q and W^Q.

Here the index ρ can take three values ($\rho=1,2,3$). To completely define this index one must arbitrarily fix three conditions. It is particularly simple to take them as

$$\langle (pp'p'')\,\|X\|\,(pp'p'')\rangle_2=\langle (pp'p'')\,\|X\|\,(pp'p'')\rangle_3$$
$$=\langle (pp'p'')\,\|V\|\,(pp'p'')\rangle_3=0. \qquad \text{(IV.18)}$$

With this choice we can calculate the six remaining reduced matrix elements for the vectors X, V, W by taking the matrix elements of the six invariant operators I_k ($i=2,3,\ldots,7$) and solving the system. We thus obtain the following results, in which $\langle I_k\rangle$ designates the eigenvalue of the invariant I_k:

$$\langle (pp'p'')\,\|X\|\,(pp'p'')\rangle_1=(\langle I_2\rangle)^{1/2},$$

$$\langle (pp'p'')\,\|V\|\,(pp'p'')\rangle_1=\frac{\langle I_3\rangle}{(\langle I_2\rangle)^{1/2}},$$

$$\langle (pp'p'')\,\|V\|\,(pp'p'')\rangle_2=\frac{(\langle I_2I_5\rangle-\langle I_3^2\rangle)^{1/2}}{\langle I_2\rangle},$$

$$\langle (pp'p'')\,\|W\|\,(pp'p'')\rangle_1=\frac{\langle I_4\rangle}{(\langle I_2\rangle)^{1/2}}, \qquad \text{(IV.19)}$$

$$\langle (pp'p'')\,\|W\|\,(pp'p'')\rangle_2=\frac{\langle I_6I_2\rangle-\langle I_3I_4\rangle}{[\langle I_2\rangle(\langle I_2I_5\rangle-\langle I_3^2\rangle)]^{1/2}},$$

$$\langle (pp'p'')\,\|W\|\,(pp'p'')\rangle_3$$
$$=\left(\frac{(\langle I_2I_5\rangle-\langle I_3^2\rangle)(\langle I_2I_7\rangle-\langle I_4^2\rangle)-(\langle I_2I_6\rangle-\langle I_3I_4\rangle)^2}{\langle I_2\rangle(\langle I_2I_5\rangle-\langle I_3^2\rangle)}\right)^{1/2}.$$

The Kronecker product of the IR $(p00)$ or $(pp\pm p)$ with the IR (110) is multiplicity free. This means that ρ can take only one value. In fact we see that:

(a) For the $(p00)$ IR, only one matrix element is different from zero

$$\langle (p00)\,\|X\|\,(p00)\rangle_1=\sqrt{p(p+4)}; \qquad \text{(IV.20)}$$

(b) For the $(pp\pm p)$ IR all the nonvanishing matrix elements correspond to the same value of the index ρ:

$$\langle (pp\pm p)\,\|X\|\,(pp\pm p)\rangle_1=\sqrt{3p(p+2)},$$
$$\langle (pp\pm p)\,\|V\|\,(pp\pm p)\rangle_1=\pm(p+1)\sqrt{3p(p+2)}, \qquad \text{(IV.21)}$$
$$\langle (pp\pm p)\,\|W\|\,(pp\pm p)\rangle_1=2(p+1)^2\sqrt{3p(p+2)}.$$

It follows that the semireduced matrix elements of V^Q and W^Q are zero for $(p00)$:

$$\langle (p00)\,\theta'S'T'\,\|V^Q\|\,(p00)\,\theta ST\rangle=0,$$
$$\langle (p00)\,\theta'S'T'\,\|W^Q\|\,(p00)\,\theta ST\rangle=0. \qquad \text{(IV.22)}$$

In the case of the $(pp\pm p)$ IR a proportionality relation can be written between the semireduced matrix elements of Q, V^Q, and W^Q:

$$\langle (pp\pm p)\,\theta'S'T'\,\|V^Q\|\,(pp\pm p)\,\theta ST\rangle$$
$$=\pm(p+1)\,\langle (pp\pm p)\,\theta'S'T'\,\|Q\|\,(pp\pm p)\,\theta ST\rangle,$$
$$\langle (pp\pm p)\,\theta'S'T'\,\|W^Q\|\,(pp\pm p)\,\theta ST\rangle \qquad \text{(IV.23)}$$
$$=2(p+1)^2\,\langle (pp\pm p)\,\theta'S'T'\,\|Q\|\,(pp\pm p)\,\theta ST\rangle.$$

V. CALCULATION OF THE MATRIX ELEMENTS $\Sigma_\theta\langle (pp'p'')\theta ST\,\|\,Q\,\|\,(pp'p'')\,\theta ST\rangle$ AND $\Sigma_{\theta\theta'}\langle (pp'p'')\,\theta'S'T'\,\|Q\|\,(pp'p'')\,\theta ST\rangle^2$

The eigenvalues of the operators Φ and Ω, or s and t, can be obtained on the Gel'fand and Zetlin basis, by a diagonalization process. But such a method is very cumbersome and can only be achieved with the aid of a computer. On the contrary, it is fairly easy to calculate $\sum_\theta\langle\theta ST\,\|Q\|\,\theta ST\rangle$ when one writes the conservation of the trace of the matrix representation in the Gel'fand, and physical basis.

The operators S_0 and T_0 are simultaneously diagonal in both bases. For an arbitrary operator, Z, we have equality of the traces of its matrix representation in both bases. These traces are taken for a given $(pp'p'')$ IR, at the level of each submatrix labeled by a given M_S and M_T.

Let us introduce the positive integers α, β, γ, and w; we can write the equation expressing the equality of the traces of any Z, in the two basis, for given M_S and M_T as follows:

$$\sum_{\alpha\beta\gamma}^{x}\left\langle \begin{matrix} p+p' & p-p'' & p'-p'' & 0 \\ p+p'-\alpha & p-p''-\beta & p'-p''-\gamma \\ p+p'-x & M_S-p''+x \\ M_S+M_T+\alpha+\beta+\gamma \end{matrix} \middle| Z \middle| \begin{matrix} p+p' & p-p'' & p'-p'' & 0 \\ p+p'-\alpha & p-p''-\beta & p'-p''-\gamma \\ p+p'-x & M_S-p''+x \\ M_S+M_T+\alpha+\beta+\gamma \end{matrix} \right\rangle$$

$$= \sum_{\substack{\theta \\ S\geq |M_S| \\ T\geq |M_T|}} \langle (pp'p'')\,\theta S M_S T M_T\,|Z|\,(pp'p'')\,\theta S M_S T M_T\rangle.$$

The variables α, β, γ, and x, must satisfy the betweeness conditions:

$$\begin{aligned} &p'+p''\geq\alpha\geq 0, \quad p'+p''+\beta\geq x\geq\alpha, \\ &p-p'\geq\beta\geq 0, \quad p-M_S-\beta\geq x\geq p'-M_S-\gamma, \\ &p'-p''\geq\gamma\geq 0, \quad p+p'-M_S-M_T-x\geq\alpha+\beta+\gamma\geq -M_T-p''+x. \end{aligned} \tag{V.2}$$

For the particular case $M_S=p$, let us take $M_T\geq 0$. Equation (V.1) becomes

$$\sum_{\gamma=0}^{\min\{p'-p'',\, p'-M_T\}}\left\langle \begin{matrix} p+p' & p-p'' & p'-p'' & 0 \\ p+p' & p-p'' & p'-p''-\gamma \\ p+p' & p-p'' \\ p+M_T+\gamma \end{matrix} \middle| Z \middle| \begin{matrix} p+p' & p-p'' & p'-p'' & 0 \\ p+p' & p-p'' & p'-p''-\gamma \\ p+p' & p-p'' \\ p+M_T+\gamma \end{matrix} \right\rangle$$

$$= \sum_{\substack{\theta \\ T\geq M_T}} \langle (pp'p'')\,\theta ppTM_T\,|Z|\,(pp'p'')\,\theta ppTM_T\rangle. \tag{V.3}$$

If Z is Q_{00}, which is diagonal in the Gel'fand basis, we have to calculate

$$\sum_{\gamma=0}^{\min\{p'-p'',\, p'-M_T\}} \{M_T-(p'-p'')+2\gamma\} = \sum_{T\geq M_T} \frac{pM_T}{\sqrt{p(p+1)T(T+1)}} \langle (pp'p'')\,\theta pT\,\|Q\|\,(pp'p'')\,\theta pT\rangle. \tag{V.4}$$

Here θ is redundant, because $N_{pT}(pp'p'')=1$ if $S\geq p''$ and 0 if $S<p''$.

Now we can get, by taking the difference between the two sums $\sum_{T\geq M_T}-\sum_{T\geq M_T+1}$, the matrix element

$$\langle (pp'p'')pT\,\|Q\|\,(pp'p'')pT\rangle = \begin{cases} 0 \quad \text{if } T<p'', \\ p''(p'+1)\left(\dfrac{p+1}{pT(T+1)}\right)^{1/2} \quad \text{if } T\geq p''. \end{cases} \tag{V.5}$$

This result is still valid when we interchange S and T.

Let $\{T^Q_{\alpha i}, T^S_\alpha, T^T_i\}$ be any vector following the same transformation rules as the generators $\{Q_{\alpha i}, S_\alpha, T_i\}$ and consider the commutation relation

$$[Q_{\mu i}, T^Q_{\alpha j}] = \sqrt{2}\,(-)^{\mu+i}\{\delta_{i+j,0}\,c(111;\mu+\alpha,-\mu)\,T^S_{\mu+\alpha}+\delta_{\mu+\alpha,0}\,c(111;i+j,-i)\,T^T_{i+j}\}. \tag{V.6}$$

Taking the matrix element of the two sides of this equation, let us calculate

$$\sum_{\substack{ijM'_T \\ \alpha\mu M'_S}} c(11f;j,i,m_f)\,c(11f';\alpha,\mu,m'_f)\,\langle\theta'S'M'_ST'M'_T|[Q_{\mu i},T^Q_{\alpha j}]|\theta SM_STM_T\rangle. \tag{V.7}$$

We get

$$\begin{aligned} &\sum_{\theta_1S_1T_1}\sqrt{(2S_1+1)(2T_1+1)}\,W(SS'11;f'S_1)\,W(TT'11;fT_1)\times\{\langle\theta'S'T'\,\|Q\|\,\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\,\|T^Q\|\,\theta ST\rangle \\ &\quad -(-)^{f+f'}\langle\theta'S'T'\,\|T^Q\|\,\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\,\|Q\|\,\theta ST\rangle\} \\ &= -\sqrt{2}\,\{\delta_{f,0}\delta_{f',1}\delta_{TT'}\langle\theta'S'T'\,\|T^S\|\,\theta ST\rangle+\delta_{f,1}\delta_{f',0}\delta_{SS'}\langle\theta'S'T'\,\|T^T\|\,\theta ST\rangle\}, \end{aligned} \tag{V.8}$$

where f and f' can only be 0, 1, 2.

Consider now the particular case $T^Q = Q$, $\theta' = \theta$, $S' = S$, $T' = T$. In order to simplify our notation we write only N_{ST} instead of $N_{ST}(pp'p'')$ for the multiplicity of the states of given ST in the $(pp'p'')$ IR and we define

$$\langle S'T' | Q | ST\rangle^2 = \sum_{\theta\theta'} \langle \theta' S'T' \| Q \| \theta ST\rangle^2. \tag{V.9}$$

The invariant operator I_2, and Eq. (V.8) give the following five equations between the matrix elements (V.9):

$$\begin{aligned}
&\text{(a)}\ \sum_{S_1T_1} \langle ST | Q | S_1T_1\rangle^2 = N_{ST}\{\langle I_2\rangle - S(S+1) - T(T+1)\},\\
&\text{(b)}\ \sum_{S_1T_1} (-)^{S-S_1+T-T_1} W(TT11;2T_1)\, W(SS11;1S_1)\langle ST|Q|S_1T_1\rangle^2 = 0,\\
&\text{(c)}\ \sum_{S_1T_1} (-)^{S-S_1+T-T_1} W(TT11;1T_1)\, W(SS11;2S_1)\langle ST|Q|S_1T_1\rangle^2 = 0,\\
&\text{(d)}\ \sum_{S_1T_1} (-)^{S-S_1+T-T_1} W(TT11;1T_1)\, W(SS11;0S_1)\langle ST|Q|S_1T_1\rangle^2 = -N_{ST}\left(\frac{T(T+1)}{2(2S+1)(2T+1)}\right)^{1/2},\\
&\text{(e)}\ \sum_{S_1T_1} (-)^{S-S_1+T-T_1} W(TT11;0T_1)\, W(SS11;1S_1)\langle ST|Q|S_1T_1\rangle^2 = -N_{ST}\left(\frac{S(S+1)}{2(2S+1)(2T+1)}\right)^{1/2}.
\end{aligned} \tag{V.10}$$

The notation (V.9) has a great advantage; it leads to a sixth equation derived from the Hermitian conjugate relation

$$\langle ST|Q|S'T'\rangle^2 = \frac{(2S'+1)(2T'+1)}{(2S+1)(2T+1)} \langle S'T'|Q|ST\rangle^2. \tag{V.11}$$

From (a), (c), (d), and (e) we can derive the following relation

$$\begin{aligned}
S\langle ST|Q|S-1T\rangle^2 - (S+1)\langle ST|Q|S+1T\rangle^2 &= -\frac{T+1}{T}\{\langle I_2\rangle - 2S(S+1) - T(T+4)\}N_{ST}\\
&+\frac{2T+1}{T(2T+3)}\{2T+S+3)\langle ST|Q|S+1T+1\rangle^2 + 2(T+1)\langle ST|Q|ST+1\rangle^2 + (2T-S+2)\langle ST|Q|S-1T+1\rangle^2\}\ .
\end{aligned} \tag{V.12}$$

We can now use the Hermitian conjugate equation to write (V.12) in a form which explicitly shows a recurrence in the variable S,

$$\begin{aligned}
&S(2S-1)\langle S-1T|Q|ST\rangle^2 - (S+1)(2S+1)\langle ST|Q|S+1T\rangle^2\\
&\quad= -\frac{(T+1)(2S+1)}{T}\{\langle I_2\rangle - 2S(S+1) - T(T+4)\}N_{ST} + \frac{(2T+1)(2S+1)}{T(2T+3)}\\
&\qquad\times\{(2T+S+3)\langle ST|Q|S+1T+1\rangle^2 + 2(T+1)\langle ST|Q|ST+1\rangle^2 + (2T-S+2)\langle ST|Q|S-1T+1\rangle^2\}.
\end{aligned} \tag{V.13}$$

Suppose we know, for a given T, all the matrix elements $\langle ST|Q|S_\alpha T+1\rangle^2$ ($S_\alpha = S,\ S+1,\ S-1$). Then it is possible to calculate the six matrix elements $\langle ST|Q|S_\alpha T\rangle^2$ and $\langle ST|Q|S_\alpha T-1\rangle^2$.

First we use the recurrent relation (V.13) in the calculation of the matrix element $\langle S-1\ T|Q|ST\rangle^2$,

$$\begin{aligned}
\langle S-1\ T|Q|ST\rangle^2 &= -\frac{T+1}{ST(2S-1)} \sum_{S'\geq S}(2S'+1)\{\langle I_2\rangle - 2S'(S'+1) - T(T+1)\}N_{S'T} + \frac{2T+1}{T(2T+3)S(2S-1)}\sum_{S'\geq S}(2S'+1)\\
&\quad\times\{(2T+S'+3)\langle S'T|Q|S'+1T+1\rangle^2 + 2(T+1)\langle S'T|Q|S'T+1\rangle^2 + (2T-S+2)\langle S'T|Q|S'-1T+1\rangle^2\}.
\end{aligned} \tag{V.14}$$

When we change S for $S+1$ we obtain the matrix element $\langle ST|Q|S+1\ T\rangle^2$. It we now use the relation (V.11) we can derive the matrix element $\langle ST|Q|S-1\ T\rangle^2$

$$\langle ST|Q|S-1\ T\rangle^2 = \frac{2S-1}{2S+1}\langle S-1\ T|Q|ST\rangle^2. \tag{V.15}$$

Now with the five equations of the set (V.10) we can calculate the four other matrix elements, resulting in:

$$\begin{aligned}
\langle ST|Q|ST\rangle^2 &= \frac{(T+1)(S+1)}{TS}\{\langle I_2\rangle - S(S+2) - T(T+4)\}N_{ST} - \frac{2S+1}{S}\langle ST|Q|S+1\ T\rangle^2 - \frac{(2T+1)(2TS+4S+2T+3)}{TS(2T+3)}\\
&\quad\times\langle ST|Q|S+1\ T+1\rangle^2 - \frac{(2T+1)(2TS+3S+2T+2)}{TS(2T+3)}\langle ST|Q|S\ T+1\rangle^2\\
&\quad-2\,\frac{(2T+1)(S+1)(T+1)}{TS(2T+3)}\langle ST|Q|S-1\ T+1\rangle^2,
\end{aligned}$$

$$\langle ST|Q|S+1\ T-1\rangle^2 = \frac{(2S+3)(T-S)}{2S+1}N_{ST} - \frac{1}{T+1}\langle ST|Q|S+1\ T\rangle^2 - \frac{S(2S+3)-T(2T+3)}{(2T+3)(2S+1)(T+1)(S+1)}\langle ST|Q|S+1\ T+1\rangle^2$$

$$+\frac{(2T+1)(2S+3)}{(2T+3)(2S+1)(S+1)}\langle ST|Q|S\ T+1\rangle^2+\frac{(2T+1)(2S+3)}{(2T+3)(2S+1)}\langle ST|Q|S-1\ T+1\rangle^2, \tag{V.16}$$

$$\langle ST|Q|S-1\ T-1\rangle^2=\frac{N_{ST}}{ST(2S+1)}\{(2S+1)[\langle I_2\rangle-2S(S+1)-T(T+4)]+ST(2S-1)(S+T+1)\}-\frac{(S+1)}{S(T+1)}\langle ST|Q|S+1\ T\rangle^2$$
$$-(2T+1)\left\{\frac{(2S+1)(2T+S+3)-ST(T+1)(2S-1)}{ST(2S+1)(T+1)(2T+3)}\right\}\langle ST|Q|S+1\ T+1\rangle^2$$
$$-(2T+1)\left\{\frac{T(2S-1)+2(2S+1)}{ST(2S+1)(2T+3)}\right\}\langle ST|Q|S\ T+1\rangle^2-\frac{8ST-2S^2-2T^2+3S+3T+2}{ST(2S+1)(2T+3)}\langle ST|Q|S-1\ T+1\rangle^2,$$

$$\langle ST|Q|S\ T-1\rangle^2=-\frac{N_{ST}}{ST(S+1)}\{(S+1)^2\langle I_2\rangle+S(S+1)(S^2+2S+2+2T^2+5T)+3T(S+1)(T+4)-T(3T+8)\}$$
$$+\frac{2S+1}{S}\langle ST|Q|S+1\ T\rangle^2+\frac{(2T+1)(2S-T+2)}{ST(2T+3)}\langle ST|Q|S-1\ T+1\rangle^2$$
$$+(2T+1)\left\{\frac{(2S+1)(2T+S+3)+S(T+1)(2S+T)}{ST(S+1)(T+1)(2T+3)}\right\}\langle ST|Q|S+1\ T+1\rangle^2$$
$$+\frac{(2T+1)[S^2+4S+2ST+2-T]+S(T+1)[2S+1-2T]}{ST(S+1)(2T+3)}\langle ST|Q|S\ T+1\rangle^2.$$

So we know all the matrix elements $\langle ST-1|Q|S_\alpha T\rangle^2$, for a given T, and $S_\alpha=S$, $S\pm1$ in terms of $\langle ST|Q|S_\alpha\ T+1\rangle^2$ and $\langle ST|Q|S+1\ T\rangle^2$. Using the same method it is possible to calculate the six matrix elements $\langle S\ T-1|Q|S_\alpha\ T-1\rangle^2$ and $\langle S\ T-1|Q|S_\alpha\ T-2\rangle^2$, and so on.

In the particular case $T=p$, we known that $\langle Sp|Q|S_\alpha\ p+1\rangle^2=0$ and the second members of the various results (V.14) and (V.16) are completely known. We recall here that $N_{Sp}(pp'p'')=1$ when $p'\geq S\geq p''$ and 0 if $S<p''$. The result is the following:

$$\begin{aligned}
&\langle Sp|Q|S-1\ p\rangle^2=\frac{p+1}{pS(2S+1)}[S^2-p''^2][(p'+1)^2-S^2],\\
&\langle Sp|Q|S+1\ p\rangle^2=\frac{p+1}{p(S+1)(2S+1)}[(S+1)^2-p''^2][(p'+1)^2-(S+1)^2],\\
&\langle Sp|Q|Sp\rangle^2=\frac{p+1}{pS(S+1)}(p'+1)^2p''^2,\\
&\langle Sp|Q|S+1\ p-1\rangle^2=(p-S)\frac{2S+3}{2S+1}-\frac{[(S+1)^2-p''^2][(p'+1)^2-(S+1)^2]}{p(S+1)(2S+1)},\\
&\langle Sp|Q|S\ p-1\rangle^2=\frac{p(p+1)S(S+1)-(p'+1)^2p''^2}{pS(S+1)},\\
&\langle Sp|Q|S-1\ p-1\rangle^2=\frac{2S-1}{2S+1}(p+S+1)+\frac{p''^2(p'+1)^2+S^2[S^2-(p'+1)^2-p''^2]}{pS(2S+1)}.
\end{aligned} \tag{V.17}$$

It is now possible, with the equations (V.14), (V.15), and (V.16) to calculate the other matrix elements with $T=p-1,\ p-2,\cdots$, and completely obtain all the $\langle ST|Q|S'T'\rangle^2$.

If we multiply the two members of Eq. (V.8) by $(2f'+1)\,[(2\bar S_1+1)]^{1/2}W(SS'11;f'\bar S_1)$ and sum over f', we can use the orthogonality relation of the Racah coefficients, and obtain a relation in which the number of summation indices is reduced by one,

$$\sum_{\theta_1S_1T_1}\sqrt{(2S_1+1)(2T_1+1)}\,W(SS'11;1S_1)\,W(TT'11;fT_1)\,\langle\theta'S'T'\|T^Q\|\,\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\|Q\|\,\theta ST\rangle$$
$$=\tfrac{1}{6}\sum_{\theta_1T_1}\left(\frac{2T_1+1}{2\bar S_1+1}\right)^{1/2}\cdot\frac{W(TT'11;fT_1)}{W(SS'11;1\bar S_1)}\{\langle\theta'S'T'\|T^Q\|\,\theta_1\bar S_1T_1\rangle\langle\theta_1\bar S_1T_1\|Q\|\,\theta ST\rangle-(-)^f\langle\theta'S'T'\|Q\|\,\theta_1\bar S_1T_1\rangle\langle\theta_1\bar S_1T_1\|T^Q\|\,\theta ST\rangle\}$$
$$-\frac{1}{\sqrt2}\delta_{f,0}\,\delta_{TT'}\langle\theta'S'T'\|T^S\|\,\theta ST\rangle+\frac{1}{3\sqrt2}\delta_{f,1}\,\delta_{SS'}\frac{W(SS11;0\bar S_1)}{W(SS11;1\bar S_1)}\langle\theta'S'T'\|T^T\|\,\theta ST\rangle. \tag{V.18}$$

Due to the symmetry of the roles played by S and T, the spin and isospin variables, we can exchange them in the above relation.

Equation (V.18) gives for the semireduced matrix elements of V^Q a simpler expression than Eq. (IV.3c). Let us take the particular case when $f=1$, $T^Q=Q$, $T^T=T$, $T^S=S$ and let $\bar T_1$ be one or the other value T or T'. We obtain

$$\langle\theta'S'T'\|V^Q\|\,\theta ST\rangle=\sum_{\theta_1S_1}\left(\frac{2S_1+1}{2\bar T_1+1}\right)^{1/2}\frac{W(SS'11;1S_1)}{W(TT'11;1\bar T_1)}\langle\theta'S'T'\|Q\|\,\theta_1S_1\bar T_1\rangle\langle\theta_1S_1\bar T_1\|Q\|\,\theta ST\rangle. \tag{V.19}$$

In the same way, we obtain, with $\bar{S}_1$, equal to either S or S',

$$\langle\theta'S'T'\|V^Q\|\theta ST\rangle=\sum_{\theta_1T_1}\left(\frac{2T_1+1}{2\bar{S}_1+1}\right)^{1/2}\frac{W(TT'11;1T_1)}{W(SS'11;1\bar{S}_1)}\langle\theta'S'T'\|Q\|\theta_1\bar{S}_1T_1\rangle\langle\theta_1\bar{S}_1T_1\|Q\|\theta ST\rangle. \tag{V.20}$$

By the main of the commutator $[Q_{\mu i}, T^Q_{\alpha j}]$ we obtain Eq. (V.8). In the same way, using the commutators $[V^Q_{\mu i}, V^Q_{\alpha j}]$, $[V^Q_{\mu i}, W^Q_{\alpha j}]$, $[W^Q_{\mu i}, W^Q_{\alpha j}]$ we obtain the equations

$$\sum_{\theta_1S_1T_1}[1-(-)^{f+f'}]\sqrt{(2S_1+1)(2T_1+1)}\,W(SS'11;f'S_1)\,W(TT'11;fT_1)\langle\theta'S'T'\|V^Q\|\theta ST\rangle$$

$$=(\tfrac{2}{3})^{1/2}\begin{cases}\delta_{1,f}\sqrt{S'(S'+1)(2S'+1)}\,W(SS'11;f'S')\langle\theta'S'T'\|V^Q\|\theta ST\rangle\\ +\delta_{1,f'}\sqrt{T'(T'+1)(2T'+1)}\,W(TT'11;fT')\langle\theta'S'T'\|V^Q\|\theta ST\rangle\\ +\delta_{1,f}(-)^{f'}\dfrac{4-f(f+1)}{2}\sqrt{T'(T'+1)}\displaystyle\sum_{\theta_1S_1}\sqrt{2S_1+1}\,W(SS'11;f'S_1)\langle\theta'S'T'\|Q\|\theta_1S_1T'_1\rangle\langle\theta_1S_1T'_1\|Q\|\theta ST\rangle\\ +\delta_{1,f'}(-)^{f'}\dfrac{4-f(f+1)}{2}\sqrt{s'(s'+1)}\displaystyle\sum_{\theta_1T_1}\sqrt{2T_1+1}\,W(TT'11;fT_1)\langle\theta'S'T'\|Q\|\theta_1S'T_1\rangle\langle\theta_1S'T_1\|Q\|\theta ST\rangle.\end{cases} \tag{V.21}$$

The same method when applied to $[V^Q_{\mu i}, W^Q_{\alpha j}]$ leads to the result

$$\sum_{\theta_1S_1T_1}\sqrt{(2S_1+1)(2T_1+1)}\,W(SS'11;f'S_1)\,W(TT'11;fT_1)\{\langle\theta'S'T'\|W^Q\|\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\|V^Q\|\theta ST\rangle$$

$$-(-)^{f+f'}\langle\theta'S'T'\|V^Q\|\theta_1S_1T_1\rangle\langle\theta_1S_1T_1\|W^Q\|\theta ST\rangle\}$$

$$=\sqrt{\tfrac{2}{3}}\begin{cases}\delta_{1,f}\sqrt{S'(S'+1)(2S'+1)}\,W(SS'11;f'S')\langle\theta'S'T'\|W^Q\|\theta ST\rangle\\ +\delta_{1,f'}\sqrt{T'(T'+1)(2T'+1)}\,W(TT'11;fT')\langle\theta'S'T'\|W^Q\|\theta ST\rangle\\ +\delta_{1f}(-)^{f'}\dfrac{4-f'(f'+1)}{2}\displaystyle\sum_{\theta_1S_1}\sqrt{2S_1+1}\,W(SS'11;f'S_1)\langle\theta'S'T'\|W^S\|\theta_1S_1T'\rangle\langle\theta_1S_1T'\|Q\|\theta ST\rangle\\ +\delta_{1f'}(-)^{f}\dfrac{4-f(f+1)}{2}\displaystyle\sum_{\theta_1T_1}\sqrt{2T_1+1}\,W(TT'11;fT_1)\langle\theta'S'T'\|W^T\|\theta_1S'T_1\rangle\langle\theta_1S'T_1\|Q\|\theta ST\rangle.\end{cases} \tag{V.22}$$

We do not give here the result derived from the commutator $[W^Q_{\mu i}, W^Q_{\alpha j}]$ due to its great complexity.

Now using Eqs. (V.8) (with $T^Q=V^Q$ and $T^Q=W^Q$), (V.21), and (V.22), the invariant operators I_3, I_4, I_5, and I_6, and generalizing the method used for the calculation of $\langle ST|Q|S'T'\rangle^2$, we can obtain the quantities:

$$\sum_{\theta\theta'}\langle\theta'S'T'\|Q\|\theta ST\rangle\langle\theta'S'T'\|V^Q\|\theta ST\rangle,\quad \sum_{\theta\theta'}\langle\theta'S'T'\|Q\|\theta ST\rangle\langle\theta'S'T'\|W^Q\|\theta ST\rangle,\quad \sum_{\theta\theta'}\langle\theta'S'T'\|V^Q\|\theta ST\rangle^2,$$

$$\times\sum_{\theta\theta'}\langle\theta'S'T'\|V^Q\|\theta ST\rangle\langle\theta'S'T'\|W^Q\|\theta ST\rangle. \tag{V.23}$$

In principle, using the invariant operator I_7 and the equation derived from the commutator $[W^Q_{\mu i}, W^Q_{\alpha j}]$ we can also calculate the quantity

$$\sum_{\theta\theta'}\langle\theta'S'T'\|W^Q\|\theta ST\rangle^2. \tag{V.24}$$

VI. SEARCH FOR A COMPLETE DETERMINATION OF EACH MATRIX ELEMENT $\langle(p'p'p'')\theta S'T'\|Q\|(pp'p'')\theta ST\rangle^2$ WITH $\theta=(\omega,\varphi)$ OR (σ,τ)

Up to now all the equations we have written contain the index θ and do not need it to be more precise. But from now on, we shall distinguish the two cases: $\theta=(\omega,\varphi)$ and $\theta=(\sigma,\tau)$. In each case we have specific equations satisfied by the matrix elements of Q, V^Q, and W^Q.

(a) First case $\theta=(\omega\varphi)$: The operators Ω and Φ are diagonal and we can write

$$\sqrt{S(S+1)T(T+1)}\langle\varphi'\omega'ST\|Q\|\varphi\omega ST\rangle$$

$$=\omega\,\delta_{\varphi\varphi'}\,\delta_{\omega\omega'}, \tag{VI.1}$$

$$2\sqrt{S(S+1)T(T+1)}\langle\varphi'\omega'ST\|V^Q\|\varphi\omega ST\rangle$$

$$+S(S+1)\sum_{\varphi_1\omega_1T_1}\langle\varphi'\omega'ST\|Q\|\varphi_1\omega_1ST_1\rangle$$

$$\times\langle\varphi\omega ST\|Q\|\varphi_1\omega_1ST_1\rangle$$

$$+T(T+1)\sum_{\varphi_1\omega_1S_1}\langle\varphi'\omega'ST\|Q\|\varphi_1\omega_1S_1T\rangle$$

$$\times\langle\varphi\omega ST\|Q\|\varphi_1\omega_1S_1T)$$

$$=\varphi\delta_{\varphi\varphi'}\,\delta_{\omega\omega'}. \tag{VI.2}$$

Furthermore, the operators Φ and Ω give a relation between the nondiagonal matrix elements of Q, V^Q, and W^Q. For example, let us calculate the commutation relation of Ω with $Q_{\mu i}$,

$$[\Omega, Q_{\mu i}] = \sum_\alpha (-)^\alpha [V_\alpha^S S_{-\alpha}, Q_{\mu i}]$$

$$= \sqrt{2} \sum_\alpha c(111; \alpha+\mu, -\alpha)\{V^Q_{\alpha+\mu\, i} S_{-\alpha} + V^S_{-\alpha} Q_{\alpha+\mu\, i}\}.$$

Using relation (III.9) we can write this again,

$$[\Omega, Q_{\mu i}] = \sqrt{2} \sum_\alpha c(111; \alpha+\mu, -\alpha) V^Q_{\alpha+\mu\, i} S_{-\alpha} + \sqrt{2} \sum_j c(111; i+j, -i) T_{-j} V^Q_{\mu\, i+j}.$$

Now we take the matrix elements of this equation in the base where Ω is diagonal, and obtain

$$(\omega' - \omega)\langle \varphi'\omega'S'T' \| Q \| \varphi\omega ST\rangle = \frac{T'(T'+1) - T(T+1) + S'(S'+1) - S(S+1)}{2} \times \langle \varphi'\omega'S'T' \| V^Q \| \varphi\omega ST\rangle. \quad \text{(VI. 3)}$$

We can apply the same calculus for Φ. We have first for the commutator

$$[\Phi, Q_{\mu i}] = \sum_\alpha (-)^\alpha W^S_{-\alpha}[S_\alpha, Q_{\mu i}] + \sum_\alpha (-)^\alpha [W^S_\alpha, Q_{\mu i}] S_{-\alpha} + \sum_j (-)^j W^T_{-j}[T_j, Q_{\mu i}] + \sum_j (-)^j [W^T_j, Q_{\mu i}] T_{-j}.$$

We introduce the commutators, explicitly, and then use the formula (III.9) and get

$$[\Phi, Q_{\mu i}] = \sqrt{2} \sum_\alpha c(111; \alpha+\mu, -\alpha)\{W^Q_{\alpha+\mu\, i} S_{-\alpha} + S_{-\alpha} W^Q_{\alpha+\mu\, i}\} + \sqrt{2} \sum_j c(111; i+j, -j)\{W^Q_{\mu\, i+j} T_{-j} + T_{-j} W^Q_{\mu\, i+j}\}.$$

Finally we take the matrix elements of this last equation in the basis where Φ is diagonal and get

$$(\varphi' - \varphi)\langle \varphi'\omega'S'T' \| Q \| \varphi\omega ST\rangle = \{S'(S'+1) - S(S+1) + T'(T'+1) - T(T+1)\} \times \langle \varphi'\omega'S'T' \| W^Q \| \varphi\omega ST\rangle. \quad \text{(VI. 4)}$$

Observe that the matrix element, when diagonal in S and T, is necessarily diagonal in Φ and Ω. Furthermore (VI.1) gives $\omega = 0$ when S and/or $T = 0$. This last result has been independently given in a recent paper by Quesne.[15]

Relation (VI.2) can be given a new form if one expresses the matrix element of V^Q,

$$T(T+1) \sum_{\varphi_1\omega_1 S_1} \frac{S(S+1) - S_1(S_1+1) + 4}{2} \langle \varphi'\omega'ST \| Q \| \varphi_1\omega_1 S_1 T\rangle \times \langle \varphi\omega ST \| Q \| \varphi_1\omega_1 S_1 T\rangle + S(S+1) \times \sum_{\varphi_1\omega_1 T_1} \frac{T(T+1) - T_1(T_1+1) + 4}{2} \times \langle \varphi'\omega'ST \| Q \| \varphi_1\omega_1 ST_1\rangle \langle \varphi\omega ST \| Q \| \varphi_1\omega_1 ST_1\rangle = \varphi\delta_{\varphi\varphi'}\delta_{\omega\omega'}. \quad \text{(VI. 5)}$$

Using this result we can see that $\varphi \geq 0$ if $S = 0$ (or $T = 0$) and $\varphi = 0$ when $S = T = 0$.

(b) Second case $\theta = (\sigma\tau)$: Here the operators s and t are diagonal and we can write

$$T(T+1) \sum_{\sigma_1\tau_1 S_1} \langle \sigma'\tau'ST \| Q \| \sigma_1\tau_1 S_1 T\rangle \times \langle \sigma\tau ST \| Q \| \sigma_1\tau_1 S_1 T\rangle = \sigma\delta_{\sigma\sigma'}\delta_{\tau\tau'}, \quad \text{(VI. 6)}$$

$$S(S+1) \sum_{\sigma_1\tau_1 T_1} \langle \sigma'\tau'ST \| Q \| \sigma_1\tau_1 ST_1\rangle \langle \sigma\tau ST \| Q \| \sigma_1\tau_1 ST_1\rangle = \tau\delta_{\tau\tau'}\delta_{\sigma'\sigma}.$$

We immediately see that the eigenvalues σ and τ are always positive, and $\sigma = 0$ ($\tau = 0$) when $T = 0$ (resp. $S = 0$).

Finally the commutators of s and t with the generator $Q_{\mu i}$ lead to the pair of equations:

$$(\sigma' - \sigma)\langle \sigma'\tau'S'T' \| Q \| \sigma\tau ST\rangle = \sum_{\sigma_1\tau_1 S_1} \sqrt{3(2S_1+1)}\, W(SS'11; 1S_1) \times \{\sqrt{T'(T'+1)}\langle \sigma'\tau'S'T' \| Q \| \sigma_1\tau_1 S_1 T'\rangle \times \langle \sigma_1\tau_1 S_1 T' \| V^Q \| \sigma\tau ST\rangle - \sqrt{T(T+1)}\langle \sigma'\tau'S'T' \| V^Q \| \sigma_1\tau_1 S_1 T\rangle \times \langle \sigma_1\tau_1 S_1 T \| Q \| \sigma\tau ST\rangle\}$$

$$(\tau' - \tau)\langle \sigma'\tau'S'T' \| Q \| \sigma\tau ST\rangle = \sum_{\sigma_1\tau_1 T_1} \sqrt{3(2T_1+1)}\, W(TT'11; 1T_1) \times \{\sqrt{S'(S'+1)}\langle \sigma'\tau'S'T' \| Q \| \sigma_1\tau_1 S'T_1\rangle \times \langle \sigma_1\tau_1 S'T_1 \| V^Q \| \sigma\tau ST\rangle - \sqrt{S(S+1)}\langle \sigma'\tau'S'T' \| V^Q \| \sigma_1\tau_1 ST_1\rangle \langle \sigma_1\tau_1 ST_1 \| Q \| \sigma\tau ST\rangle\}. \quad \text{(VI. 7)}$$

We emphasize the remarkable symmetry of the two labels σ and τ, which lengthens the same symmetry of the quantum numbers (SM_S) and (TM_T).

Using the results (VI.3) and (VI.5) we can obtain the eigenvalues of Ω and Φ in the particular case when $T = p$ and any S,

$$\varphi_{pS} = p''^2\{(p+1)^2 + (p'+1)^2\} + p'(p'+2)(p+1)^2 + S(S+1)(p+1)(p+2), \quad \text{(VI. 8)}$$

$$\omega_{pS} = p''(p'+1)(p+1).$$

We have found that in particular simple and nontrivial cases, the sets of equations given in Sec. V are sufficient to calculate completely the eigenvalues of ω and φ (or σ and τ) and the matrix elements of Q. As an example, we have completely solved the case of the (320) IR with $\theta = (\omega, \varphi)$. N_{ST} is easily calculated with the aid of the formulas given in Sec. III, and we have also the nontrivial multiplicity values $N_{12}(320) = N_{21}(320) = 3$. In Tables I, II, and III, we give the calculated values of:

TABLE I. This table gives the values of $\Sigma_{\theta\theta'}\langle(320)\theta\, ST \| Q \| (320)\theta' S'T'\rangle^2$ in the upper part. The rows are labeled by ST and the columns by $S'T'$. We give the value of the multiplicity $N_{ST}(320)$ between brackets when it exceeds one. In the lower two rows are given the values of $\Sigma_{\omega\varphi}\omega^2$ and $\Sigma_{\omega\varphi}\varphi$ for every $S'T'$.

ST \ S'T'	32	31	30	23	22	21	20	13	12	11	10	03	02	01
32	0	$\frac{8}{3}$	0	$\frac{7}{5}$	4	$\frac{44}{15}$	0	0	0	0	0	0	0	0
31	$\frac{40}{9}$	0	$\frac{32}{9}$	0	$\frac{20}{9}$	4	$\frac{7}{9}$	0	0	0	0	0	0	0
30	0	$\frac{32}{3}$	0	0	0	$\frac{19}{3}$	0	0	0	0	0	0	0	0
23	$\frac{7}{5}$	0	0	0	4	0	0	$\frac{8}{3}$	$\frac{44}{15}$	0	0	0	0	0
(2) 22	$\frac{28}{5}$	$\frac{28}{15}$	0	$\frac{28}{5}$	$\frac{1}{2}$	$\frac{247}{30}$	0	$\frac{28}{15}$	$\frac{247}{30}$	$\frac{21}{10}$	0	0	0	0
(3) 21	$\frac{308}{45}$	$\frac{28}{5}$	$\frac{133}{45}$	0	$\frac{247}{18}$	$\frac{19}{2}$	$\frac{52}{9}$	0	$\frac{233}{30}$	$\frac{79}{10}$	$\frac{44}{15}$	0	0	0
20	0	$\frac{49}{15}$	0	0	0	$\frac{52}{3}$	0	0	0	$\frac{12}{5}$	0	0	0	0
13	0	0	0	$\frac{40}{9}$	$\frac{20}{9}$	0	0	0	4	0	0	$\frac{32}{9}$	$\frac{7}{9}$	0
(3) 12	0	0	0	$\frac{308}{45}$	$\frac{247}{18}$	$\frac{233}{30}$	0	$\frac{28}{5}$	$\frac{19}{2}$	$\frac{79}{10}$	0	$\frac{133}{45}$	$\frac{52}{9}$	$\frac{44}{15}$
(2) 11	0	0	0	0	$\frac{35}{6}$	$\frac{79}{6}$	$\frac{4}{3}$	0	$\frac{79}{6}$	$\frac{25}{2}$	$\frac{4}{3}$	0	$\frac{4}{3}$	$\frac{4}{3}$
10	0	0	0	0	0	$\frac{44}{3}$	0	0	0	4	0	0	0	$\frac{25}{3}$
03	0	0	0	0	0	0	0	$\frac{32}{3}$	$\frac{19}{3}$	0	0	0	0	0
02	0	0	0	0	0	0	0	$\frac{49}{15}$	$\frac{52}{3}$	$\frac{12}{5}$	0	0	0	0
01	0	0	0	0	0	0	0	0	$\frac{44}{3}$	4	$\frac{25}{3}$	0	0	0
$\sum_{\omega\varphi}\omega^2$	0	0	0	0	18	114	0	0	114	50	0	0	0	0
$\sum_{\omega\varphi}\varphi$	248	168	128	248	170	308	104	168	308	58	8	128	104	8

TABLE II. This table gives the values of $\Sigma_{\theta\theta'}\langle(320)\theta\, ST \| V^Q \| (320)\theta' S'T'\rangle^2$. The rows are labeled by ST, the columns by $S'T'$.

ST \ S'T'	32	31	30	23	22	21	20	13	12	11	10	03	02	01
32	128	0	0	0	4	4	0	0	0	0	0	0	0	0
31	0	$\frac{128}{3}$	0	0	20	$\frac{76}{3}$	0	0	0	0	0	0	0	0
30	0	0	0	0	0	64	0	0	0	0	0	0	0	0
23	0	0	0	128	4	0	0	0	4	0	0	0	0	0
22	$\frac{28}{5}$	$\frac{84}{5}$	0	$\frac{28}{5}$	98	$\frac{152}{5}$	0	$\frac{84}{5}$	$\frac{152}{5}$	$\frac{42}{5}$	0	0	0	0
21	$\frac{28}{3}$	$\frac{532}{15}$	$\frac{448}{15}$	0	$\frac{152}{3}$	$\frac{274}{3}$	$\frac{28}{3}$	0	78	$\frac{176}{5}$	$\frac{84}{5}$	0	0	0
20	0	0	0	0	0	28	0	0	0	60	0	0	0	0
13	0	0	0	0	20	0	0	$\frac{128}{3}$	$\frac{76}{3}$	0	0	0	0	0
12	0	0	0	$\frac{28}{3}$	$\frac{152}{3}$	78	0	$\frac{532}{15}$	$\frac{274}{3}$	$\frac{176}{5}$	0	$\frac{448}{15}$	$\frac{28}{3}$	$\frac{84}{5}$
11	0	0	0	0	$\frac{70}{3}$	$\frac{176}{3}$	$\frac{100}{3}$	0	$\frac{176}{3}$	2	$\frac{100}{3}$	0	$\frac{100}{3}$	$\frac{100}{3}$
10	0	0	0	0	0	84	0	0	0	100	0	0	0	0
03	0	0	0	0	0	0	0	0	64	0	0	0	0	0
02	0	0	0	0	0	0	0	0	28	60	0	0	0	0
01	0	0	0	0	0	0	0	0	84	100	0	0	0	0

TABLE III. The rows are labeled by ST, the columns by $S'T'$. It gives the values of $\Sigma_{\theta\theta'}\langle(320)\theta ST \| Q \| (320)\theta' S'T'\rangle\langle(320)\theta ST \| W^Q \| (320)\theta' S'T'\rangle$.

ST \ S'T'	32	31	30	23	22	21	20	13	12	11	10	03	02	01
32	0	$\frac{160}{3}$	0	$-\frac{56}{5}$	52	$\frac{628}{15}$	0	0	0	0	0	0	0	0
31	$\frac{800}{9}$	0	$\frac{640}{9}$	0	$-\frac{20}{9}$	52	$\frac{56}{9}$	0	0	0	0	0	0	0
30	0	$\frac{640}{3}$	0	0	0	$\frac{128}{3}$	0	0	0	0	0	0	0	0
23	$\frac{56}{5}$	0	0	0	52	0	0	$\frac{160}{3}$	$\frac{628}{15}$	0	0	0	0	0
22	$\frac{364}{5}$	$-\frac{28}{15}$	0	$\frac{364}{5}$	17	$\frac{1798}{15}$	0	$-\frac{28}{15}$	$\frac{1798}{15}$	$\frac{147}{5}$	0	0	0	0
21	$\frac{4396}{45}$	$\frac{364}{5}$	$\frac{896}{45}$	0	$\frac{1798}{9}$	197	$\frac{596}{9}$	0	$\frac{803}{15}$	$\frac{476}{5}$	$\frac{628}{15}$	0	0	0
20	0	$\frac{392}{15}$	0	0	0	$\frac{596}{3}$	0	0	0	$\frac{276}{5}$	0	0	0	0
13	0	0	0	$\frac{800}{9}$	$-\frac{20}{9}$	0	0	0	52	0	0	$\frac{640}{9}$	$\frac{56}{9}$	0
12	0	0	0	$\frac{4396}{45}$	$\frac{1798}{9}$	$\frac{803}{15}$	0	$\frac{364}{5}$	197	$\frac{476}{5}$	0	$\frac{896}{45}$	$\frac{596}{9}$	$\frac{628}{15}$
11	0	0	0	0	$\frac{245}{3}$	$\frac{476}{3}$	$\frac{92}{3}$	0	$\frac{476}{3}$	125	$\frac{100}{3}$	0	$\frac{92}{3}$	$\frac{100}{3}$
10	0	0	0	0	0	$\frac{628}{3}$	0	0	0	100	0	0	0	$\frac{200}{3}$
03	0	0	0	0	0	0	0	$\frac{640}{3}$	$\frac{128}{3}$	0	0	0	0	0
02	0	0	0	0	0	0	0	$\frac{392}{15}$	$\frac{596}{3}$	$\frac{276}{5}$	0	0	0	0
01	0	0	0	0	0	0	0	0	$\frac{628}{3}$	100	$\frac{200}{3}$	0	0	0

$$\sum_{\theta\theta'} \langle(320)\,\theta ST \,\|Q\|\, (320)\,\theta' S'T'\rangle^2,$$

$$\sum_{\theta\theta'} \langle(320)\,\theta ST \,\|V^Q\|\, (320)\,\theta' S'T'\rangle^2,$$

$$\sum_{\theta\theta'} \langle(320)\,\theta ST \,\|Q\|\, (320)\,\theta' S'T'\rangle \times \langle(320)\,\theta ST \,\|W^Q\|\, (320)\,\theta' S'T'\rangle,$$

respectively. Note the particular result (characteristic of the IR with $p''=0$)

$$\sum_{\theta\theta'} \langle(320)\,\theta ST \,\|Q\|\, (320)\theta' S'T'\rangle \times \langle(320)\,\theta ST \,\|V^Q\|\, (320)\,\theta' S'T'\rangle = 0.$$

Finally in Table IV we give the eigenvalues of the operators Ω and Φ and the matrix elements $\langle(320)\,\omega\varphi ST \,\|Q\|\, (320)\,\omega'\varphi' S'T'\rangle^2$. Note that these matrix elements can be irrational, but φ is always rational.

CONCLUDING REMARKS

We want to point that we have been able to directly derive (that is without the medium of the Gel'fand basis) the eigenvalues of the operators Ω and Φ and all the semireduced matrix elements of Q, in the particular case, taken as an example of the [320] IR in which the multiplicity N_{ST}, of the (ST) states of SU(2)⊗SU(2) exceeds 2, and reaches 3 for $(S=1,\ T=2)$ and $(S=2,\ T=1)$. Some of the eigenvalues of these operators are irrational numbers; but we have shown that the sums over the square of the semireduced matrix elements $\langle ST \,|Q|\, S'T'\rangle^2 = \sum_{\theta\theta'} \langle\theta ST \|Q\| \theta' S'T'\rangle^2$ are rational numbers. Moreover we have given a method which allows the same calculation for any $(pp'p'')$ IR.

Recently a work has been published by Quesne,[15] who has computed the eigenvalues of Ω and Φ for a number of IR. We are in exact agreement with the results of this author, if we make the following correspondence denoting the operators of Ref. 16 with a prime to avoid any possibility of confusion:

$$Q'_{qk} = \tfrac{1}{2} Q_{\alpha i},$$

$$C^{(111)} = \tfrac{1}{2} \sum_{\alpha i} S_\alpha Q_{-\alpha -i} T_i,$$

$$C^{(202)} = \tfrac{1}{4} \sum_i (-)^i V_i^T V_{-i}^T - \tfrac{3}{4} \sum_\alpha (-)^\alpha S_\alpha S_{-\alpha},$$

$$C^{(022)} = \tfrac{1}{4} \sum_\alpha (-)^\alpha V_\alpha^S V_{-\alpha}^S - \tfrac{3}{4} \sum_i (-)^i T_i T_{-i},$$

$$C^{(112)} = -\tfrac{1}{2} \sum_{\alpha\mu ij} (-)^{\alpha+\mu+i+j} c(111;\alpha,-\mu) \times c(111;i,-j) S_{-\alpha} T_{-i} Q_{\mu j} Q_{\alpha-\mu\, i-j},$$

$$\Omega' = \tfrac{1}{2}\Omega,$$

$$\Phi' = \tfrac{1}{4}\Phi - \tfrac{3}{4} \sum_i (-)^i T_i T_{-i} - \tfrac{3}{4} \sum_\alpha (-)^\alpha S_\alpha S_{-\alpha} - \tfrac{1}{2} \sum_{\alpha i} (-)^{\alpha+i} S_\alpha S_{-\alpha} T_i T_{-i}.$$

With the vectors V and W defined in Sec. III, we can easily build a set of SU(2)⊗SU(2) invariant operators, in the enveloping algebra of SU(4):

$$\sum_\alpha (-)^\alpha S_\alpha V_{-\alpha}^S, \quad \sum_i (-)^i T_i V_{-i}^T, \quad \sum_\alpha (-)^\alpha W_\alpha^S S_{-\alpha},$$

$$\sum_i (-)^i W_i^T T_{-i}, \quad \sum_\alpha (-)^\alpha V_\alpha^S V_{-\alpha}^S, \quad \sum_i (-)^i V_i^T V_{-i}^T,$$

$$\sum_\alpha (-)^\alpha V_\alpha^S W_{-\alpha}^S, \quad \sum_i (-)^i V_i^T W_{-i}^T, \quad \sum_\alpha (-)^\alpha W_\alpha^S W_{-\alpha}^S,$$

$$\sum_i (-)^i W_i^T W_{-i}^T.$$

TABLE IV. The rows (and columns) of this table are labeled by the values of φ (first row and column), ω (second row and column), and ST (third row and column) relative to the (320) IR. The other numbers are the values of the square of the semireduced matrix elements of Q, i.e., $\langle(320)\ \omega\varphi ST\|Q\|(320)\omega'\varphi'S'T'\rangle^2$.

φ			248	168	128	248	170	170	90	90	128	104	168	90	90	128	58	58	8	128	104	8
	ω		0	0	0	0	+3	−3	$+\sqrt{57}$	$-\sqrt{57}$	0	0	0	$+\sqrt{57}$	$-\sqrt{57}$	0	+5	−5	0	0	0	0
		ST	32	31	30	23	22	22	21	21	21	20	13	12	12	12	11	11	10	03	02	01
248	0	32	0	$\frac{8}{3}$	0	$\frac{7}{5}$	2	2	$\frac{50}{57}$	$\frac{50}{57}$	$\frac{112}{95}$	0	0	0	0	0	0	0	0	0	0	0
168	0	31	$\frac{40}{9}$	0	$\frac{32}{9}$	0	$\frac{10}{9}$	$\frac{10}{9}$	2	2	0	$\frac{7}{9}$	0	0	0	0	0	0	0	0	0	0
128	0	30	0	$\frac{32}{3}$	0	0	0	0	$\frac{128}{57}$	$\frac{128}{57}$	$\frac{35}{19}$	0	0	0	0	0	0	0	0	0	0	0
248	0	23	$\frac{7}{5}$	0	0	0	2	2	0	0	0	0	$\frac{8}{3}$	$\frac{50}{57}$	$\frac{50}{57}$	$\frac{112}{95}$	0	0	0	0	0	0
170	+3	22	$\frac{14}{5}$	$\frac{14}{15}$	0	$\frac{14}{5}$	$\frac{1}{4}$	0	$\frac{401+51\sqrt{57}}{456}$	$\frac{401-51\sqrt{57}}{456}$	$\frac{224}{95}$	0	$\frac{14}{15}$	$\frac{401+51\sqrt{57}}{456}$	$\frac{401-51\sqrt{57}}{456}$	$\frac{224}{95}$	0	$\frac{21}{20}$	0	0	0	0
170	−3	22	$\frac{14}{5}$	$\frac{14}{15}$	0	$\frac{14}{5}$	0	$\frac{1}{4}$	$\frac{401-51\sqrt{57}}{456}$	$\frac{401+51\sqrt{57}}{456}$	$\frac{224}{95}$	0	$\frac{14}{15}$	$\frac{401-51\sqrt{57}}{456}$	$\frac{401+51\sqrt{57}}{456}$	$\frac{224}{95}$	$\frac{21}{20}$	0	0	0	0	0
90	$+\sqrt{57}$	21	$\frac{350}{171}$	$\frac{14}{5}$	$\frac{896}{855}$	0	$\frac{5}{1368}(401+51\sqrt{57})$	$\frac{5}{1368}(401-51\sqrt{57})$	$\frac{19}{4}$	0	0	$\frac{14}{171}$	0	$\frac{15}{4}$	0	0	$\frac{21}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{21}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{126}{95}$	0	0	0
90	$-\sqrt{57}$	21	$\frac{350}{171}$	$\frac{14}{5}$	$\frac{896}{855}$	0	$\frac{5}{1368}(401-51\sqrt{57})$	$\frac{5}{1368}(401+51\sqrt{57})$	0	$\frac{19}{4}$	0	$\frac{14}{171}$	0	0	$\frac{15}{4}$	0	$\frac{21}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{21}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{126}{95}$	0	0	0
128	0	21	$\frac{784}{285}$	0	$\frac{49}{57}$	0	$\frac{224}{57}$	$\frac{224}{57}$	0	0	0	$\frac{320}{57}$	0	0	0	$\frac{4}{15}$	$\frac{32}{19}$	$\frac{32}{19}$	$\frac{16}{57}$	0	0	0
104	0	20	0	$\frac{49}{15}$	0	0	0	0	$\frac{14}{57}$	$\frac{14}{57}$	$\frac{320}{19}$	0	0	0	0	0	$\frac{6}{5}$	$\frac{6}{5}$	0	0	0	0
168	0	13	0	0	0	$\frac{40}{9}$	$\frac{10}{9}$	$\frac{10}{9}$	0	0	0	0	0	2	2	0	0	0	0	$\frac{32}{9}$	$\frac{7}{9}$	0
90	$+\sqrt{57}$	12	0	0	0	$\frac{350}{171}$	$\frac{5}{1368}(401+51\sqrt{57})$	$\frac{5}{1368}(401-51\sqrt{57})$	$\frac{15}{4}$	0	0	0	$\frac{14}{5}$	$\frac{19}{4}$	0	0	$\frac{21}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{21}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	0	$\frac{896}{855}$	$\frac{14}{171}$	$\frac{126}{95}$
90	$-\sqrt{57}$	12	0	0	0	$\frac{350}{171}$	$\frac{5}{1368}(401-51\sqrt{57})$	$\frac{5}{1368}(401+51\sqrt{57})$	0	$\frac{15}{4}$	0	0	$\frac{14}{5}$	0	$\frac{19}{4}$	0	$\frac{21}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{21}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	0	$\frac{896}{855}$	$\frac{14}{171}$	$\frac{126}{195}$
128	0	12	0	0	0	$\frac{784}{285}$	$\frac{224}{57}$	$\frac{224}{57}$	0	0	$\frac{4}{15}$	0	0	0	0	0	$\frac{32}{19}$	$\frac{32}{19}$	0	$\frac{49}{57}$	$\frac{320}{57}$	$\frac{16}{57}$
58	+5	11	0	0	0	0	0	$\frac{35}{12}$	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}-\frac{41}{95}\right)$	$\frac{160}{57}$	$\frac{2}{3}$	0	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{35}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{160}{57}$	$\frac{25}{4}$	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$
58	−5	11	0	0	0	0	$\frac{35}{12}$	0	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}-\frac{41}{95}\right)$	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{160}{57}$	$\frac{2}{3}$	0	$\frac{35}{8}\left(-\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{35}{8}\left(\sqrt{\frac{3}{19}}+\frac{41}{95}\right)$	$\frac{160}{57}$	0	$\frac{25}{4}$	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$
8	0	10	0	0	0	0	0	0	$\frac{126}{19}$	$\frac{126}{19}$	$\frac{80}{57}$	0	0	0	0	0	2	2	0	0	0	$\frac{25}{3}$
128	0	03	0	0	0	0	0	0	0	0	0	0	$\frac{32}{3}$	$\frac{128}{57}$	$\frac{128}{57}$	$\frac{35}{19}$	0	0	0	0	0	0
104	0	02	0	0	0	0	0	0	0	0	0	0	$\frac{49}{15}$	$\frac{14}{57}$	$\frac{14}{57}$	$\frac{320}{19}$	$\frac{6}{5}$	$\frac{6}{5}$	0	0	0	0
8	0	01	0	0	0	0	0	0	0	0	0	0	0	$\frac{126}{19}$	$\frac{126}{19}$	$\frac{80}{57}$	2	2	$\frac{25}{3}$	0	0	0

All of these operators are not independent [for example $\sum_\alpha(-)^\alpha S_\alpha V^S_{-\alpha}=\sum_i(-)^i T_i V^T_{-i}$]. In her paper Quesne states that among all of these, only seven operators are independent, and form an integrity basis for the SU(2) $\otimes$SU(2) scalars belonging to the enveloping algebra of SU(4).

Using them we have written the pair of operators Ω and Φ, first introduced by Nagel and Moshinsky,[9] and we have furthermore shown that another pair of operators, which we have called s and t, allows the solution of the state labeling problem.[17]

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Poincaré is a subgroup of Galilei in one space dimension more

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Through an imaginary change of coordinates, the ordinary Poincaré algebra is shown to be a subalgebra of the Galilei one in four space dimensions. Through a subsequent contraction the remaining Lie generators are eliminated in a natural way. An application of these results to connect Galilean and relativistic field equations is discussed.

1. INTRODUCTION

Some papers have been issued[1] which deal with the connection between the usual relativistic field equations (those of Dirac, Bargmann and Wigner, Proca, Rarita and Schwinger, and Singh and Hagen)[2] and the nonrelativistic ones (of Lévy-Leblond and Hagen and Hurley).[3] In particular, for example, from the Dirac eqution one obtains that of Lévy-Leblond[3] for a spin-$\frac{1}{2}$ particle, and subsequently the Schrödinger–Pauli equation, and starting with the Bargmann–Wigner equation for an arbitrary spin particle one obtains the $6s+1$ Galilean invariant theory of Hagen and Hurley.[3]

These connections have been established by means of a general change of the coordinates of the Minkowski space, which has the property of showing up a Galilean $(2+1)$-dimensional Lie subalgebra in the ordinary Poincaré algebra.[4] The usual light-cone frame[2] and the nonorthogonal one of Bell and Ruegg[5] are interesting particular cases of this general coordinate transformation. Moreover, making use of a convenient parametrization of this coordinate transformation a $(2+1)$-dimensional Poincaré algebra can be reobtained from the Galilean subalgebra in a continuous way and for a particular value of the parameter. Thus the circle is closed, making possible, in particular, the calculation of higher-order terms of the Schrödinger–Pauli equation derived before.[4]

It is the purpose of the present paper to go one step ahead of this program by studing in a more general manner the connection between the Galilei and Poincaré algebras. It will be shown by means of a certain imaginary coordinate transformation of the $(4+1)$-dimensional space–time frame—which changes the fourth spatial and the time coordinates—that the ordinary Poincaré algebra is a subalgebra of that of Galilei in four space dimensions. With a subsequent contraction of the Lie group, the rest of the generators will be consistently eliminated, and what will remain are exactly the commutation relations of the ordinary Poincaré group. We think that this procedure will be of much use in the derivation of relativistic field equations for any spin starting from Galilean invariant ones, in a process inverse to the one which has been employed before.[1] Anyway, we are not going to develop these possibilities here, where we only concern ourselves with the mathematics of the problem.

Finally, we want to point out that the results of the present paper confirm the feeling[1,4] that the loss of one space dimension in the light–cone frame—and in the Galilean equations thereby obtained—is not of much relevance. In other words, it has been conjectured[1,4] that the correct ordinary nonrelativistic expressions in three space dimensions would be obtained in the light-cone frame provided one started with the corresponding relativistic ones in one more space coordinate. Although in the contrary direction, this is also proved in this work.

2. FROM THE GALILEI TO THE POINCARE ALGEBRA

Let $x^\mu=(x^0, x^1, x^2, x^3, x^4)$ denote a point in a $(4+1)$-dimensional space–time frame, x^0 being the time coordinate, and let the coordinates of the same point in a new frame $\bar{x}^\mu=(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)$ be defined as

$$\begin{aligned} \bar{x}^0 &= ax^0+bx^4, \\ \bar{x}^i &= x^i \quad (i=1,2,3), \\ \bar{x}^4 &= cx^0+dx^4, \end{aligned} \qquad (2.1)$$

where a, b, c, and d are some constants, to be determined in order that the commutation relations of the Lie algebra of the Galilei group in the old frame be transformed into those of the ordinary Poincaré group in the new system.

Before going on let us recall that the Galilei group G in $4+1$ dimensions can be put into the form[6]

$$G=[\mathrm{SO}(4)\times T_4^{(v)}]\times[T_1\otimes T_4], \qquad (2.2)$$

where $T_4^{(v)}$ is the subgroup of the generators of Galilean boosts and T_4 that of the tranlations in 4-space. As is usual, $\times$ means semidirect and $\otimes$ direct product.

Now let

$$\begin{aligned} x^0 &= A\bar{x}^0+B\bar{x}^4, \quad A=\frac{d}{ad-bc}, \quad B=\frac{-b}{ad-bc}, \\ x^4 &= C\bar{x}^0+D\bar{x}^4, \quad C=\frac{-c}{ad-bc}, \quad D=\frac{a}{ad-bc} \end{aligned} \qquad (2.3)$$

be the inverse transformation of (2.1), and let l_i, λ_i $(i=1,2,3)$ be the generators[7] of the subgroup SO(4) of G, g_r $(r=1,2,3,4)$ those of the Galilean boosts, and d_μ $(\mu=0, 1, 2, 3, 4)$ the generators of the time–space translations. If we denote with a bar the corresponding generators in the new coordinate system, we have

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$$\bar{l}_i = l_i, \quad \bar{d}_i = d_i, \quad \bar{g}_i = Ag_i - C\lambda_i,$$
$$\bar{d}_0 = Ad_0 + Cd_4 \quad (i=1, 2, 3). \tag{2.4}$$

As it has already been pointed out,[4,8] the choice of the new generators of boosts (which we call k_i for a reason that will become obvious in a moment) is not so clear. In general, let us put $k_i = \alpha g_i + \beta\lambda_i$ ($i=1, 2, 3$), $\bar{g}_i$ being one of the possibilities, while the other two are $\bar{\lambda}_i$ (Bjorken, Kogut, and Soper)[8] and M^{0i} (quasi-light-cone frame)[4] where $M_{\mu\nu}$ are the generators of $SO(4) \times T_4^{(v)}$.

The commutation relations of the generators of the Galilei group in $4+1$ dimensions are the following. In the first place, for the rotation group SO(4) we have

$$[R_{rs}, R_{uv}] = i(\delta_{rv}R_{su} + \delta_{su}R_{rv} - \delta_{ru}R_{sv} - \delta_{sv}R_{ru})$$
$$(r, s, u, v = 1, 2, 3, 4)$$

or putting $l_i = -\frac{1}{2}\epsilon_{ijk}R_{jk}$, $\lambda_i = R_{4i}$ ($i, j, k = 1, 2, 3$), the equivalent ones

$$[l_i, l_j] = i\epsilon_{ijk}l_k,$$
$$[l_i, \lambda_j] = i\epsilon_{ijk}\lambda_k \quad (i, j, k = 1, 2, 3), \tag{2.5}$$
$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}l_k.$$

Moreover,

$$[\lambda_i, g_j] = i\delta_{ij}g_4, \quad [\lambda_i, d_j] = i\delta_{ij}d_4, \quad [\lambda_i, d_0] = 0,$$
$$[\lambda_i, g_4] = -ig_i, \quad [\lambda_i, d_4] = -id_i, \tag{2.6}$$
$$[l_i, g_4] = [l_i, d_4] = 0 \quad (i, j = 1, 2, 3).$$

And finally, the Lie algebra of the ordinary Galilei group,

$$[l_i, l_j] = i\epsilon_{ijk}l_k, \quad [l_i, g_j] = i\epsilon_{ijk}g_k, \quad [l_i, d_j] = i\epsilon_{ijk}d_k$$
$$[g_i, g_j] = [d_i, d_j] = [l_i, d_0] = [d_i, d_0] = 0, \quad [g_i, d_0] = id_i, \tag{2.7}$$
$$[g_i, d_j] = i\delta_{ij}\mu, \quad [l_i, \mu] = [d_i, \mu] = [g_i, \mu] = [d_0, \mu] = 0$$

in a true eleven-parameter group representation.[6]

In Eqs. (2.5)–(2.7) we have listed the whole set of commutation relations of our Galilei group in the original frame. After the coordinate transformation has been made, the new set of generators is given by

$$M_{\mu\nu} = \begin{pmatrix} 0 & Ag_1 - Cd_1 & Ag_2 - C\lambda_2 & Ag_3 - C\lambda_3 & (AD-BC)g_4 \\ -Ag_1 + C\lambda_1 & 0 & l_3 & -l_2 & -Bg_1 + D\lambda_1 \\ -Ag_2 + C\lambda_2 & -l_3 & 0 & l_1 & -Bg_2 + D\lambda_2 \\ -Ag_3 + C\lambda_3 & l_2 & -l_1 & 0 & -Bg_3 + D\lambda_3 \\ -(AD-BC)g_4 & Bg_1 - D\lambda_1 & Bg_2 - D\lambda_2 & Bg_3 - D\lambda_3 & 0 \end{pmatrix}, \tag{2.8}$$
$$d_\mu = (Ad_0 + Cd_4, d_1, d_2, d_3, Bd_0 + Dd_4),$$

and the possible choices for k_i which we have mentioned above are the following:

$$k_i = \bar{g}_i = Ag_i - C\lambda_i \qquad \text{(natural)},$$
$$k_i = \bar{l}_i = -Bg_i + D\lambda_i \qquad \text{(light-cone frame)}, \tag{2.9}$$
$$k_i = \bar{M}^{0i} = -(ag_i + b\lambda_i) \qquad \text{(quasi-light-cone frame)},$$

It is easy to see that in the transformed frame, the commutation relations of the generators l_i, k_i, d_i ($i=1, 2, 3$), and $h \equiv d_0$, are given by

$$[l_i, l_j] = i\epsilon_{ijk}l_k, \quad [l_i, k_j] = i\epsilon_{ijk}k_k, \quad [l_i, d_j] = i\epsilon_{ijk}d_k,$$
$$[k_i, k_j] = i\beta^2\epsilon_{ijk}l_k, \quad [d_i, d_j] = [l_i, h] = [d_i, h] = 0 \tag{2.10}$$
$$[k_i, h] = i(\alpha A - \beta C)d_i, \quad [k_i, d_j] = i(\alpha\mu + \beta bh + \beta d\bar{d}_4)\delta_{ij}.$$

These equalities constitute the Lie algebra of the ordinary Poincaré group, provided we put

$$\alpha = 0, \quad \beta^2 = -1, \quad \beta b = 1, \quad \beta d = 0, \quad \alpha A - \beta C = 1 \tag{2.11}$$

or, equivalently,

$$\alpha = 0, \quad \beta = \pm i, \quad b = \mp i, \quad d = 0, \quad C = \pm i. \tag{2.12}$$

The first choice of k_i is consistent with these conditions when $A=0$ and $C = \pm i$, B and D remaining arbitrary. In this case $k_i = \mp i\lambda_i$ ($i=1, 2, 3$). Also the second choice is consistent, taking $A=0$, B arbitrary, $C = D = \pm i$, and $k_i = \pm i\lambda_i$. Finally, for the third choice we have $a = d = 0$, $b = \pm i$, c arbitrary, and $k_i = \mp i\lambda_i$. This is the one we are going to study in more detail.[4]

Summing up, through the change of coordinates

$$\bar{x}^0 = \pm ix^4, \qquad x^0 = (1/c)\bar{x}^4,$$
$$\bar{x}^i = x^i \quad (i=1, 2, 3) \qquad x^i = \bar{x}^i, \tag{2.13}$$
$$\bar{x}^4 = cx^0 \quad (c \text{ arbitrary}) \qquad x^4 = \mp i\bar{x}^0,$$

the commutation relations for the transformed generators l_i, k_i, d_i ($i=1, 2, 3$), and h, are those of the ordinary Poincaré algebra. The rest of the transformed generators are easily seen to be given by

$$\bar{\lambda}_i = -(1/c)g_i, \quad \bar{g}_4 = (i/c)g_4, \quad \bar{d}_4 = (1/c)d_0 \tag{2.14}$$

and their commutation relations by

$$[\bar{\lambda}_i, \bar{\lambda}_j] = 0, \quad [l_i, \bar{\lambda}_j] = i\epsilon_{ijk}\bar{\lambda}_k, \quad [\bar{\lambda}_i, k_j] = -i\delta_{ij}g_4,$$
$$[\bar{\lambda}_i, g_4] = 0, \quad [\bar{\lambda}_i, d_j] = -(i/c)\delta_{ij}\mu, \quad [\bar{\lambda}_i, h] = 0,$$
$$[\bar{\lambda}_i, \bar{d}_4] = -(i/c)d_i, \tag{2.15}$$
$$[l_i, \bar{g}_4] = 0, \quad [k_i, \bar{g}_4] = i\bar{\lambda}_i, \quad [\bar{g}_4, d_i] = 0, \quad [\bar{g}_4, h] = (i/c)\mu,$$
$$[\bar{g}_4, \bar{d}_4] = (i/c^2)h, \quad [l_i, \bar{d}_4] = [k_i, \bar{d}_4] = [d_i, \bar{d}_4] = [h, \bar{d}_4] = 0.$$

Notice that when $c \to \infty$ $\bar{d}_4$ becomes a neutral element of the transformed Lie algebra. On the other hand, as the value of c is arbitrary we can make it go to infinity and, then, the only commutation relations of (2.15) which remain different from zero are the following:

$$[l_i, \bar{\lambda}_j] = i\epsilon_{ijk}\bar{\lambda}_k, \quad [\bar{\lambda}_i, k_j] = -i\delta_{ij}\bar{g}_4, \quad [k_i, \bar{g}_4] = i\bar{\lambda}_i. \qquad (2.16)$$

At the same time, observe that in (2.14) we also can make c as large as we like and, in this way, the generators $\bar{\lambda}_i$, $\bar{g}_4$, and $\bar{d}_4$ become negligible.

3. CONCLUSIONS

Starting with the Galilei group in four space/one time dimensions and making the imaginary change of coordinates given by (2.13), we have seen that the commutation relations satisfied by the generators transformed of l_i, g_i, d_i ($i = 1, 2, 3$), and d_0 are exactly those of the ordinary Poincaré algebra in the Minkowski space. That the transformation must be imaginary is clear if we notice that the Euclidean matrix $\delta_{\mu\nu}$ must be converted into the Lorentz's $g_{\mu\nu}$. Moreover, the transformed of the other five generators can be made as little as we like without affecting in the least the commutation relations which define the Poincaré algebra. Therefore, the transformation which has been carried out here can be defined as an imaginary change of coordinates followed by a contraction of the resulting Lie algebra with respect to the subalgebra of the generators (2.10) which satisfy the Poincaré relations.

Following a procedure parallel to the one developed elsewhere,[1,4] we presume that this result may be of much use in relating Galilean field equations with relativistic invariant ones and, particularly, to obtain the latter from the former, in just the reciprocal way to the one employed till now. The contraction of one space dimension which takes place in the light-cone frame,[1,8] i.e., the ordinary relativistic equations in the Minkowski space give rise in this frame to Galilean invariant ones in $2+1$ dimensions, also occurs here. In order to prove that this contraction does not depend on the particular number $3+1$ of dimensions of the Minkowski space, we have started here with a $(4+1)$-dimensional Galilei algebra. As expected, the spatial contraction has carried us to a $(3+1)$-dimensional Poincaré world. Extrapolating this procedure to an arbitrary number n of space dimensions, it is plausible to believe that the method develped here would transform a $(n+1)$-dimensional Galilei algebra into a $[(n-1)+1]$-dimensional Poincaré one while, at the same time, it looks appealing to think that the light-cone frame procedure would lead from a $(n+1)$-dimensional Poincaré algebra to a Galilei one in $(n-1)+1$ dimensions. Naturally this will have its parallel counterpart at the level of the wave equations, which always appear in the number of space–time dimensions of the corresponding Lie group.

Let us finish by saying that the ultimate purpose of this paper has been to find new and deeper connections between the mathematical structure of relativistic and Galilean Lie groups, connections which we hope will be useful in order to throw some light into the rather complicated world of their corresponding field equations for different spin particles.

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$SU_6 \times SU_3^c$ scalars in E_7 irreps

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The branching rules for $E_7 \to SU_6 \times SU_3^c$ have been determined for all irreps of E_7 of dimension < 52000. Among these irreps only those of dimension 0, 1463, 1539, 7371, and 8645 contain an $SU_6 \times SU_3^c$ scalar. The Kronecker square and cube of the 133-dimensional adjoint irrep of E_7 are resolved, and the number of symmetric second and third order operators that transform as scalars under $SU_6 \times SU_3^c$ determined.

We have recently used the theory of Schur functions (S functions[1]) to greatly simplify the calculation of Kronecker products and branching rules for the irreps of the five exceptional Lie groups.[2]

A systematic labeling scheme for the irreps of the exceptional groups, based on the maximal weights of their maximal subgroups, has been established. The relationship of our labels (λ) to the customary Dynkin labels[3] are given for a number of relevant irreps of E_7 in Table I.

Most algorithms for calculating the properties of the exceptional groups make use of projection onto the one-dimensional weight subspaces of the representations and as a consequence are unable to treat the properties of high dimensional irreps efficiently. Our techniques basically involve projection onto the irreps of the largest maximal subgroup followed by use of Schur functions (effectively Young tableaux) to systematically compute Kronecker products and branching rules for the exceptional groups and various relevant subgroups. In this way it has been possible to handle irreps even of dimension greater than 30 000 000 by simple hand calculation. In the particular case of E_7 the branching rules for $E_7 \to SU_6 \times SU_3^c$ were derived for all irreps of dimension less than 52 000.

The group structure $E_7 \supset SU_6 \times SU_3^c$ where SU_3^c is the quark color group and SU_6 the quark flavor group has been proposed[3–7] as a spontaneously broken gauge underlying a unified field theory of strong electromagnetic and weak interactions. The elementary fermions are ascribed to the (1^6) irrep which decomposes under $E_7 \to SU_6 \times SU_3^c$ as[8–10]

$$(1^6) \to \{1^5\}\{1^2\}^c + \{1\}\{1\}^c + \{1^3\}\{0\}^c.$$

The adjoint irrep (21^6) decomposes as

$$(21^6) \to \{0\}\{21\}^c + \{1^2\}\{1^2\}^c + \{1^4\}\{1\}^c + \{21^4\}\{0\}^c.$$

Ramond[6,7] has looked for irreps of E_7 that do not couple to the fermion mass matrix and which are capable of providing the vector bosons with their required Goldstone companions. Irreps of E_7 that can be associated with vacuum expectation values that preserve flavor and color must contain an $SU_6 \times SU_3^c$ singlet[7] (i.e., the $\{0\}\{0\}^c$ irrep).

Using the techniques outlined earlier[1,2] it is a simple task to identify the irreps of E_7 that contain a $SU_6 \times SU_3^c$ singlet. For the irreps appearing in Table I the singlet state occurs only in the

$$(0),\ (2^5 1^2),\ (2^6),\ (3^2 2^5),\ \text{and}\ (42^6) \tag{1}$$

irreps once and not at all in the remaining irreps listed in Table I. The branching rules for $(2^5 1^2)$ and (2^6) are known.[2,3,7] We also obtain

$$\begin{aligned}
(3^2 2^5) \to\ & \{1^2\}\{32\}^c + \{1^4\}\{31\}^c + \{0\}\{3^2\}^c + \{0\}\{3\}^c \\
& + (\{2^2 1^2\} + 2\{21^4\} + \{0\})\{21\}^c + (\{21^2\} + \{1^4\})\{2^2\}^c \\
& + (\{2^3 1^2\} + \{1^2\})\{2\}^c \\
& + (\{321^3\} + \{2^4\} + \{2^3 1^2\} + \{2\} + 2\{1^2\})\{1^2\}^c \\
& + (\{32^3 1\} + \{2^2\} + \{21^2\} + \{2^5\} + 2\{1^4\})\{1\}^c \\
& + (\{3^2 2^3\} + \{31^3\} + \{21^4\} + \{2^2 1^2\} + \{0\})\{0\}^c, \\
(42^6) \to\ & \{0\}\{42\}^c + \{1^2\}\{32\}^c + \{1^4\}\{31\}^c \\
& + (\{2^2 1^2\} + \{21^4\} + \{0\})\{21\}^c \\
& + (\{2^2\} + \{1^4\})\{2^2\}^c + (\{2^4\} + \{1^2\})\{2\}^c \\
& + (\{321^3\} + \{2^3 1^2\} + \{1^2\})\{1^2\}^c \\
& + (\{32^3 1\} + \{2^5 1^2\} + \{1^4\})\{1\}^c \\
& + (\{42^4\} + \{2^2 1^2\} + \{21^4\} + \{0\})\{0\}^c.
\end{aligned}$$

TABLE I. Irreps of E_7.

(λ)	Dynkin label	$D_{(\lambda)}$
(0)	(0000000)	1
(1^6)	(0000010)	56
(21^6)	(1000000)	133
$(2^5 1^2)$	(0000100)	1539
(2^6)	(0000020)	1463
(2^7)	(0000001)	912
$(32^5 1)$	(1000010)	6480
$(3^2 2^5)$	(0100000)	8645
$(3^4 2^3)$	(0001000)	27664
$(3^5 21)$	(0000110)	51072
(3^6)	(0000030)	24320
$(3^6 2)$	(0000011)	40755
(42^6)	(2000000)	7371
$(43^4 2^2)$	(1000100)	152152
$(4^3 3^4)$	(0010000)	365750
$(4^6 2)$	(0000011)	885248
(543^5)	(1100000)	573440
(63^6)	(3000000)	238602

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The identification of the $SU_6 \times SU_3^c$ singlets in E_7 irreps is also important in the construction of operators that break E_7 while preserving $SU_6 \times SU_3^c$ symmetry. The generators of E_7 belong to the adjoint irrep (21^6). The Kronecker squares and cubes of the adjoint irrep may be readily evaluated to give

$$(21^6) \otimes \{2\} = (42^6) + (2^5 1^2) + (0),$$

$$(21^6) \otimes \{1^2\} = (3^2 2^5) + (21^6),$$

$$(21^6) \otimes \{3\} = (63^6) + (43^4 2^2) + (3^2 2^5) + (2^6) + (21^6),$$

$$(21^6) \otimes \{21\} = (543^5) + (43^4 2^2) + (4^6 2) + (3^6 2) + (3^2 2^5) + (2^5 1^2) + 2(21^6),$$

$$(21^6) \otimes \{1^3\} = (4^3 3^4) + (42^6) + (3^2 2^5) + (2^5 1^2) + (0).$$

These results show that there is just one second-order symmetric E_7 scalar operator (the usual second-order Casimir invariant[9]) and no third-order symmetric E_7 scalar. There are two symmetric second-order E_7 symmetry breaking $SU_6 \times SU_3^c$ scalars transforming under E_7 as (42^6) and $(2^5 1^2)$ respectively while there are four symmetric third-order E_7 symmetry breaking $SU_6 \times SU_3$ scalars transforming as (63^6), $(43^4 2^2)$, $(3^2 2^5)$, and $(2^5 1^2)$, respectively. There is no difficulty in determining the number of $SU_6 \times SU_3^c$ scalars appearing in higher dimensional irreps of E_7 or determining the relevant branching rules.

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[8] We use ordered partitions[1,9] of $n-1$ integers f_i to label the irreps of SU_n where the f_i are related to the corresponding Dynkin integers a_i by $f_i = \sum_{j=i}^{n-1} a_j$.
[9] B.G. Wybourne, *Classical Groups for Physicists* (Wiley-Interscience, New York, 1974).
[10] Those wishing to see these results in terms of dimensions may easily translate them by using Tables A-2 and A-5 of Ref. 1.

Poisson's formulas for wave propagation in a superfluid

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The general solution of the initial-value problem for wave propagation in a superfluid is established. A uniqueness theorem is also proved for the associated initial-boundary value problem.

1. INTRODUCTION

The solution of the initial-value problem for the wave equation due to Poisson is a classical result in the literature. In the context of the acoustic problem for an inviscid gas the formula provides the solution for each of the thermodynamic variables in terms of the initial values of the variable and its time derivative. At low temperatures however, when the gas exhibits superfluidity properties, the wave equation is inadequate in describing the acoustic phenomenon and it becomes necessary to use two coupled equations for the thermodynamic variables.

It appears from the literature that Poisson type formulas for transient wave motion in a superfluid have not been obtained and it is the purpose of this note to establish these. The two- and three-dimensional cases are studied and extensions of the resulting formulas are used to derive a uniqueness theorem for the associated initial-boundary value problem.

2. THE INITIAL-VALUE PROBLEM

In the Euclidean space R^3 the propagation of sound in a superfluid is governed by the equations[1]

$$\alpha \frac{\partial^2 p}{\partial t^2} - \Delta p - \gamma \frac{\partial^2 T}{\partial t^2} = 0, \tag{2.1}$$

and

$$\beta \frac{\partial^2 T}{\partial t^2} - \Delta T - \mu \frac{\partial^2 p}{\partial t^2} = 0, \tag{2.2}$$

where p and T represent the small changes in pressure and temperature from their constant equilibrium values. The positive coefficients α, β, γ, μ denote the constants

$$\left(\frac{\partial \rho_e}{\partial p_e}\right)_{T_e}, \quad \frac{\rho_{n_e}}{\rho_{s_e} s_e^2}\left(\frac{\partial s_e}{\partial T_e}\right)_{p_e}, \quad -\left(\frac{\partial \rho_e}{\partial T_e}\right)_{p_e},$$

$$-\frac{\rho_{n_e}}{\rho_{s_e} s_e^2}\left(\frac{\partial s_e}{\partial p_e}\right)_{T_e},$$

where $\rho_{n_e} + \rho_{s_e} = \rho_e$ and $\rho_{n_e}, \rho_{s_e}, s_e, T_e, p_e$ represent the equilibrium values of normal density, superfluid density, entropy, temperature, and pressure. The disturbance variables p and T are functions of position $\underline{r}[\equiv (x, y, z)]$ and time t and Δ is the three-dimensional Laplacian operator.

To complete the formulation of the initial-value problem we impose the following initial conditions on p and T:

$$p(\underline{r}, 0) = p_0(\underline{r}), \quad T(\underline{r}, 0) = T_0(\underline{r}), \tag{2.3}$$

$$\frac{\partial p}{\partial t}(\underline{r}, 0) = p_1(\underline{r}), \quad \frac{\partial T}{\partial t}(\underline{r}, 0) = T_1(r). \tag{2.4}$$

Throughout the analysis which follows we will assume that p, T, p_{i-1}, T_{i-1}, $i = 1, 2$ are sufficiently well behaved functions of $\underline{r}$ and t to justify the mathematical operations employed.

3. GREEN'S MATRIX

To solve the initial-value problem we introduce a Green's function G which is a two by two matrix $\{G_{ij}(\underline{r}, \underline{r}'; t)\}$ defined by

$$AG = \delta(\underline{r} - \underline{r}')\delta(t) I, \tag{3.1}$$

where

$$A \equiv \begin{pmatrix} \alpha \dfrac{\partial^2}{\partial t^2} - \Delta, & -\gamma \dfrac{\partial^2}{\partial t^2} \\ -\mu \dfrac{\partial^2}{\partial t^2}, & \beta \dfrac{\partial^2}{\partial t^2} - \Delta \end{pmatrix},$$

I is the unit matrix, and δ represents the Dirac delta function. The initial conditions imposed on the elements G_{ij} of G are

$$G_{ij}(\underline{r}, \underline{r}'; 0) = \frac{\partial}{\partial t} G_{ij}(\underline{r}, \underline{r}'; 0) = 0, \quad i = 1, 2, \quad j = 1, 2. \tag{3.2}$$

To solve for the functions G_{ij} we transform to the polar coordinate system centered at $\underline{r}'$ and seek solutions of the form $G_{ij}(R, t)$ where $R = |\underline{r} - \underline{r}'|$. Using a Laplace transform we set $\bar{G}_{ij} = \int_0^\infty e^{-st} G_{ij}\, dt$ so that the transformed equation (3.1) has the form

$$\begin{pmatrix} \alpha s^2 - \Delta, & -\gamma s^2 \\ -\mu s^2, & \beta s^2 - \Delta \end{pmatrix} \{\bar{G}_{ij}\} = \frac{\delta(R)}{4\pi R^2} I,$$

where (see Ref. 2)

$$\Delta \bar{G}_{ij} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \bar{G}_{ij}}{\partial R^2}\right) + \frac{\bar{G}_{ij}^{(1)}}{R^2}\delta'(R) + \frac{\bar{G}_{ij}^{(0)}}{R^2}\delta(R),$$

$$\bar{G}_{ij}^{(0)} = \lim_{R \to 0}\left(R^2 \frac{\partial \bar{G}_{ij}}{\partial R}\right) \quad \text{and} \quad \bar{G}_{ij}^{(1)} = \lim_{R \to 0}(R^2 \bar{G}_{ij}). \tag{3.3}$$

If $\{\bar{G}_{ij}\}$ is a solution of (3.3) we must have

$$\begin{pmatrix} \alpha s^2 - \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2 \dfrac{\partial}{\partial R}\right), & -\gamma s^2 \\ -\mu s^2, & \beta s^2 - \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2 \dfrac{\partial}{\partial R}\right) \end{pmatrix} \{\bar{G}_{ij}\} = 0, \tag{3.4}$$

together with

$$\bar{G}_{ij}^{(0)} = -\delta_{ij}/4\pi, \quad \bar{G}_{ij}^{(1)} = 0, \quad i = 1, 2, \quad j = 1, 2, \tag{3.5}$$

where δ_{ij} is the Kronecker delta symbol. Solutions to (3.4) of the form $\bar{G}_{ij} = h_{ij} e^{-\lambda R}/R$ which remain bounded at infinity can be obtained, provided λ satisfies

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$$\begin{vmatrix} \alpha s^2-\lambda^2, & -\tau s^2 \\ -\mu s^2, & \beta s^2-\lambda^2 \end{vmatrix}=0 \tag{3.6}$$

or

$$(\alpha\beta-\gamma\mu)\left(\frac{s}{\lambda}\right)^4-(\alpha+\beta)\left(\frac{s}{\lambda}\right)^2+1=0. \tag{3.7}$$

The solutions of this quadratic for $(s/\lambda)^2$ are given by[1] u_1^2, and u_2^2 $(u_1>u_2)$, where u_i, $i=1,2$ are the two velocities of sound propagation in a superfluid. The general solution of (3.4) can then be written in the form

$$\bar{G}_{ij}=h_{ij}^{(1)}\frac{e^{-sR/u_1}}{R}+h_{ij}^{(2)}\frac{e^{-sR/u_2}}{R} \tag{3.8}$$

with

$$(\alpha u_{(k)}^2-1)h_{ij}^{(k)}=\gamma u_{(k)}^2 h_{2j}^{(k)}, \quad k=1,2, \quad j=1,2. \tag{3.9}$$

The constants $h_{ij}^{(k)}$ can be determined such that conditions (3.5) and Eqs. (3.9) are satisfied, and by inverting (3.8) we obtain

$$G=\frac{1}{4\pi(u_1^2-u_2^2)R}\sum_{i=1}^{2}(-1)^{i-1}\delta(t-R/u_i)A_i, \tag{3.10}$$

where

$$A_i=\begin{pmatrix} u_i^2-\alpha(u_1u_2)^2, & \gamma(u_1u_2)^2 \\ \mu(u_1u_2)^2, & \alpha(u_1u_2)^2-u_{3-i}^2 \end{pmatrix}. \tag{3.11}$$

4. SOLUTION OF THE INITIAL-VALUE PROBLEM

The convolution of two scalar-valued functions $f(\underline{r},t)$ and $g(\underline{r},t)$ is defined in the usual manner by

$$f*g(\underline{r},t)=\int_0^t f(\underline{r},\tau)g(\underline{r},t-\tau)\,d\tau. \tag{4.1}$$

For two vector-valued functions $\underline{a}(\underline{r},t)$ and $\underline{b}(\underline{r},t)$ we define

$$\underline{a}*\cdot\underline{b}(\underline{r},t)=a_x^*b_x(\underline{r},t)+a_y^*b_y(\underline{r},t)+a_z^*b_z(\underline{r},t). \tag{4.2}$$

The algebraic properties of the convolution are well known and need not be stated here. For the purposes of our analysis we will need the following relations:

$$\delta(t)*f(\underline{r},t)=f(\underline{r},t), \tag{4.3a}$$

$$t*\frac{\partial^2 f(\underline{r},t)}{\partial t^2}=f(\underline{r},t)-t\frac{\partial f}{\partial t}(\underline{r},0)-f(\underline{r},0), \tag{4.3b}$$

$$\frac{\partial}{\partial t}(t*f(\underline{r},t))=1*f(\underline{r},t), \tag{4.3c}$$

$$\frac{\partial^2}{\partial t^2}(t*f(\underline{r},t))=\frac{\partial}{\partial t}(1*f(\underline{r},t))=f(\underline{r},t). \tag{4.3d}$$

Equations (2.1)—(2.2) may be written in the form

$$AF=\begin{pmatrix}0\\0\end{pmatrix}, \tag{4.4}$$

where $F\equiv\binom{p}{T}$. Also, if G_1 and G_2 represent the column vectors of the Green's matrix and e_1, e_2 the unit orthogonal vectors $\binom{1}{0}$ and $\binom{0}{1}$, then

$$AG_i=\delta(\underline{r}-\underline{r}')\delta(t)e_i, \quad i=1,2. \tag{4.5}$$

Using (4.3a) together with (4.4) and (4.5) we can write

$$\mu(t*(AG_1)*\cdot(pe_1)-t*(AF)*\cdot(G_{11}e_1))$$

$$=\mu t*p(\underline{r},t)\delta(\underline{r}-\underline{r}'), \tag{4.6}$$

and

$$\gamma(t*(AG_1)*\cdot(Te_2)-t*(AF)*\cdot(G_{21}e_2))=0. \tag{4.7}$$

By adding (4.6) and (4.7) and integrating the result over the interior of a large sphere with radius L $(\gg u,t)$ centered at $\underline{r}'$ we find, with the aid of (3.2) and (4.3b)

$$\mu t*p(\underline{r}',t)$$

$$=\mu t*\int(G_{11}^*\Delta p-p^*\Delta G_{11})\,dV+\gamma t*\int(G_{21}^*\Delta T-T^*\Delta G_{21})\,dV$$

$$+\mu\int(p_0+tp_1)^*(\alpha G_{11}-\gamma G_{21})\,dV$$

$$+\gamma\int(T_0+tT_1)^*(\beta G_{21}-\mu G_{11})\,dV. \tag{4.8}$$

The first two volume integrals in (4.8) can be transformed into surface integrals over the surface of the sphere by using Green's formula. These surface integrals vanish since G_{ij}; $i=1,2$, $j=1,2$ and their derivatives vanish for $L\gg u_1t$. As $L\to\infty$ the remaining volume integrals in (4.8) can be evaluated by using the polar coordinate system centered at $\underline{r}'$ and the fundamental property of the delta function. By differentiating the result twice with respect to t we obtain, with the aid of (3.10), (4.3c), and (4.3d),

$$p(\underline{r}',t)=\frac{1}{4\pi(u_1^2-u_2^2)}$$

$$\times\sum_{i=1}^{2}(-1)^{i-1}((u_i^2-\alpha(u_1u_2)^2)P^{(i)}$$

$$+\gamma(u_1u_2)^2T^{(i)}), \tag{4.9}$$

where

$$P^{(i)}=\frac{\partial}{\partial t}\left(t\int p_0(R=u_it)\,d\Omega\right)$$

$$+t\int p_1(R=u_it)\,d\Omega, \quad i=1,2, \tag{4.10}$$

and

$$T^{(i)}=\frac{\partial}{\partial t}\left(t\int T_0(R=u_it)\,d\Omega\right)$$

$$+t\int T_1(R=u_it)\,d\Omega, \quad i=1,2, \tag{4.11}$$

with $d\Omega$ denoting the element of solid angle. In a similar manner, by adding the equations

$$\gamma(t*(AG_2)*\cdot(Te_2)-t*(AF)*\cdot(G_{22}e))$$

$$=\gamma t*T(\underline{r},t)\delta(\underline{r}-\underline{r}'), \tag{4.12}$$

and

$$\mu(t*(AG_2)*\cdot(pe)-t*(AF)*\cdot(G_{12}e_1))=0, \tag{4.13}$$

and integrating the result over R^3 we obtain

$$T(\underline{r}'t)=\frac{1}{4\pi(u_1^2-u_2^2)}\sum_{i=1}^{2}(-1)^{i-1}((u_i^2-\beta(u_1u_2)^2)T^{(i)}$$

$$+\mu(u_1u_2)^2P^{(i)}). \tag{4.14}$$

Each of the functions $p(\underline{r}',t)$ and $T(\underline{r}',t)$ is determined uniquely from (4.9) and (4.14) in terms of the initial values of p, T, $\partial p/\partial t$, and $\partial T/\partial t$ on two spheres of radii $u_1 t$ and $u_2 t$ centered at $\underline{r}'$. The solutions obtained are a superposition of the two types of sound wave which occur in a superfluid. If the initial disturbances vanish everywhere in R^3 except in a finite region B then, for $\underline{r}' \in B \cup \partial B$ and $D \equiv \sup_{\underline{r} \in B} |\underline{r} - \underline{r}'|$ the subsequent disturbances at $\underline{r}'$ due to the first sound wave occur during the time interval $(0, D/u_1)$ and cease thereafter. Similarly the disturbances at $\underline{r}'$ due to the second sound wave occur only during the interval $(0, D/u_2)$. Also during the interval $(0, D/u_1)$ disturbances due to both types of sound waves occur at $\underline{r}'$. If $\underline{r}' \notin B \cup \partial B$ and $d \equiv \inf_{\underline{r} \in B} |\underline{r} - \underline{r}'|$, then the disturbances at $\underline{r}'$ due to the first and second sound waves occur only during the time intervals $(d/u_1, D/u_1)$ and $(d/u_2, D/u_2)$, respectively, and during the interval $(d/u_2, D/u_1)$ the disturbances due to both types of sound waves occur at $\underline{r}'$.

In concluding this section we note that formulas (4.9) and (4.14) are consistent with the classical result due to Poisson in the limit $\rho_{s_e} = 0$. Using (3.7) and the thermodynamic relations it can readily be shown that when $\rho_{s_e} = 0$ the quantities $u_2^2, u_1^2, \beta u_2^2, \mu u_2^2$ assume the values $0, c^2, 1$ and $-(\partial s_e/\partial p_e)_{T_e}/(\partial s_e/\partial T_e)_{p_e}$ respectively where $c^2 = (\partial p_e/\partial s_e)_{s_e}$. Also $P^{(2)}$ and $T^{(2)}$ reduce to $p_0 + tp_1$ and $T_0 + tT_1$ so that (4.9) and (4.14) have the form

$$p(\underline{r}', t) = \frac{1}{4\pi}\left(\frac{\partial}{\partial t}\left(t \int p_0(R = ct)\, d\Omega\right) + t \int p_1(R = ct)\, d\Omega\right) \tag{4.15}$$

and

$$\left(\frac{\partial s_e}{\partial T_e}\right)_{p_e} (T(\underline{r}', t) - T_0 - tT_1) = -\left(\frac{\partial s_e}{\partial p_e}\right)_{T_e} (p(\underline{r}', t) - p_0 - tp_1). \tag{4.16}$$

Equation (4.15) is Poisson's equation and (4.16) is consistent with the isentropic motion of a gas for which

$$\left(\frac{\partial s_e}{\partial T_e}\right)_{p_e} T(\underline{r}' t) + \left(\frac{\partial s_e}{\partial p_e}\right)_{T_e} p(\underline{r}' t)$$

and its time derivative vanish initially and throughout the subsequent motion.

5. THE TWO-DIMENSIONAL CASE

In the two-dimensional case it can be shown that $\bar{G}_{ij} = h_{ij} K_0(\lambda \bar{R})$ with $\lim_{R \to 0}(\bar{R}\partial \bar{G}_{ij}/\partial \bar{R}) = -\delta_{ij}/2\pi$ and $\lim_{R \to 0}(R\bar{G}_{ij}) = 0$, where K_0 is the modified Bessel function and $\bar{R}^2 = (x' - x)^2 + (y' - y)^2$. The constants h_{ij} are found as before and after an inversion we obtain the Green's matrix in the form

$$G = \frac{1}{2\pi(u_1^2 - u_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \frac{u_i H(u_i t - \bar{R})}{(u_i^2 t^2 - \bar{R}^2)^{1/2}} A_i, \tag{5.1}$$

where H is the Heaviside function.

The solutions for p and T may be written

$$p(\underline{r}', t) = \frac{1}{2\pi(u_1^2 - u_2^2)} \times \sum_{i=1}^{2} (-1)^{i-1} \left((u_i^2 - \alpha(u_1 u_2)^2)\bar{P}^{(i)} + \gamma(u_1 u_2)^2 \bar{T}^{(i)}\right), \tag{5.2}$$

and

$$T(\underline{r}', t) = \frac{1}{2\pi(u_1^2 - u_2^2)} \times \sum_{i=1}^{2} (-1)^{i-1} \left((u_i^2 - \beta(u_1 u_2)^2)\bar{T}^{(i)} + \mu(u_1 u_2)^2 \bar{P}^{(i)}\right), \tag{5.3}$$

where

$$\bar{P}^{(i)} = \frac{1}{u_i}\left(\frac{\partial}{\partial t} \iint_{D_i} \frac{p_0\, dx\, dy}{(u_i^2 t^2 - \bar{R}^2)^{1/2}} + \iint_{D_i} \frac{p_1\, dx\, dy}{(u_i^2 t^2 - \bar{R}^2)^{1/2}}\right), \quad i = 1, 2, \tag{5.4}$$

and

$$\bar{T}^{(i)} = \frac{1}{u_i}\left(\frac{\partial}{\partial t} \iint_{D_i} \frac{T_0\, dx\, dy}{(u_i^2 t^2 - \bar{R}^2)^{1/2}} + \iint_{D_i} \frac{T_1\, dx\, dy}{(u_i^2 t^2 - \bar{R}^2)^{1/2}}\right), \quad i = 1, 2, \tag{5.5}$$

with D_i, $i = 1, 2$ denoting the interior of the circles of radii $u_i t$, $i = 1, 2$, centered at $\underline{r}'$.

In this case if the initial disturbances vanish everywhere in R^2 except in a finite region $\bar{B}$, then, for $\underline{r}' \in \bar{B} \cup \partial \bar{B}$, the subsequent disturbances at $\underline{r}'$ due to both types of sound waves occur during the time interval $(0, \infty)$. In contrast to the three-dimensional case there is no finite time at which the disturbances due to the sound waves cease. Again, if $\underline{r}' \notin \bar{B} \cup \partial \bar{B}$ and $\bar{d} \equiv \inf_{x, y \in B} \bar{R}$, then the disturbances at $\underline{r}'$ due to the first and second sound waves begin at the times $\bar{d}/u_1$ and $\bar{d}/u_2$ respectively and persist thereafter.

As in the three-dimensional case it can be shown that formulas (5.2) and (5.3) are consistent with the classical solution in the limit $\rho_{s_e} = 0$. For brevity we will omit the details.

6. A UNIQUENESS THEOREM

In Ref. 3 a uniqueness theorem was derived for the initial-boundary value problem associated with the classical wave equation which was valid both for bounded and unbounded regions. A uniqueness proof was also given in Ref. 4 for the analogous problem associated with superfluid acoustics. However this latter proof was valid only for bounded regions. It is our purpose here to show that the uniqueness proof can be established for infinite regions by using an extension of the Poisson formulas derived above.

The solution of the initial-boundary problem will be unique if $p \equiv T \equiv 0$ is the solution of the initial-boundary value problem consisting of Eqs. (2.1)–(2.2) together with the initial conditions

$$p(\underline{r}, 0) = T(\underline{r}, 0) = \frac{\partial p}{\partial t}(\underline{r}, 0) = \frac{\partial T}{\partial t}(\underline{r}, 0), \tag{6.1}$$

and the boundary conditions

$$\left.\begin{aligned}\frac{\partial p}{\partial \nu}+kp=0 \quad (6.2)\\ \frac{\partial T}{\partial \nu}+lT=0 \quad (6.3)\end{aligned}\right\} \text{ on } S_1$$

and

$$\left.\begin{aligned}p=0 \quad (6.4)\\ T=0 \quad (6.5)\end{aligned}\right\} \text{ on } S_2,$$

where S_i, $i=1,2$ are bounded closed surfaces in R^3 with an inward normal denoted by ν and $k(\underline{r})$, $l(\underline{r}) \geq 0$.

By integrating the sum of (4.6) and (4.7) throughout the region exterior to S_1 and S_2 and interior to a large sphere centered at $\underline{r}'$ and differentiating the result twice with respect to time we obtain, by virtue of (4.3d) and (6.1),

$$p(\underline{r}', t)=\int (G_{11}^* \Delta p - p^* \Delta G_{11})\, dV + \frac{\gamma}{\mu}\int (G_{21}^* \Delta T - T^* \Delta G_{21})\, dV. \quad (6.6)$$

Using Green's formula we transform the volume integrals in (6.6) into surface integrals over S_i, $i=1,2$ and the surface S of the sphere. If $\inf_{\underline{r}\in S\cup S_1 \cup S_2} |\underline{r}-\underline{r}'| \gg u_1 t$, then the surface integrals vanish since G_{ij}; $i=1,2$, $j=1,2$ and their derivatives vanish. Therefore, $p(\underline{r}'t)=0$ when $|\underline{r}'|$ is sufficiently large. By using Eqs. (4.12) and (4.13) we can show in a similar manner that $T(\underline{r}', t)=0$ when $|\underline{r}'|$ is sufficiently large. This ensures the existence of the volume integrals in the following positive definite energy function $E(t)$, where

$$E(t)=\frac{1}{2}\int \left(\beta(\mu(\nabla p)\cdot(\nabla p)+\gamma(\nabla T)\cdot(\nabla T))+\gamma\left(\mu \frac{\partial p}{\partial t} - \beta\frac{\partial T}{\partial t}\right)^2 + \mu(\alpha\beta-\gamma\mu)\left(\frac{\partial p}{\partial t}\right)^2\right) dV + \frac{1}{2}\int_{S_1} \beta(\mu k p^2+\gamma l T^2)\, d\sigma, \quad t \geq 0, \quad (6.7)$$

and ∇ denotes the gradient operator in R^3.

By differentiating (6.7) with respect to t we obtain, using the divergence theorem together with Eqs. (2.1), (2.2), and (6.2)–(6.5), $dE/dt=0$. Since $E(0)=0$, then $E(t)=0$, $t\geq 0$ and uniqueness follows. Uniqueness can be established for the two-dimensional problem in a similar manner.

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An algebraically special subclass of vacuum metrics admitting a Killing motion[a)]

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The subclass of vacuum metrics with a Killing vector field (which may be either timelike or spacelike), having two of the nonzero eigenvalues of the Ricci subtensor equal, is investigated. Invariant methods are used. A special triad is chosen for the associated space V_3 which gives rise to a special tetrad for the space–time V_4, and this choice simplifies the expressions for the Weyl tensor and Newman–Penrose coefficients for V_4. This subclass of vacuum metric reduces to two cases depending on whether or not the complex dilatation vanishes. In the first case the metric reduces to an example of plane-fronted waves. In the second case the problem reduces to a difficult pair of partial differential equations which has not been solved in the fullest generality. However, it has been shown that this case includes Robinson–Trautman metrics, Held–Robinson metrics, and some additional new Petrov type-III metrics with twisting rays.

1. INTRODUCTION

A stationary vacuum metric gives rise to an associated space V_3. The eigenvalues of the Ricci subtensor of V_3 satisfy the weak inequality[1] $\lambda_2 \leq \lambda_3 \leq \lambda_1 = 0$. The case $\lambda_1 = \lambda_3$ corresponds to either static V_4 or the Papapetrou–Ehlers class of stationary V_4. It is a reasonable question to ask now what metrics correspond to the case $\lambda_2 = \lambda_3$. To include possibly more solutions in this class, the Killing vector in V_4 is allowed to be either timelike or spacelike.

In Sec. 2 the necessary facts in space–time admitting a Killing field are reviewed. We write down the vacuum equations in complex invariant form, generalizing the form of Das.[2] Although these equations are similar to those of Perjês,[3] a different choice of frame for V_4 leads to formulas connecting the structure of V_3 and V_4 which are simplified over those of Perjês.

Section 3 describes in detail the derivation of the class of spaces defined by the equality of the two nonzero eigenvalues of the Ricci subtensor. This algebraic subclass falls into two cases according as the complex dilation is zero or not. The first case can be completely solved and the metric is transformable to the stationary pp-waves.[4] The remaining case can be reduced to the formidable pair of partial differential equations:

$$W_{,z\bar{z}} = -(z+\bar{z})e^{W},$$

$$T_{,z\bar{z}} = -(z+\bar{z})(e^{W})T,$$

where z is a complex coordinate. In fact, this particular algebraic specialization carries over to the V_4, with the result that our solutions are of Petrov types N and III. Included in the class are the stationary Robinson–Trautman metrics[5] of type III, and a new class of type III metrics with rotating rays independently discovered by Held.[6,7] Moreover, we give some new solutions, and provide further insight into the reduced equations.

In our formulation the only explicitly known examples of these metrics are nonstationary, possessing an everywhere spacelike Killing vector field. The new metrics are not asymptotically flat, and contain wire singularities. The general solution of the pair of partial differential equations still remains an open problem.

2. NOTATIONS AND FIELD EQUATIONS

The space–time manifold V_4 is chosen to have signature 2. A Killing vector is assumed to exist in V_4 and that gives rise to an associated space V_3. The indices $i, j, k, \cdots, M, N, P, \cdots$ range from 1 to 4, while $\alpha, \beta, \gamma, \cdots, A, B, C, \cdots$ range from 1 to 3; often the indices 2, 3 are replaced by 0, $\mathring{0}$ the complex conjugated indices. The orthonormal ennuples are denoted by $\Lambda_M{}^i$ in V_4 and $\lambda_A{}^\alpha$ in V_3. Fixed coordinate indices are written which symbolizes the coordinates, thus, we have $\Lambda_1{}^r$, $\Lambda_2{}^z$ as some of the components of $\Lambda_M{}^i$. The summation convention is followed on capital indices as well as on lower case indices. A comma denotes a partial or invariant derivative whereas a vertical bar denotes covariant differentiation in V_3 either with respect to the coordinates or with respect to the triad. The definitions of Ricci rotation coefficients such as $\gamma^A{}_{BC} \equiv \lambda^A{}_{\alpha|\beta}\lambda_B{}^\alpha\lambda_C{}^\beta$ in V_3, and Riemann and Ricci invariants are those of Eisenhart.[8] The definition of the covariant derivative of an invariant T_A with respect to the triad is

$$T_{A|B} \equiv T_{A,B} + \gamma^C{}_{AB}T_C = T_{\alpha|\beta}\lambda_A{}^\alpha\lambda_B{}^\beta. \qquad (2.1a)$$

Antisymmetrization is indicated by $T_{[\alpha\beta]} \equiv \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$.

In coordinates adapted to the Killing motion the metric form of V_4 is written as

$$\Phi = f^{-1}(\mathbf{x})g_{\alpha\beta}(\mathbf{x})\,dx^\alpha\,dx^\beta - f(\mathbf{x})\,[a_\alpha(\mathbf{x})\,dx^\alpha + dx^4]^2. \qquad (2.1b)$$

The metric form $\tilde{\Phi} = g_{\alpha\beta}(\mathbf{x})\,dx^\alpha dx^\beta$ defines the associated V_3. According as the Killing vector field is timelike or spacelike, the function $f(\mathbf{x})$ is positive- or negative-valued, V_3 has signature 3 or -1 and the indicator $c = +1$, or -1.

The following definition sets up a natural correspondence between orthonormal frames of the V_3 and V_4:

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$$\Lambda_{(4)}{}^{i}=(cf)^{-1/2}\delta_4{}^{i},$$
$$\Lambda_A{}^{i}=(cf)^{1/2}[\lambda_A{}^{i}-a_A\delta_4{}^{i}], \quad (2.2)$$

where $\lambda_A{}^4\equiv 0$, $a_A\equiv a_\alpha\lambda_A{}^\alpha$.

Defining $\phi^\alpha\equiv cf^2\eta^{\alpha\beta\gamma}a_{[\beta,\gamma]}$, where $\eta_{\alpha\beta\gamma}=\sqrt{cg}\,\epsilon_{\alpha\beta\gamma}$, the field equations show that $\phi_\alpha=\phi_{,\alpha}$, where ϕ is a scalar function called the twist potential.

If a complex potential F is defined by $F\equiv f-i\phi$, the vacuum equations reduce to

$$R_{AB}+\tfrac14(\mathrm{Re}F)^{-2}(F_{,A}\overline{F}_{,B}+\overline{F}_{,A}F_{,B})=0,$$
$$g^{AB}(F_{,AB}+\gamma^C{}_{AB}F_{,C}-(\mathrm{Re}F)^{-1}F_{,A}F_{,B}]=0, \quad (2.3)$$

where R_{AB} denotes the the triad components of the Ricci subtensor and a possible choice of g_{AB} is given in (2.4). One can now write down a complex version of the field equations, derived either from spinorial considerations[3] or the geometrical optics in V_3. The complex triad vectors are chosen to be $\lambda_{(0)}{}^\alpha$, $\lambda_{(\mathring{0})}{}^\alpha$, $\lambda_{(1)}{}^\alpha$, where $\lambda_{(0)}{}^\alpha\equiv(2)^{-1/2}(\lambda_{(2)}{}^\alpha+i\lambda_{(3)}{}^\alpha)$, and $g_{\alpha\beta}\lambda_{(1)}{}^\alpha\lambda_{(1)}{}^\beta=1$. Triad indices are therefore raised and lowered by the fundamental form

$$[g^{AB}]=[g_{AB}]\equiv\begin{bmatrix}0&c&0\\c&0&0\\0&0&1\end{bmatrix}. \quad (2.4)$$

The complex Ricci rotation coefficients are denoted as follows:

$$\alpha\equiv\gamma_{101}\equiv\lambda_{(1)|\beta}{}^\alpha\lambda_{(0)\alpha}\lambda_{(1)}{}^\beta,$$
$$\beta\equiv\gamma_{100},\gamma\equiv\gamma_{10\mathring{0}},\ \delta\equiv\gamma_{0\mathring{0}1},\ \epsilon\equiv\gamma_{0\mathring{0}\mathring{0}}. \quad (2.5)$$

$|\alpha|$, $|\beta|$, and γ have the significance of the first curvature, shearing, and complex dilatation of the $\lambda_{(1)}{}^\alpha$-tangent curves, respectively.

With these notations, the field equations (2.3) take the following form:

$$\alpha_{,0}-\beta_{,1}=\alpha^2+c\beta(\gamma+\overline{\gamma}-2\delta)+c\alpha\epsilon+\tfrac12 f^{-2}(F_{,0}\overline{F}_{,0}), \quad (2.6a)$$

$$\beta_{,\mathring{0}}-\gamma_{,0}=\alpha(\overline{\gamma}-\gamma)-2c\beta\overline{\epsilon}+\tfrac14 cf^{-2}(F_{,1}\overline{F}_{,0}+\overline{F}_{,1}F_{,0}), \quad (2.6b)$$

$$\gamma_{,1}-\alpha_{,\mathring{0}}=-|\alpha|^2-c|\beta|^2-c\gamma^2+c\alpha\overline{\epsilon}-\tfrac14 cf^{-2}(F_{,1}\overline{F}_{,1}), \quad (2.6c)$$

$$\epsilon_{,\mathring{0}}+\overline{\epsilon}_{,0}=|\beta|^2-|\gamma|^2+\delta(\overline{\gamma}-\gamma)-2c|\epsilon|^2$$
$$+cf^{-2}(cF_{,1}\overline{F}_{,1}-F_{,0}\overline{F}_{,\mathring{0}}-\overline{F}_{,0}F_{,\mathring{0}}), \quad (2.6d)$$

$$\epsilon_{,1}-\delta_{,0}=-\alpha(\overline{\gamma}+\delta)+\beta(\overline{\alpha}+c\overline{\epsilon})-c\epsilon(\overline{\gamma}-\delta)$$
$$+cf^{-2}(F_{,1}\overline{F}_{,0}+\overline{F}_{,1}F_{,0}), \quad (2.6e)$$

$$F_{,11}+2cF_{,0\mathring{0}}=-2c\gamma F_{,1}+(c\overline{\alpha}-2\overline{\epsilon})F_{,0}+c\alpha F_{,\mathring{0}}$$
$$+f^{-1}(F_{,1}{}^2+2cF_{,0}F_{,\mathring{0}}), \quad (2.6f)$$

where commas denote invariant derivatives. The commutation relations are, for any scalar function h,

$$h_{,10}-h_{,01}=\alpha h_{,1}+c(\overline{\gamma}-\delta)h_{,0}+c\beta h_{,\mathring{0}}, \quad (2.7a)$$

$$h_{,0\mathring{0}}-h_{,\mathring{0}0}=(\overline{\gamma}-\gamma)h_{,1}-c\overline{\epsilon}h_{,0}+c\epsilon h_{,\mathring{0}}. \quad (2.7b)$$

As noted in (2.2) there is a natural correspondence between adapted frames of V_4 and frames of the V_3. With this correspondence, with each curve of V_3 may be associated a null curve of V_4 by the following mapping of tangent vectors: $\lambda_{(1)}{}^a\to 2^{-1/2}(\Lambda_{(1)}{}^a+\Lambda_{(4)}{}^a)$. The mapping of curves so defined is one-to-one and onto. If we let our frame correspond to that of the Newman–Penrose[9] formalism by $l^a=2^{-1/2}(\Lambda_{(1)}{}^a+\Lambda_{(4)}{}^a)$, $n^a=2^{-1/2}c(-\Lambda_{(1)}{}^a+\Lambda_{(4)}{}^a)$, $m^a=2^{-1/2}(\Lambda_{(2)}{}^a+i\Lambda_{(3)}{}^a)$, and $\overline{m}^a=2^{-1/2}(\Lambda_{(2)}{}^a-i\Lambda_{(3)}{}^a)$, then we have a special choice of gauge and null rotation for l^a and n^a, respectively. The results of this choice are to simplify the expressions for the Weyl curvature spinors and the relation between the Ricci rotation coefficients of V_4 and V_3. The following formulas for the Weyl curvature spinors ψ_i, $i=0,1,2,3,4$, may be contrasted with those of Perjés[3]:

$$-2\psi_0=F_{|00}+\tfrac12 f^{-1}F_{,0}{}^2$$
$$=F_{,00}+\beta F_{,1}-c\epsilon F_{,0}+\tfrac12 f^{-1}F_{,0}{}^2, \quad (2.8a)$$

$$2\sqrt2\,c\psi_1=F_{|01}+\tfrac12 f^{-1}F_{,0}F_{,1}$$
$$=F_{,01}-c\delta F_{,0}+\alpha F_{,1}+\tfrac12 f^{-1}F_{,0}F_{,1}, \quad (2.8b)$$

$$2\psi_2=F_{|11}-cf^{-1}F_{,0}F_{,\mathring{0}}$$
$$=F_{,11}-c\alpha F_{,\mathring{0}}-c\overline{\alpha}F_{,0}-cf^{-1}F_{,0}F_{,\mathring{0}}, \quad (2.8c)$$

$$-2\sqrt2\,\psi_3=F_{|\mathring{0}1}+\tfrac12 f^{-1}F_{,\mathring{0}}F_{,1}$$
$$=F_{,\mathring{0}1}+\overline{\alpha}F_{,1}+c\delta F_{,\mathring{0}}+\tfrac12 f^{-1}F_{,\mathring{0}}F_{,1}, \quad (2.8d)$$

$$-2\psi_4=F_{|\mathring{0}\mathring{0}}+\tfrac12 f^{-1}F_{,\mathring{0}}{}^2$$
$$=F_{,\mathring{0}\mathring{0}}+\overline{\beta}F_{,1}-c\overline{\epsilon}F_{,\mathring{0}}+\tfrac12 f^{-1}F_{,\mathring{0}}{}^2. \quad (2.8e)$$

The complex Ricci rotation coefficients are also simplified. The Newman–Penrose[9] spin coefficients have equivalents in our notation; see Table I.

3. AN ALGEBRAICALLY SPECIAL SUBCLASS OF V_3

In a stationary space–time the eigenvalues of the Ricci subtensor of V_3 are given by[1]

$$\lambda_1=0,$$
$$\lambda_2=-(\mathrm{Re}F)^{-2}(\Delta_1(F,\overline{F})+|\Delta_1F|), \quad (3.1)$$
$$\lambda_3=-(\mathrm{Re}F)^{-2}(\Delta_1(F,\overline{F})-|\Delta_1F|).$$

For the static case or for the Papapetrou–Ehlers class of stationary metrics, $\lambda_1=\lambda_3$. In this section we investigate another possible algebraic specialization,

TABLE I.

N–P spin coefficient	"Stationary" equivalent
κ	$\tfrac12 f^{1/2}(\alpha+f^{-1}F_{,0})$
σ	$(1/\sqrt2)f^{1/2}(\beta)$
ρ	$(1/\sqrt2)f^{1/2}(\gamma-\tfrac12 cf^{-1}F_{,1})$
τ	$-\tfrac12 cf^{1/2}(\alpha)$
α	$\tfrac12 f^{1/2}(\overline{\epsilon}-\tfrac12 cf^{-1}F_{,\mathring{0}})$
β	$-\tfrac12 f^{1/2}(\epsilon-\tfrac12 cf^{-1}F_{,0})$
γ	$(c/2\sqrt2)f^{1/2}(\delta-\tfrac12 cf^{-1}F_{,1})$
ϵ	$(1/2\sqrt2)f^{1/2}(\overline{\delta}-\tfrac12 cf^{-1}F_{,1})$
π	$\tfrac12 cf^{1/2}(\overline{\alpha})$
μ	$(c/\sqrt2)f^{1/2}(\overline{\gamma}-\tfrac12 cf^{-1}F_{,1})$
λ	$(c/\sqrt2)f^{1/2}(\overline{\beta})$
ν	$-\tfrac12 f^{1/2}(\overline{\alpha}-cf^{-1}F_{,\mathring{0}})$

namely $\lambda_2=\lambda_3$. In this case $\Delta_1 F=0$, which may be written as $F_{,1}{}^2+2cF_{,0}F_{,\bar{0}}=0$. By choosing the direction of $\lambda_{(1)}{}^\alpha$ and the orientation of the triad, we can ensure that $F_{,1}=F_{,0}=0$. Then $\lambda_{(1)}{}^\alpha$ is along the eigenrays.

The commutation relation (2.7a) (substituting F for h) and the potential equations (2.6f) give $\alpha=\beta=0$. Hence, the eigenrays are geodesic and shear-free, and the space—time is algebraically special.

Since $\lambda_{(1)}{}^\alpha$ is along geodesics and $F_{,1}=0$, we can choose coordinates $\{z, \bar{z}, r\}$ in which r is an affine parameter along the geodesics and $\bar{z}=F$. Using the freedom in the triad, we can put $F_{,0}=Q$ where Q is real. This brings the triad into the form

$$\lambda_{(1)}{}^\alpha=c\delta_r{}^\alpha,\quad \lambda_{(0)}{}^\alpha=\eta\delta_r{}^\alpha+Q\delta_{\bar{z}}{}^\alpha, \tag{3.2}$$

where η is complex. The remaining coordinate freedom is $\hat{z}=z$, $\hat{r}=r+\lambda(z,\bar{z})$. With our choice of frame the metric for V_3 is

$$\tilde{\Phi}=(dr-\eta Q^{-1}dz-\bar{\eta}Q^{-1}d\bar{z})^2+2cQ^{-2}dzd\bar{z}. \tag{3.3}$$

The effect of choosing $F_{,\bar{0}}$ real is to make $\lambda_{(2)}{}^\alpha$ and $\lambda_{(3)}{}^\alpha$ tangent to the lines to force and lines of twist, respectively; hence they are both normal to surfaces. This leads to $\delta=\frac{1}{2}(\bar{\gamma}-\gamma)$; from (2.6b), $\gamma_{,0}=0$; hence $\delta_{,0}=\frac{1}{2}\bar{\gamma}_{,0}$. We may distinguish two cases, $\gamma=0$ and $\gamma\neq0$. When $\gamma=0$, it turns out that we can use our coordinate freedom to set $\eta=0$. When $\gamma\neq0$, we use it to make $\gamma^{-1}-r$ pure imaginary, which is fairly standard.

With the conventions made above out of the way, the solutions of the remaining equations is straightforward. The results are as follows.

Case I ($\gamma=0$): The metric of the associated space is

$$\tilde{\Phi}=dr^2+(z+\bar{z})H(z)\bar{H}(\bar{z})\,dzd\bar{z}, \tag{3.4}$$

where H is an arbitrary analytic function. H is related to Q and ϵ by

$$Q=\sqrt{2}\,|H|^{-1}(z+\bar{z})^{-1/2},\quad \epsilon=-cQ_{,z}. \tag{3.5}$$

In this case by Eqs. (2.8) all the ψ_i's vanish except for ψ_4, and the space—time[9] is of Petrov type (4). The metric of the V_4 is transformable to

$$\Phi=dx^2+dy^2+2dvdt-W(x,y)\,dt^2, \tag{3.6}$$

where $W_{,xx}+W_{,yy}=0$. We thus have a special case of the well-known plane-fronted gravitational waves with parallel rays.[4]

Case II ($\gamma\neq0$): The metric of the V_3 is

$$\tilde{\Phi}=(dr+iT_{,z}dz-iT_{,\bar{z}}d\bar{z})^2+2c(r^2+T^2)e^V dzd\bar{z}, \tag{3.7}$$

where $T=T(z,\bar{z})$ and $V=V(z,\bar{z})$ are real functions satisfying the reduced field equations

$$V_{,z\bar{z}}=-ce^V-(z+\bar{z})^{-2}, \tag{3.8a}$$

$$T_{,z\bar{z}}=-ce^V T. \tag{3.8b}$$

The complex dilatation γ is given by $\gamma=(r+iT)^{-1}$, while $Q=e^{-V/2}|\gamma|$ and $\epsilon=-ce^{-V/2}|\gamma|(iT_{,z}\bar{\gamma}-\frac{1}{2}V_{,z})$.

The metric of V_4 may be readily found by resorting to (2.2). It is

$$\Phi=(z+\bar{z})^{-1}\tilde{\Phi}-(z+\bar{z})[4c(r+U)(z+\bar{z})^{-2}d(z+\bar{z})+dt]^2, \tag{3.9}$$

where $\tilde{\Phi}$ is given by (3.7) and $U=U(z,\bar{z})$ is a real function which is a particular solution of the equation

$$U_{,z}-U_{,\bar{z}}=i(T_{,z}+T_{,\bar{z}}). \tag{3.10}$$

The metric form (3.9) for the case $\gamma\neq0$ belongs to Petrov type (3,1). This metric involves unknown functions V and T satisfying the pair of partial differential equations (3.8). At the present moment the general solutions of (3.8) are not known. Nevertheless, some special solutions of (3.8) and the consequent vacuum metrics will be mentioned.

The present case may be divided into two subcases $T=0$ and $T\neq0$. For $T=0$ the vacuum metrics have nontwisting eigenrays and are therefore of the Robinsion—Trautman class.[5] Furthermore, there is a special solution[10] with $T=0$, namely,

$$\Phi=(z+\bar{z})^{-1}dr^2-6r^2(z+\bar{z})^{-3}|dz|^2-(z+\bar{z})[-4r(z+\bar{z})^{-2}d(z+\bar{z})+dt]^2, \tag{3.11}$$

the space of "maximum mobility" of the type (3,1).

For the subcase $T\neq0$, on the other hand, the rays have twist and the metric belongs to a new class independently described by Held.[6,7] In this case some special solutions of (3.8) are given by

$$\begin{aligned} &e^V=3x^{-2},\\ &T=\sum_{n=-\infty}^{\infty}\sqrt{x}\,\mathrm{Re}[\{\alpha_n J_\nu(in\,x)+\beta_n Y_\nu(in\,x)\}e^{in\,y}],\\ &\nu=\pm\sqrt{13}/2,\\ &c=-1,\\ &x=z+\bar{z},\quad y\equiv i(\bar{z}-z), \end{aligned} \tag{3.12}$$

where α_n, β_n are arbitrary complex constants and J_ν, Y_ν are Bessel functions of 1st and 2nd kind. As a particular case (3.12) we have

$$T=x^{(1\pm\sqrt{13})/2}(A+By), \tag{3.13}$$

where A, B are arbitrary real constants. This corresponds to the solution obtained by Held.[6,7] By putting $A=B=0$ in (3.13) the metric (3.11) can be recovered.

The new solutions generated by (3.12) are a subclass of algebraically special vacuum metrics admitting a spacelike (angular) Killing motion. Moreover, these metrics contain wire singularities and thus none of these are asymptotically flat.

The equations mentioned in the introduction are obtained from (3.8) upon defining

$$e^W=ce^V(z+\bar{z})^{-1}. \tag{3.14}$$

Finally, we note that Eqs. (3.8) are invariant under a three-parameter group, based upon the following characteristic of the equations: Whenever $V=V(z,\bar{z})$ is a solution, then so is

$$V'(z,\bar{z})\equiv V(az^{-1}+ib,\ a\bar{z}^{-1}-ib)-2\ln|z|^2/a. \tag{3.15}$$

The three-parameter group leads to a two-parameter family of metrics starting from any sufficiently general metric. But the metrics defined by (3.12) are not so well-favored.

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ERRATA

Erratum: Acoustic emission and the plate Green's function [J. Math. Phys. 18, 676 (1977)]

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(Received 11 October 1977)

In the last line of Eq. (86), $p_d p_b$ should be $p_a p_b$. The matrix B in Eq. (100) should have a coefficient $i(=\sqrt{-1})$. The explicit factors of Q in Eqs. (110) and (120) should be omitted, having been absorbed into B[Eq. (B5)]. In Eq. (110), the subscripts "V" should be "L". Equation (118) should have an additional term

$$+[U_{12}(z')U_{12}(d)^{-1}]\otimes 1(H_V^\circ(\omega, Q, z-d))_L$$

and the entire resulting expression should be multiplied by -1, as should the right side of Eq. (119). In Eq. (A24) the factor "$\frac{1}{2}$" should be "2". In Eq. (B5), the 31 element should be $\frac{1}{4}Q_y$, not $\frac{1}{2}Q_y$, and the 22 element of the matrix in Eq. (B9) should involve Q_y^2 instead of Q_x^2. In Eq. (B16), $1+\sigma$ should be $1-\sigma$ and E^{-1} should be replaced by $(1-\sigma^2)E^{-1}$ throughout this equation. The prefactor of the matrix in Eq. (B19) should contain c_t instead of $V_L\,(=\sqrt{2}\,c_t)$. In Eq. (B20), Q_p should be QV_p. The right side of Eq. (B22) should be multiplied by -1. Note that the velocities defined by Eq. (B13) and (B21) are identical. Though not in error, the different choice of sign convention for the arguments of the exponentials in Eqs. (85) and (96) is potentially misleading.

CUMULATIVE AUTHOR INDEX

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In the last line of Eq. (86), $p_d p_b$ should be $p_a p_b$. The matrix B in Eq. (100) should have a coefficient $i(=\sqrt{-1})$. The explicit factors of Q in Eqs. (110) and (120) should be omitted, having been absorbed into B[Eq. (B5)]. In Eq. (110), the subscripts "V" should be "L". Equation (118) should have an additional term

$$+[U_{12}(z')U_{12}(d)^{-1}]\otimes 1(H_V^\circ(\omega, Q, z-d))_L$$

and the entire resulting expression should be multiplied by -1, as should the right side of Eq. (119). In Eq. (A24) the factor "$\frac{1}{2}$" should be "2". In Eq. (B5), the 31 element should be $\frac{1}{4}Q_y$, not $\frac{1}{2}Q_y$, and the 22 element of the matrix in Eq. (B9) should involve Q_y^2 instead of Q_x^2. In Eq. (B16), $1+\sigma$ should be $1-\sigma$ and E^{-1} should be replaced by $(1-\sigma^2)E^{-1}$ throughout this equation. The prefactor of the matrix in Eq. (B19) should contain c_t instead of $V_L(=\sqrt{2}c_t)$. In Eq. (B20), Q_p should be QV_p. The right side of Eq. (B22) should be multiplied by -1. Note that the velocities defined by Eq. (B13) and (B21) are identical. Though not in error, the different choice of sign convention for the arguments of the exponentials in Eqs. (85) and (96) is potentially misleading.

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